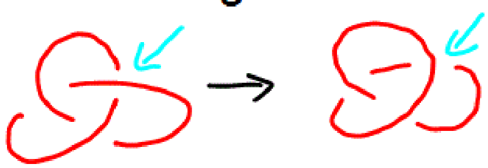
3.1: Unknotting Number

Note Title 3/2/2010

d(K, K') = the minimum number of crossing changes needed to convert K into K'.

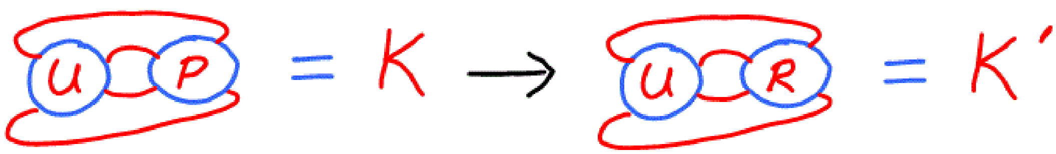
u(K) = *unknotting number* of K = d(K, 0_1).

Ex:  $\Rightarrow d(3_1, 0_1) = u(3_1) = 1$

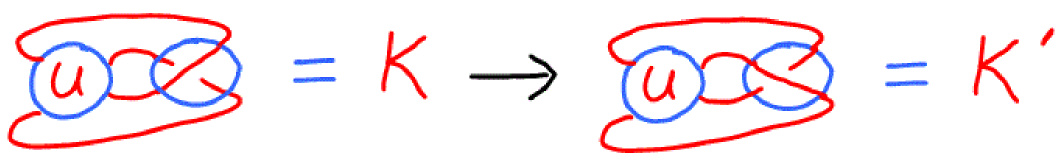
Defn: If there exists a solution for U such that

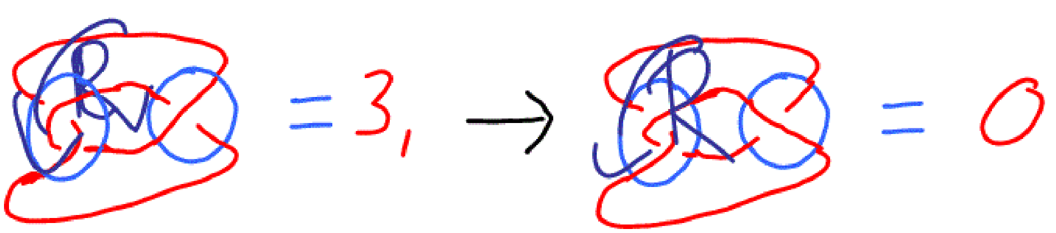
N(U + P) = K and N(U + R) = K',

then K' is said to have been obtained from K by a (P, R) move.



d(K, K') = 1 iff K ≠ K' and K' can be obtained from K by a (-1, 1) move.





Note d is a *metric* on the space of 1-component knots.

" distance

0) d(K, K') is a finite nonnegative integer.

1) d(K, K') = 0 iff K = K'.

$\Rightarrow$ 0 crossing changes to convert K to K' $\Leftrightarrow$ K = K'

2) d(K, K') = d(K', K).

$K \leftrightarrow K_1 \leftrightarrow K_2 \cdots \leftrightarrow K_n \leftrightarrow K'$

3) d(K, K') ≤ d(K, K₁) + d(K₁, K'). ← Δ ineq

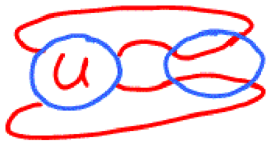
$n + l$

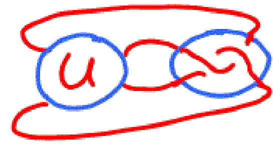
$K \to H_1 \to H_2 \to \cdots \to H_n \to K_1$

$\downarrow$

$K' \leftarrow H_l \quad \cdots \leftarrow H_{n+1}$

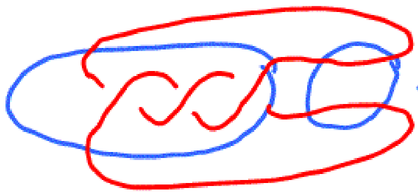
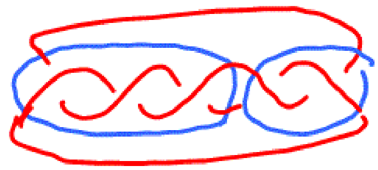
$d_2(K, K') = 1$ iff $K \neq K'$ and K' can be obtained from K by a (0, 2) move.



$= K \longrightarrow$


$= K'$

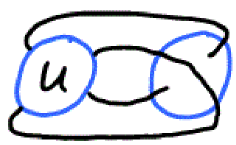
Ex:


$= 3_1^* \to$

$= 0_1$

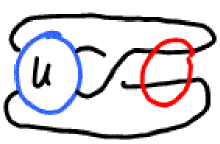
$d(K, K') = 1$ iff $d_2(K, K') = 1$

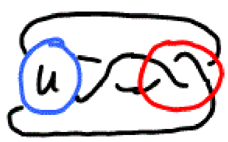
Pf ($\Rightarrow$) Suppose $d(K, K') = 1$

Then there exists U st

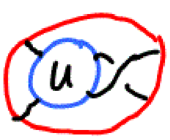
$= K$,


$= K'$

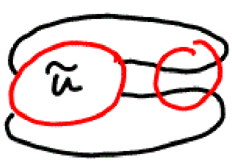
$\Rightarrow$


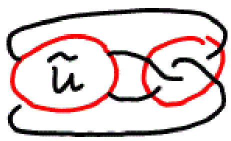
$= K$,


$= K'$

$\Rightarrow$


is a sol'n to

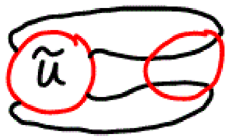
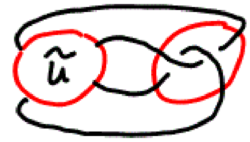


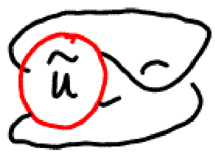
$= K$,


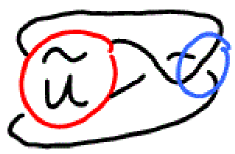
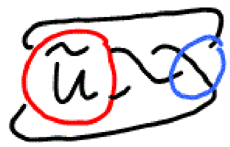
$= K' \Rightarrow d_2(K, K') = 1$

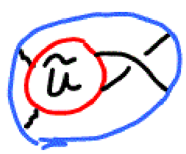
($\Leftarrow$) Suppose $d_2(K,K')=1$

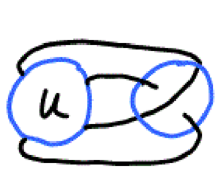
Then there ex's $\tilde{u}$ st

 $=K$,  $=K'$

$\Rightarrow$  $=K$,  $=K'$

$\Rightarrow$  $=K$,  $=K'$

$\Rightarrow$  is a sol'n to

 $=K$,  $=K'$

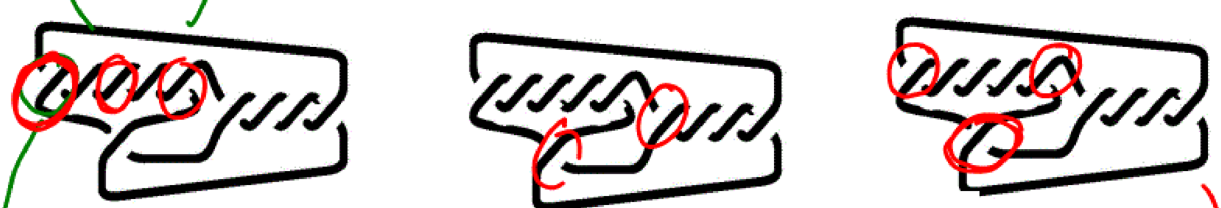
$\Rightarrow d(K,K')=1$

A P-R move is *equivalent* to a P’-R’ move
if and only if for every pair of Knots, K and K’,
the system of tangle equations
N(U + P) = K, N(U + R) = K’ has a solution
if and only if the system of tangle equations
N(U + P’) = K, N(U + R’) = K’ has a solution.

Let D = a diagram of K. N(5, 1, -1) → unknot
Claim: d(K, K’) = min{minimum number of crossing changes needed to convert D into a diagram D’ for K'}.

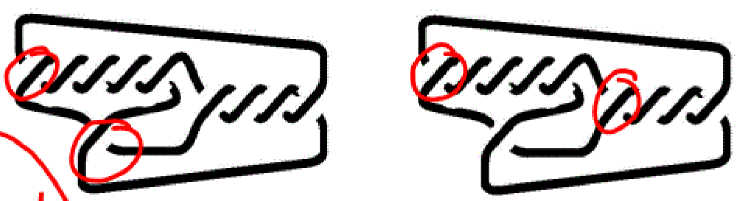
where min is taken over all diagrams D of K.

Ex: N(514)



N(3 1 4)
N(1 1 4)
N(1 -1 4) = unknot

N(-1 1 4)



$4+\frac{1}{1+-1}=\frac{1}{0}$ = unknot ⇒ ul

$\Rightarrow \boxed{u(N(5,14)) \leq 3}$

Check 2 crossings

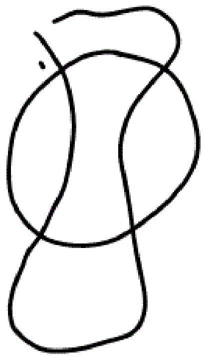
$N(5\ 14) \to\to N(1\ 1\ 4)$ ✓

$\to\to N(3\ {-1}\ 4)$ ✓

$\to\to N(3\ 1\ 2)$ ✓

$\to\to N(5\ {-1}\ 2)$

$\to\to N(5\ 1\ 0))$

 $= N\left(\frac{1}{0}\right) = \text{unknot}$

If $N\left(\frac{a}{b}\right) = \text{unknot}$

$\Rightarrow a = \pm 1$

$$N(3 \; {-1} \; 4) =$$

$$N\left(4 + \frac{1}{-1 + \frac{1}{3}}\right)$$

$$= N\left(4 + \frac{1}{-2/3}\right) = N\left(4 - \frac{3}{2}\right)$$

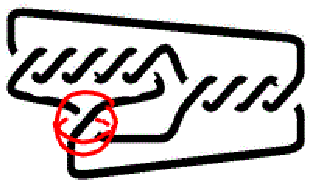
$\neq$ unKnot

$$N\left(2 + \frac{1}{-1 + \frac{1}{3}}\right) = N\left(2 + \frac{-5}{4}\right)$$

$$N\left(0 + \frac{1}{1 + \frac{1}{5}}\right) = N\left(\frac{5}{6}\right) \neq \text{unKnot}$$

d(K, K') = the minimum number of crossing changes needed to convert K into K'.

Ex: N(5 1 4)



Allow deformations btwn crossing changes

(Not Adams defn)

$N(5\ 1\ 4) \Rightarrow N(5\ {-1}\ 4)$

$$4 + \frac{1}{-1+\frac{1}{5}} \rightarrow 4 + \frac{1}{-4/5} \rightarrow 4 + \frac{-5}{4} = 11/4$$

$N(11/4)$

$$11/4 = 2 + \frac{3}{4} = 2 + \frac{1}{4/3} = 2 + \frac{1}{1+\frac{1}{3}}$$

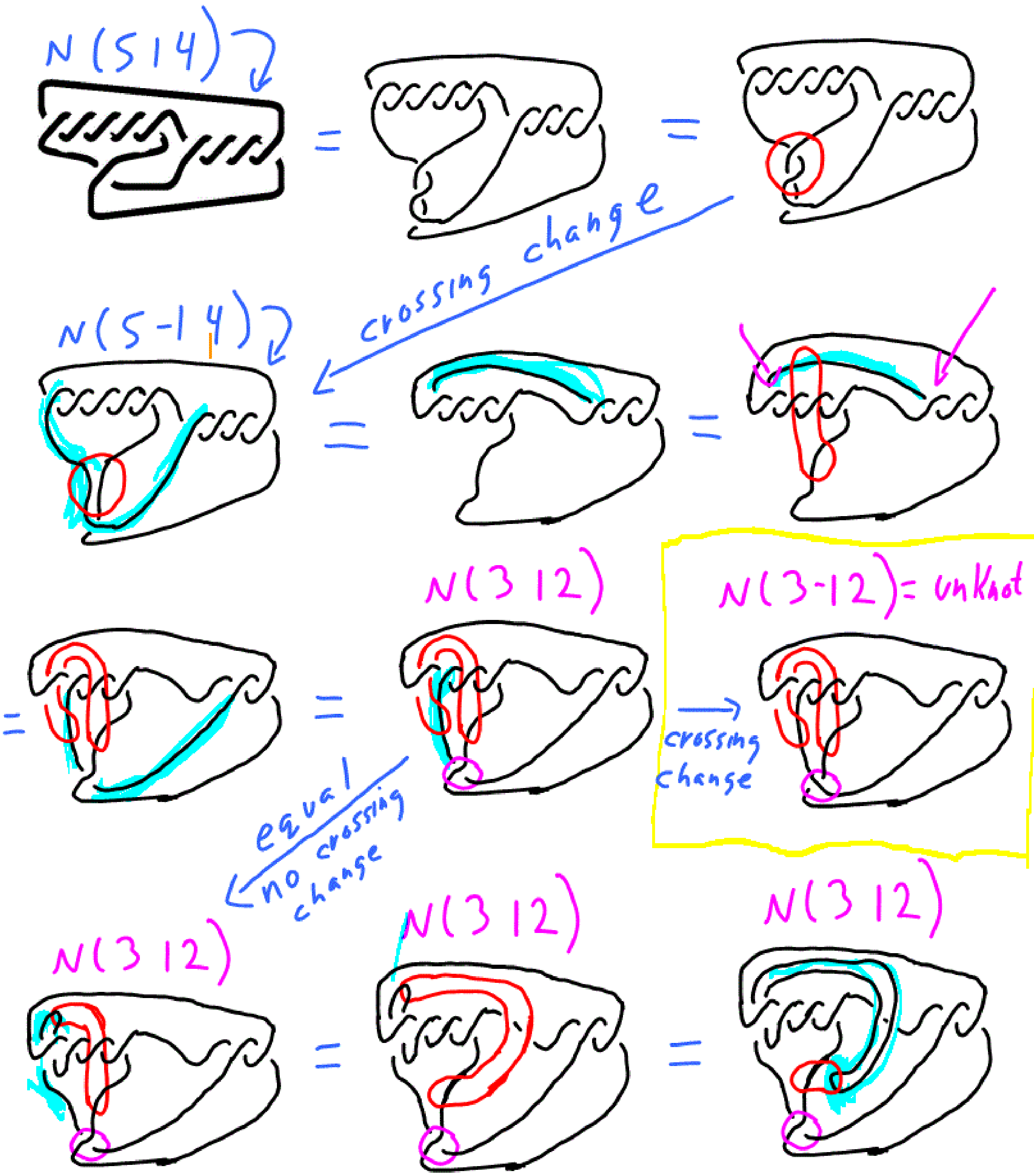
$$\Rightarrow N(11/4) = N(3\ 1\ 2)$$

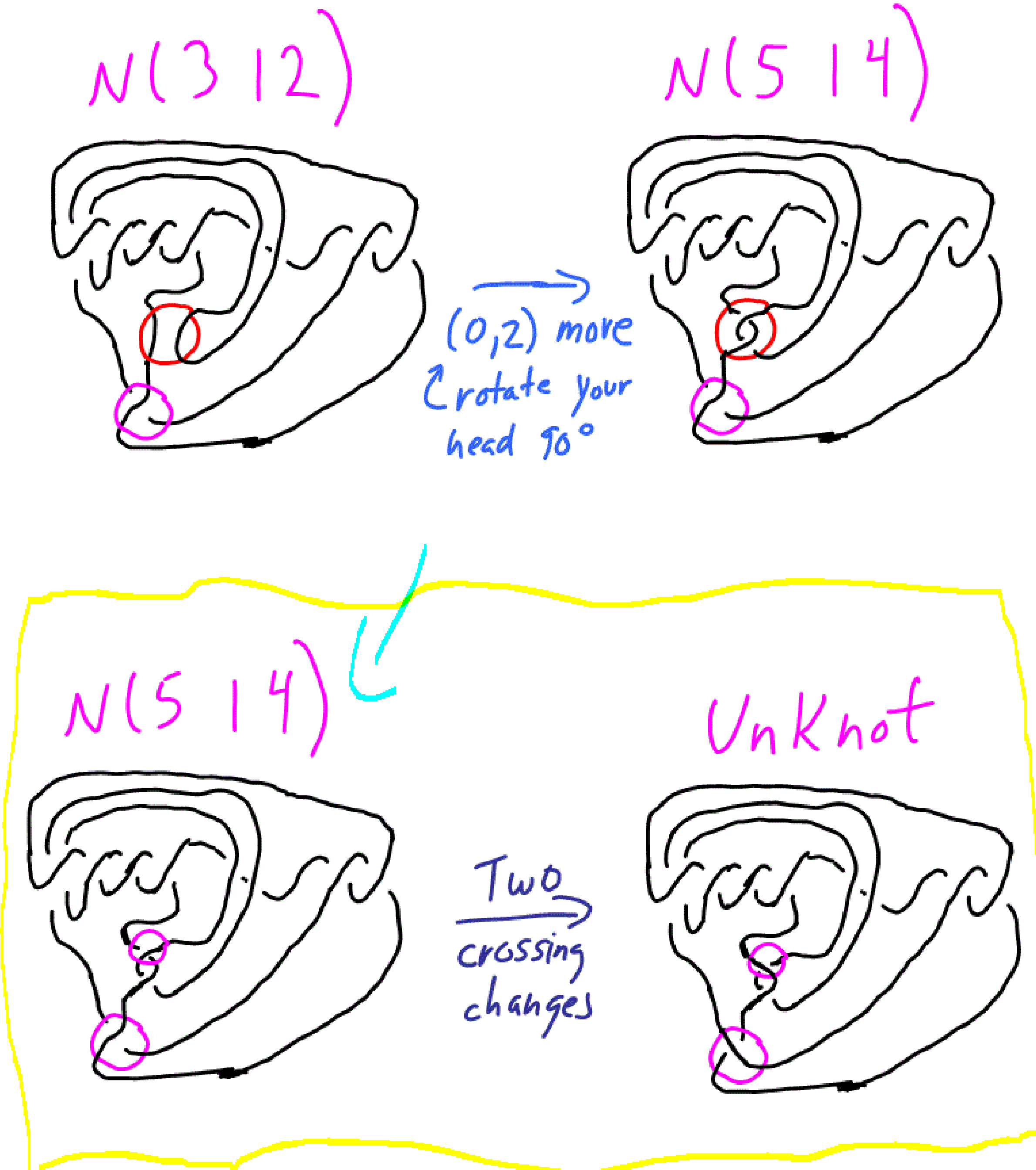
$$N(3\ 1\ 2) \to N(3\ {-1}\ 2)$$

$$2 + \frac{1}{-1+\frac{1}{3}} = 2 + \frac{-3}{2} = \frac{1}{2}$$

$N(\frac{1}{2})$ = unknot

$d(N(5/14), \text{unknot}) \leq 2$


N(5 1 4)
=
=
crossing change
N(5-1 4)
=
=
N(3 12)
N(3-12)= unknot
=
=
crossing change
equal
no crossing change
N(3 12)
N(3 12)
N(3 12)
=
=


N(3 12)
N(5 14)
(0,2) move
(rotate your head 90°
N(5 14)
Unknot
Two crossing changes


N(514)
N(312)

Rational angles:

$$a/b = c/d$$

iff $\frac{a}{b} = \frac{c}{d}$

Rational Knots