Define $h : H^n(C;G) \to Hom(H_n(C), G)$, $h([\phi]) = \overline{\phi_0}$ as follows:

If $[\phi] \in H^n(C;G)$, then $\phi \in Z^n = Ker(\delta)$.

I.e., $\phi : C_n \to G$ such that $\delta\phi = \phi\partial = 0$. Recall that $B_n = \partial(C_{n+1})$.

Note: $\phi \in Ker(\delta) = Z^n$ iff $\phi(\partial(C_{n+1})) = \phi(B_n) = 0$.

Since $\phi(B_n) = 0$, $\phi_0 = \phi|_{Z_n}$ induces quotient homomorphism:

$$\overline{\phi_0} : Z_n/B_n = H_n \to G.$$

Thus $\overline{\phi_0} \in Hom(H_n(C), G)$

Claim: h is onto:

Suppose $\phi_0 : Z_n \to G$. B_{n-1} free implies the following SES splits:

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial} B_{n_1} \longrightarrow 0$$

Thus $\exists p : C_n \to Z_n$ such that $p \circ i = id_{Z_n}$

Let $\phi = \phi_0 \circ p : C_n \to Z_n \to G$. Then $\phi|_{Z_n} = \phi_0$.

If $\phi_0(B_n) = 0$, then $\phi(B_n) = 0$. Thus $\phi \in Ker(\delta)$.

$$\begin{array}{ccccccccc} Hom(H_n(C), G) & \to & ker\delta & \to & \frac{ker(\delta)}{im(\delta)} = H^n(C, G) & \xrightarrow{h} & Hom(H_n(C), G) \\ \overline{\phi_0} & \to & \phi & \to & [\phi] & \xrightarrow{h} & \overline{\phi_0} \end{array}$$

Thus we have a split exact SES:

$$0 \to Ker(h) \to H^n(C;G) \xrightarrow{h} Hom(H_n(C),G) \to 0$$

$Ker(h) =?$

The dualization of the following chain map between split SES

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r, "i"] \arrow[d, "0"'] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"'] & B_n \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "i"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

gives us the following chain maps between split SES:

$$\begin{tikzcd}
0 & Z^*_{n+1} \arrow[l] & C^*_{n+1} \arrow[l, "i^*"'] & B^*_n \arrow[l, "\delta"'] & 0 \arrow[l] \\
0 & Z^*_n \arrow[l] \arrow[u, "0"] & C^*_n \arrow[l, "i^*"'] \arrow[u, "\delta"] & B^*_{n-1} \arrow[l, "\delta"'] \arrow[u, "0"] & 0 \arrow[l]
\end{tikzcd}$$

Note that the following is a chain complex with $\frac{ker}{im} = X$

$$... \xleftarrow{0} X \xleftarrow{0} X \xleftarrow{0} X \xleftarrow{0} ...$$

Thus the chain maps above imply the LES:

$$... \longleftarrow B^*_n \xleftarrow{i^*_n} Z^*_n \xleftarrow{\imath^*_*} H^n(C;G) \xleftarrow{\delta_*} B^*_{n-1} \longleftarrow Z^*_{n-1} \longleftarrow ...$$

LES implies SES:

$$0 \longleftarrow Ker(i^*_n) \longleftarrow H^n(C;G) \longleftarrow Coker(i^*_{n-1}) = \frac{B^*_{n-1}}{Z^*_{n-1}} \longleftarrow 0$$

$$Ker(i_n^*) = \{\phi_0 : Z_n \to G \;\mid\; \phi_0(B_n) = 0\}$$
$$\cong \{\overline{\phi_0} : Z_n/B_n = H_n \to G\} = Hom(H_n(C), G)$$

$$0 \to Coker(i_{n-1}^*) = \frac{B_{n-1}^*}{Z_{n-1}^*} \to H^n(C;G) \to Hom(H_n(C), G) \to 0$$

The following is a free resolution of $H_{n-1}(C)$:

$$0 \longrightarrow B_{n-1} \xrightarrow{i} Z_{n-1} \xrightarrow{\partial} H_{n-1}(C) \longrightarrow 0$$

with (non-exact) dualization

$$0 \longleftarrow B_{n-1}^* \xleftarrow{i^*} Z_{n-1}^* \xleftarrow{\partial^*} H_{n-1}^*(C) \longleftarrow 0$$

Thus $Ext(H_{n-1}(C), G) = \frac{B_{n-1}^*}{im(i^*)} = \frac{B_{n-1}^*}{Z_{n-1}^*}$

Thus we have the SES:

$$0 \to Ext(H_{n-1}(C), G) \to H^n(C;G) \to Hom(H_n(C), G) \to 0$$

$Ext(H_{-1}(C), G) = 0$ since $H_{-1} = 0$.
$Ext(H_0(C), G) = 0$ since H_0 is free abelian

Thus for $n = 0, 1$: $H^n(C;G) \cong Hom(H_n(C), G)$ since

$$0 \to 0 \to H^n(C;G) \to Hom(H_n(C), G) \to 0$$

Section 2.3: The Formal Viewpoint (Homology)

Definition 1. *A* (reduced) homology theory *is a sequence of covariant functors* $\widetilde{h}_n$ *from the category of CW complexes to the category of abelian groups which satisfy the following axioms.*

(1) If $f \simeq g$*, then* $f_* = g_* \colon \widetilde{h}_n(X) \to \widetilde{h}_n(Y)$.

(2) There are boundary homomorphisms $\partial \colon \widetilde{h}_n(X/A) \to \widetilde{h}_{n-1}(A)$ *defined for each CW pair* (X, A)*, fitting into an exact sequence*

$$\cdots \xrightarrow{\partial} \widetilde{h}_n(A) \xrightarrow{i_*} \widetilde{h}_n(X) \xrightarrow{q_*} \widetilde{h}_n(X/A) \xrightarrow{\partial} \widetilde{h}_{n-1}(A) \xrightarrow{i_*} \cdots ,$$

where $i \colon A \to X$ *and* $q \colon X \to X/A$ *are respectively the evident inclusion and quotient maps. Furthermore, the boundary mas are natural: for* $f \colon (X, A) \to (Y, B)$ *inducing a quotient map* $\overline{f} \colon X/A \to Y/B$*, the diagrams*

$$\begin{tikzcd} \widetilde{h}_n(X/A) \arrow[r, "\partial"] \arrow[d, "\overline{f}_*"'] & \widetilde{h}_{n-1}(A) \arrow[d, "f_*"] \\ \widetilde{h}_n(Y/B) \arrow[r, "\partial"] & \widetilde{h}_{n-1}(B) \end{tikzcd}$$

commute.

(3) For a wedge sum $X = \bigvee_\alpha X_\alpha$ *with inclusions* $i_\alpha \colon X_\alpha \hookrightarrow X$*, the direct sum map*

$$\bigoplus_\alpha (i_\alpha)_* \colon \bigoplus_\alpha \widetilde{h}_n(X_\alpha) \to \widetilde{h}_n(X)$$

is an isomorphism for all n.