Thm 2.10: If $f, g : X \to Y$ are homotopic,

then $f_* = g_* : H_(X) \to H_n(Y)$.

Proof: Let $F : X \times I \to Y$ be a homotopy from f to g.

Let $\sigma \in C_n(X)$. I.e., $\sigma : \Delta^n \to X$.

Note $F \circ (\sigma \times id) : \Delta \times I \xrightarrow{\sigma \times id} X \times I \xrightarrow{F} Y$

But $F \circ (\sigma \times id)$ is not a singular simplex.

Thus define prism operator $P : C_n(X) \to C_{n+1}(Y)$.

$$P(\sigma) = \sum_i (-1)^i F(\sigma \times id)|_{[v_0,\ldots,v_i,w_i,\ldots w_n]} \in C_{n+1}(Y)$$

Claim: P is a chain homotopy from $g_\#$ to $f_\#$.

That is $\partial P + P\partial = g_\# - f_\#$.

$$(\partial \circ P)(\sigma) = \partial(\sum_i (-1)^i F(\sigma \times id)|_{[v_0,\ldots,v_i,w_i,\ldots w_n]})$$

$$= \sum_{j \leq i} (-1)^i (-1)^j F(\sigma \times id)|_{[v_0,\ldots,\widehat{v_j},\ldots,v_i,w_i,\ldots w_n]}$$
$$+ \sum_{j \geq i} (-1)^i (-1)^{j+1} F(\sigma \times id)|_{[v_0,\ldots,v_i,w_i,\ldots \widehat{w_j},\ldots,w_n]}$$

$$P(\partial(\sigma)) = P\left(\sum_{j=0}^{n} (-1)^j \sigma|_{[v_0,\ldots,\widehat{v_j},\ldots,v_n]}\right)$$

$$= \sum_{j<i} (-1)^{i-1} (-1)^j F(\sigma \times id)|_{[v_0,\ldots,\widehat{v_j},\ldots,v_i,w_i,\ldots w_n]}$$
$$+ \sum_{j>i} (-1)^i (-1)^j F(\sigma \times id)|_{[v_0,\ldots,v_i,w_i,\ldots \widehat{w_j},\ldots,w_n]}$$

Thus $\partial P + P\partial = \sum_{i=0}^{n}(-1)^i(-1)^i F(\sigma \times id)|_{[v_0,\dots,v_{i-1},w_i,\dots w_n]}$

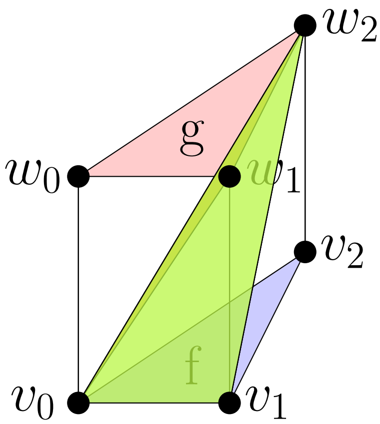
$$+ \sum_{i=0}^{n}(-1)^i(-1)^{i+1} F(\sigma \times id)|_{[v_0,\dots,v_i,w_{i+1},\dots,w_n]}$$

$$= F\circ(\sigma\times id)|_{[w_0,\dots,w_n]} - F\circ(\sigma\times id)|_{[v_0,w_1,\dots,w_n]}$$

$$+ F\circ(\sigma\times id)|_{[v_0,w_1,\dots,w_n]} - F\circ(\sigma\times id)|_{[v_0,v_1,w_2,\dots,w_n]}$$

$$+ F\circ(\sigma\times id)|_{[v_0,v_1,w_2,\dots,w_n]} - \cdots - F\circ(\sigma\times id)|_{[v_0,\dots,v_{n-1},w_n]}$$

$$+ F\circ(\sigma\times id)|_{[v_0,\dots,v_{n-1},w_n]} - F\circ(\sigma\times id)|_{[v_0,\dots,v_n]} = g_\# - f_\#.$$



$\xrightarrow{\sigma\times id} X \xrightarrow{F} Y \ \in\ C_3(Y)$

Defn: $\cdots \to G_1 \xrightarrow{f} G_2 \xrightarrow{h} G_3 \to \cdot$

This sequence is **exact at** G_2 if $im(f) = ker(h)$.

If the sequence is everywhere exact, then the sequence is said to be an **exact sequence**.

A **long exact sequence** is an exact sequence indexed by the set of integers.

If the sequence $0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{h} G_3 \to 0$ is exact, then it is a **short exact sequence**.

1.) $G_2 \xrightarrow{h} G_3 \to 0$ is exact iff h is onto.

2.) $0 \to G_1 \xrightarrow{f} G_2$ is exact iff f is 1:1.

3.) Given the short exact sequence $0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{h} G_3 \to 0$

$$G_2/f(G_1) = G_2/ker(h) \cong G_3$$

Example of a short exact sequence if h is onto:

$$0 \to ker(h) \hookrightarrow G_2 \xrightarrow{h} G_3 \to 0$$

4.) If $G_1 \xrightarrow{f} G_2 \xrightarrow{h} G_3 \xrightarrow{k} G_4$ is exact

TFAE

(i) f is onto (epimorphism).

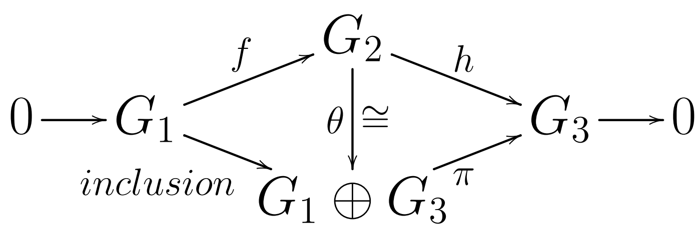
(iii) h is the 0-map.

(ii) k is 1:1 (monomorphism).

5.) The exact sequence $G_1 \xrightarrow{f} G_2 \xrightarrow{\alpha} G_3 \xrightarrow{\beta} G_4 \xrightarrow{h} G_5$ induces short exact sequence ($G_2/Im(f) = cok(f) =$ cokernel of f):

$$0 \to cok(f) \xrightarrow{\alpha'} G_3 \xrightarrow{\beta'} ker(h) \to 0$$

Defn: The short exact sequence $0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{h} G_3 \to 0$ **splits** if $G_2 = f(G_1) \oplus B$ for some group B.



$\theta(g_2) = (f^{-1}(g_2), h(g_2))$.

Thm: If $0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{h} G_3 \to 0$ is exact, then TFAE

i) The sequence splits.

ii.) $\exists p : G_2 \to G_1$ such that $p \circ f = id_{G_1}$

iii.) $\exists j : G_3 \to G_2$ such that $h \circ j = id_{G_3}$

Cor: Let $0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{h} G_3 \to 0$ be exact. If G_3 is free abelian, then the sequence splits.

Defn: Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be chain complexes. Let $f : \mathcal{C} \to \mathcal{D}$ and $h : \mathcal{D} \to \mathcal{E}$ be chain maps. Then the sequence

$$0 \to \mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{h} \mathcal{E} \to 0$$

is a **short exact sequence of chain complexes** if in each dimension n, the sequence

$$0 \to C_n \xrightarrow{f} D_n \xrightarrow{h} E_n \to 0$$

is an exact sequence of groups.