Singular homology

Generators of $C_n(X) = \{\sigma_\alpha : \Delta^n_\alpha \to X \mid \sigma_\alpha \text{ is continuous}\}$.

Let $\sigma : (v_0, ..., v_n) \to X$ be continuous.

$$\partial_n(\sigma) = \sum_{i=1}^{n} (-1)^i \sigma|(v_0, ..., \widehat{v_i}, ..., v_n) \in C_{n-1}(X)$$

Thus $\partial^2 = 0$ and $H_n(X) = Z_n(X)/B_n(X)$ is well defined where

$Z_n = ker(\partial_n) =$ cycles and $B_n = im(\partial_{n+1}) =$ boundaries.

Suppose $f : X \to Y$ is continuous.

f induces the homomorphism $f_\# : C_n(X) \to C_n(Y)$

$f_\#(\sigma : \Delta \to X) = f \circ \sigma : \Delta \to Y$ and extend linearly.

Note: $f_\# \circ \partial = \partial \circ f_\#$

$$\begin{aligned} f_\#(\partial_n(\sigma)) &= f_\#(\sum_{i=1}^{n} (-1)^i \sigma|(v_0, ..., \widehat{v_i}, ..., v_n)) \\ &= \sum_{i=1}^{n} (-1)^i (f_\# \sigma|(v_0, ..., \widehat{v_i}, ..., v_n)) \\ &= \partial_n(f_\#(\sigma)) \end{aligned}$$

If σ is a cycle, then $f_\#(\sigma)$ is a cycle. Thus $f_\#(Z_n(X)) \subset Z_n(Y)$.

If $\sigma = \partial(\beta)$, then $f_\#(\sigma) = f_\#(\partial(\beta)) = \partial(f_\#(\beta))$

Thus $f_\#(B_n(X)) \subset B_n(Y)$.

Hence $f_\# : C_n(X) \to C_n(Y)$ induces

a homomorphism $f_* : H_n(X) \to H_n(Y)$

If $f : X \to Y$ is a homeomorphism, then

$f_\# : C_n(X) \to C_n(Y)$ is an isomorphism

and $f_* : H_n(X) \to H_n(Y)$ is an isomorphism

Thus singular homology is a topological invariant.

Prop 2.6: Suppose X_α are the path components of X. Then $C_n(X) = \bigoplus_\alpha C_n(X_\alpha)$ and $H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$

Prop 2.7: If X is non-empty, path-connected, then $H_0(X) = \mathbb{Z}$.

Prop 2.8: $H_n(point) = 0$ for $n > 0$ and $H_0(point) = \mathbb{Z}$.

Reduced homology

The reduced homology groups $\widetilde{H}_n(X)$ are the homology groups of the augmented chain complex:

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $\epsilon(\sum_i n_i\sigma_i) = \sum_i n_i$. Note $\epsilon\partial_1 = 0$.

Thus ϵ induces a map $H_0 = C_0/Im(\partial_1) \to \mathbb{Z}$ w/ kernel $\widetilde{H}_0(X)$.

Thus $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$. $[\sigma] \to ([\sigma - \epsilon(\sigma)x], \epsilon(\sigma))$

For $n > 0$, $\widetilde{H}_n(X) = ker(\partial_n)/Im(\partial_{n+1}) = H_n(X)$.

Hence $\widetilde{H}_n(point) = 0$ for all n.

$$... \to G_n \xrightarrow{\partial} ... \to G_1 \to G_0 \to 0.$$

Category

A chain complex is a sequence of homomorphisms of abelian groups:

$$... \to G_n \to ... \to G_1 \to G_0 \to 0.$$

Suppose $f : X \to Y$ is continuous.

f induces the homomorphism $f_\# : C_n(X) \to C_n(Y)$

$f_\#(\sigma : \Delta \to X) = f \circ \sigma : \Delta \to Y$ and extend linearly.

$f_\# \partial(\sigma) = f_\#(= \partial f_\#(\sigma)$

A *chain map* $\phi : (C_\bullet, \partial_\bullet) \to (D_\bullet, \partial'_\bullet)$ is a collection of homomorphisms $\phi_n : C_n \to D_n$ such that the following diagram commutes.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & 0 \\
& \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \| \\
\cdots \longrightarrow & D_2 & \xrightarrow{\partial'_2} & D_1 & \xrightarrow{\partial'_1} & D_0 & \xrightarrow{\partial'_0} & 0
\end{array}$$

That is, such that $\phi_{n-1} \circ \partial_n = \partial'_n \circ \phi_n$ for all $n \geq 0$.

Objects: Chain complexes

Morphisms: Chain maps

Since $f_\# \partial = \partial f_\#$, $f_\#$ induces a homomorphism

$$f_* : H_n(X) \to H_n(Y).$$

Categories, Functors, Natural Transformations (modified from

Defn 1 . *A* **category** C *consists of:*

- *a collection* $\mathrm{Ob}(C)$ *of* **objects***.*
- *for any pair of objects* x, y*, a set* $\hom(x, y)$ *of* **morphisms** *from* x *to* y*. (If* $f \in \hom(x, y)$ *we write* $f\colon x \to y$*.)*

equipped with:

- *for any object* x*, an* **identity morphism** $1_x\colon x \to x$*.*
- *for any pair of morphisms* $f\colon x \to y$ *and* $g\colon y \to z$*, a morphism* $fg\colon x \to z$ *called the* **composite** *of* f *and* g*.*

such that:

- *for any morphism* $f\colon x \to y$*, the* **left and right unit laws** *hold:* $1_x f = f = f 1_y$*.*
- *for any triple of morphisms* $f\colon w \to x$*,* $g\colon x \to y$*,* $h\colon y \to z$*, the* **associative law** *holds:* $(fg)h = f(gh)$*.*

We usually write $x \in C$ as an abbreviation for $x \in \mathrm{Ob}(C)$. An **isomorphism** is a morphism $f\colon x \to y$ with an **inverse**, i.e. a morphism $g\colon y \to x$ such that $fg = 1_x$ and $gf = 1_y$.

Defn 2 . *Given categories C, D, a* **functor** $F\colon C \to D$ *consists of:*

- *a function* $F\colon \mathrm{Ob}(C) \to \mathrm{Ob}(D)$.
- *for any pair of objects* $x, y \in \mathrm{Ob}(C)$,
  a function $F\colon \hom(x,y) \to \hom(F(x), F(y))$.

such that:

- F **preserves identities***: for any object* $x \in C$, $F(1_x) = 1_{F(x)}$.
- F **preserves composition***: for any pair of morphisms* $f\colon x \to y$, $g\colon y \to z$ *in* C, $F(fg) = F(f)F(g)$.

It's not hard to define identity functors & composition of functors, & to check the left & right unit law & associative law for these.

Defn 3 . *Given functors $F, G\colon C \to D$, a* **natural transformation** $\alpha\colon F \Rightarrow G$ *consists of:*

- *a function* α *mapping each object* $x \in C$ *to a morphism* $\alpha_x\colon F(x) \to G(x)$

such that:

- *for any morphism* $f\colon x \to y$ *in* C, *this diagram commutes:*

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle \alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ G(x) & \xrightarrow[G(f)]{} & G(y) \end{array}$$

With a little thought you can figure out how to compose natural transformations $\alpha\colon F \to G$ and $\beta\colon G \Rightarrow H$ and get a natural transformation $\alpha\beta\colon F \Rightarrow H$. We can also define identity natural transformations. Again, it's not hard to check the left and right unit law and associativity for these.

Defn 4 . *Given functors $F, G\colon C \to D$, a* **natural isomorphism** *$\alpha\colon F \Rightarrow G$ is a natural transformation that has an* **inverse**, *i.e. a natural transformation $\beta\colon G \Rightarrow F$ such that $\alpha\beta = 1_F$ and $\beta\alpha = 1_G$.*

It's not hard to see that a natural transformation $\alpha\colon F \Rightarrow G$ is a natural isomorphism iff for every object $x \in C$, the morphism α_x is invertible.

Defn 5 . *A functor $F\colon C \to D$ is an* **equivalence** *if it has a* **weak inverse**, *that is, a functor $G\colon D \to C$ such that there exist natural isomorphisms $\alpha\colon FG \Rightarrow 1_C$, $\beta\colon GF \Rightarrow 1_D$.*