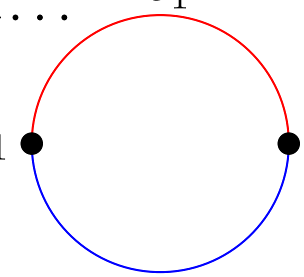
$$\cdots \longrightarrow H_1(A) \oplus H_1(B) \longrightarrow H_1(S^1) \xrightarrow{\partial_*} H_0(A \cap B) \xrightarrow{\phi_*} H_0(A) \oplus H_0(B) \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \oplus 0 \longrightarrow H_1(S^1) \xrightarrow{\partial_*} \mathbb{Z}^2 \xrightarrow{\phi_*} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \cdots$$



$0 \oplus 0 \to H_1(S^1) \xrightarrow{\partial_*} \mathbb{Z}^2$ implies ∂ is 1:1. Thus $H_1(S^1) \cong im\partial_* = ker\phi_*$.

$\phi_*([n_1v_1 + n_2v_2]) = ([n_1v_1 + n_2v_2], [n_1v_1 + n_2v_2]) = \mathbf{0}$
iff $n_1v_1 + n_2v_2 = \partial(\sigma_i)$ for $\sigma_1 \in C_1(A)$ and $\sigma_2 \in C_1(B)$

$C_1(A) = \{ne_1 \mid n \in \mathbb{Z}\}$ and $\partial(e_1) = v_2 - v_1$. Thus $B_1(A) = \{n(v_2 - v_1) \mid n \in \mathbb{Z}\}$

Thus in $H_1(A)$, $[v_1] = [v_2]$.

Thus in $H_1(A)$,
$[n_1v_1 + n_2v_2] = [n_1v_1 + n_2v_1] = [(n_1 + n_2)v_1] = [0]$ iff $n_1 + n_2 = 0$. I.e, $n_2 = -n_1$.

Similarly in $H_1(B)$, $[n_1v_1 + n_2v_2] = [0]$ iff $n_2 = -n_1$.

Thus $H_1(S^1) \cong ker\phi_* = \{n_1v_1 - n_1v_2 \mid n_1 \in Z\} = \{n_1(v_1 - v_2) \mid n_1 \in Z\} \cong \mathbb{Z}$.

Recall reduced homology for $X \neq \emptyset$

$$\to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0 \qquad \text{where } \epsilon(\sum n_i v_i) = \sum n_i$$

$\widetilde{H_n}(X) = H_n(X)$ for $n > 0$ and $\widetilde{H_0}(X) = \frac{ker(\epsilon)}{im(\partial_1)} = H_0(X) \oplus \mathbb{Z}$

Thus $\widetilde{H_0}(X) =$ (the number of components of X) - 1 when X can be triangulated.

Using reduced homology:

$$\cdots \longrightarrow H_1(A) \oplus H_1(B) \longrightarrow H_1(S^1) \xrightarrow{\partial_*} \widetilde{H_0}(A \cap B) \xrightarrow{\phi_*} \widetilde{H_0}(A) \oplus \widetilde{H_0}(B) \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \oplus 0 \longrightarrow H_1(S^1) \xrightarrow{\partial_*} \mathbb{Z} \xrightarrow{\phi_*} 0 \oplus 0 \longrightarrow \cdots$$

$0 \oplus 0 \to H_1(S^1) \xrightarrow{\partial_*} \mathbb{Z}$ implies ∂ is 1:1. Thus $H_1(S^1) \cong im\partial_* = ker\phi_* \cong \mathbb{Z}$.

6.) Finish the proof of the zig-zag lemma. In particular, show that ∂_* is a homomorphism and that the sequence is exact at $H_n(\mathcal{E})$ and $H_{n-1}(\mathcal{C})$

7.) Use reduced homology and Meyer Vietoris to calculate $\widetilde{H_n}(S^k)$ for all $k, n \in \mathbb{N}$