Defn: Let $\mathcal{C} = \{C_p, \partial_C\}$, $\mathcal{D} = \{D_p, \partial_D\}$, $\mathcal{E} = \{E_p, \partial_E\}$ be chain complexes. Let $f : \mathcal{C} \to \mathcal{D}$ and $h : \mathcal{D} \to \mathcal{E}$ be chain maps. Then the sequence

$$0 \to \mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{h} \mathcal{E} \to 0$$

is a **short exact sequence of chain complexes** if in each dimension n, the sequence

$$0 \to C_n \xrightarrow{f} D_n \xrightarrow{h} E_n \to 0$$

is an exact sequence of groups.

In other words, the following diagram commutes:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \to \cdots \\
 & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & \\
\cdots \to & D_{n+1} & \xrightarrow{\delta_{n+1}} & D_n & \xrightarrow{\delta_n} & D_{n-1} & \to \cdots \\
 & \downarrow{\scriptstyle g_{n+1}} & & \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle g_{n-1}} & \\
\cdots \to & E_{n+1} & \xrightarrow{d_{n+1}} & E_n & \xrightarrow{d_n} & E_{n-1} & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

Lemma. The Zig-Zag Lemma: Given chain complexes, $\mathcal{C} = \{C_n, \partial_C\}$, $\mathcal{D} = \{D_n, \partial_D\}$, $\mathcal{E} = \{E_n, \partial_E\}$ and chain maps f and g such that the following sequence is exact:

$$0 \to \mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{h} \mathcal{E} \to 0$$

Then $\exists$ long exact homology sequence:

$$\cdots \to H_n(\mathcal{C}) \xrightarrow{f_*} H_n(\mathcal{D}) \xrightarrow{h_*} H_n(\mathcal{E}) \xrightarrow{\partial_*} H_{n-1}(\mathcal{C}) \xrightarrow{f_*} H_{n-1}(\mathcal{D}) \to \cdots$$

where ∂_* is induced by ∂_D. That is, $\partial_*([e_n]) = [c_{n-1}]$ where $h(d_n) = e_n$ and $f(c_{n-1}) = \partial_D(d_n)$.

Proof. By diagram chasing. □

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow \partial_C & & \downarrow \partial_D & & \downarrow \partial_E & & \\
0 & \to & C_{n+1} & \xrightarrow[f_{n+1}]{1:1} & D_{n+1} & \xrightarrow[h_{n+1}]{\textit{onto}} & E_{n+1} & \to & 0 \\
 & & \downarrow \partial_C & & \downarrow \partial_D & & \downarrow \partial_E & & \\
0 & \to & C_n & \xrightarrow[f_n]{} & D_n & \xrightarrow[h_n]{} & E_n & \to & 0 \\
 & & \downarrow \partial_C & & \downarrow \partial_D & & \downarrow \partial_E & & \\
0 & \to & C_{n-1} & \xrightarrow[f_{n-1}]{} & D_{n-1} & \xrightarrow[h_{n-1}]{} & E_{n-1} & \to & 0 \\
 & & \downarrow \partial_C & & \downarrow \partial_D & & \downarrow \partial_E & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \partial_C} & & \downarrow{\scriptstyle \partial_D} & & \downarrow{\scriptstyle \partial_E} & & \\
0 \to & & C_{n+1} & \xrightarrow[f_{n+1}]{1:1} & D_{n+1} & \xrightarrow[h_{n+1}]{onto} & E_{n+1} & & \to 0 \\
 & & \downarrow{\scriptstyle \partial_C} & & \downarrow{\scriptstyle \partial_D} & & \downarrow{\scriptstyle \partial_E} & & \\
0 \to & & C_{n} & \xrightarrow[f_{n}]{} & D_{n} & \xrightarrow[h_{n}]{} & E_{n} & & \to 0 \\
 & & \downarrow{\scriptstyle \partial_C} & & \downarrow{\scriptstyle \partial_D} & & \downarrow{\scriptstyle \partial_E} & & \\
0 \to & & C_{n-1} & \xrightarrow[f_{n-1}]{} & D_{n-1} & \xrightarrow[h_{n-1}]{} & E_{n-1} & & \to 0 \\
 & & \downarrow{\scriptstyle \partial_C} & & \downarrow{\scriptstyle \partial_D} & & \downarrow{\scriptstyle \partial_E} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \partial_C} & & \downarrow{\scriptstyle \partial_D} & & \downarrow{\scriptstyle \partial_E} & & \\
0 \to & & C_{n+1} & \xrightarrow[f_{n+1}]{1:1} & D_{n+1} & \xrightarrow[h_{n+1}]{onto} & E_{n+1} & & \to 0 \\
 & & \downarrow{\scriptstyle \partial_C} & & \downarrow{\scriptstyle \partial_D} & & \downarrow{\scriptstyle \partial_E} & & \\
0 \to & & C_{n} & \xrightarrow[f_{n}]{} & D_{n} & \xrightarrow[h_{n}]{} & E_{n} & & \to 0 \\
 & & \downarrow{\scriptstyle \partial_C} & & \downarrow{\scriptstyle \partial_D} & & \downarrow{\scriptstyle \partial_E} & & \\
0 \to & & C_{n-1} & \xrightarrow[f_{n-1}]{} & D_{n-1} & \xrightarrow[h_{n-1}]{} & E_{n-1} & & \to 0 \\
 & & \downarrow{\scriptstyle \partial_C} & & \downarrow{\scriptstyle \partial_D} & & \downarrow{\scriptstyle \partial_E} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$
\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \partial_C \big\downarrow & & \partial_D \big\downarrow & & \partial_E \big\downarrow & \\
0 \to & C_{n+1} & \xrightarrow[f_{n+1}]{1:1} & D_{n+1} & \xrightarrow[h_{n+1}]{onto} & E_{n+1} & \to 0 \\
 & \partial_C \big\downarrow & & \partial_D \big\downarrow & & \partial_E \big\downarrow & \\
0 \to & C_n & \xrightarrow[f_n]{} & D_n & \xrightarrow[h_n]{} & E_n & \to 0 \\
 & \partial_C \big\downarrow & & \partial_D \big\downarrow & & \partial_E \big\downarrow & \\
0 \to & C_{n-1} & \xrightarrow[f_{n-1}]{} & D_{n-1} & \xrightarrow[h_{n-1}]{} & E_{n-1} & \to 0 \\
 & \partial_C \big\downarrow & & \partial_D \big\downarrow & & \partial_E \big\downarrow & \\
 & \vdots & & \vdots & & \vdots &
\end{array}
$$

$$
\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \partial_C \big\downarrow & & \partial_D \big\downarrow & & \partial_E \big\downarrow & \\
0 \to & C_{n+1} & \xrightarrow[f_{n+1}]{1:1} & D_{n+1} & \xrightarrow[h_{n+1}]{onto} & E_{n+1} & \to 0 \\
 & \partial_C \big\downarrow & & \partial_D \big\downarrow & & \partial_E \big\downarrow & \\
0 \to & C_n & \xrightarrow[f_n]{} & D_n & \xrightarrow[h_n]{} & E_n & \to 0 \\
 & \partial_C \big\downarrow & & \partial_D \big\downarrow & & \partial_E \big\downarrow & \\
0 \to & C_{n-1} & \xrightarrow[f_{n-1}]{} & D_{n-1} & \xrightarrow[h_{n-1}]{} & E_{n-1} & \to 0 \\
 & \partial_C \big\downarrow & & \partial_D \big\downarrow & & \partial_E \big\downarrow & \\
 & \vdots & & \vdots & & \vdots &
\end{array}
$$

THM (Mayer-Vietoris sequence). Let X be a topological space and suppose $X = int(A) \cup int(B)$ or
Let X be a complex and A, B subcomplexes of X such that $X = A \cup B$.
Then there is a long exact sequence as follows.

$$\longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \longrightarrow \cdots$$

Proof: $0 \to \mathcal{C}(A \cap B) \xrightarrow{f} \mathcal{C}(A) \oplus \mathcal{C}(B) \xrightarrow{h} \mathcal{C}(A \cup B) \to 0$ is a short exam sequence of chain complexes.

1.) Given the following commutative diagram

$$\begin{array}{ccccccc} 0 \to \mathcal{C} & \xrightarrow{\quad f \quad} & \mathcal{D} & \xrightarrow{\quad h \quad} & \mathcal{E} \to 0 \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ 0 \to \mathcal{C}' & \xrightarrow{\quad f' \quad} & \mathcal{D}' & \xrightarrow{\quad h' \quad} & \mathcal{E}' \to 0 \end{array}$$

where the horizontal sequence are exact sequences of chain complexes and the α, β, γ are chain maps, show that the following diagram commutes as well:

$$\begin{array}{ccccccccc} \cdots \to H_n(\mathcal{C}) & \xrightarrow{f_*} & H_n(\mathcal{D}) & \xrightarrow{h_*} & H_n(\mathcal{E}) & \xrightarrow{\partial_*} & H_{n-1}(\mathcal{C}) \to \cdots \\ \alpha_* \downarrow & & \beta_* \downarrow & & \gamma_* \downarrow & & \alpha_* \downarrow \\ \cdots \to H_n(\mathcal{C}') & \xrightarrow{f'_*} & H_n(\mathcal{D}') & \xrightarrow{h'_*} & H_n(\mathcal{E}') & \xrightarrow{\partial_*} & H_{n-1}(\mathcal{C}) \to \cdots \end{array}$$

2a.) Prove the Steenrod five-lemma: Given the following commutative diagram of abelian groups where the horizontal sequence are exact, show that if f_1, f_2, f_4, f_5 are isomorphisms, so is f_3.

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

2b.) Supppose one is given a commutative diagram of abelian groups as in the Five-lemma. Consider the following 8 hypotheses:

$$f_i \text{ is a monomorphism, for } i = 1, 2, 4, 5$$
$$f_i \text{ is a epimorphism, for } i = 1, 2, 4, 5$$

Which of these hypotheses will suffice to prove that f_3 is a monomorphism? Which of these hypotheses will suffice to prove that f_3 is a epimorphism?

3.) Prove the serpent lemma: Given a homomorphism of short exact sequences of abelian groups,

$$\begin{array}{ccccccc} 0 \to A & \xrightarrow{f} & B & \xrightarrow{h} & C \to 0 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 \to A' & \xrightarrow{f'} & B' & \xrightarrow{h'} & C' \to 0 \end{array}$$

show that there is an exact sequence

$$0 \to ker\alpha \to ker\beta \to ker\gamma \to cok\alpha \to cok\beta \to cok\gamma \to 0$$

4.) Given a complex K and a short exact sequence of abelian groups

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

show the following is a short exact sequence of chain complexes:

$$0 \to C_n(K; G_1) \to C_n(K; G_2) \to C_n(K; G_3) \to 0$$

This induces a long exact sequence in homology. The zig-zag homomorphism, $\beta_* : H_n(K; G_3) \to H_{n-1}(K; G_1)$ is called the **Bockstein homomorphism** associated with the given coefficient sequence.

(a.) Compute β_* for the coefficient sequences

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

where $|K|$ equals $\mathbb{R}\mathrm{P}^2$.

(b.) Repeat (a) when $|K| =$ Klein bottle.

5.) State and prove a Mayer-Vietoris Theorem for reduced homoloy. What condition does $A \cap B$ need to satisfy?

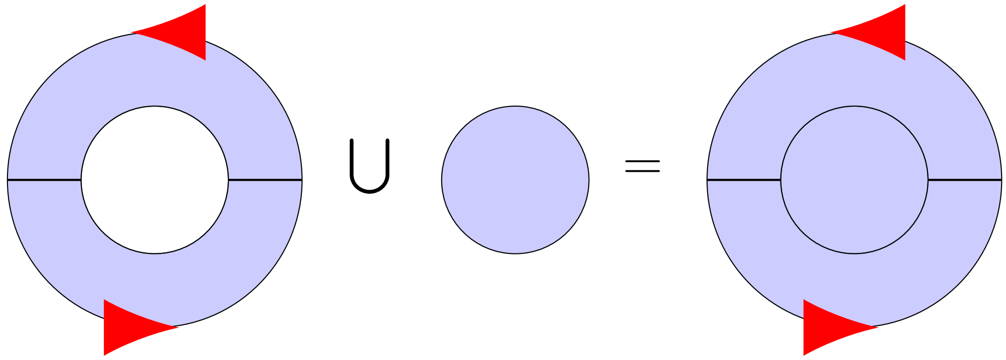

Figure 1: Mobius band ∪ disk = projective plane = $\mathbb{R}P^2$

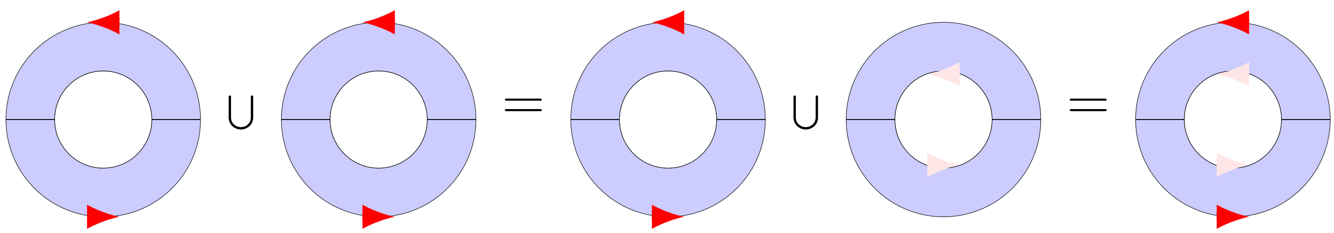

Figure 2: $\mathbb{R}P^2 \# \mathbb{R}P^2$ = Mobius band $\bigcup$ Mobius band = Klein bottle

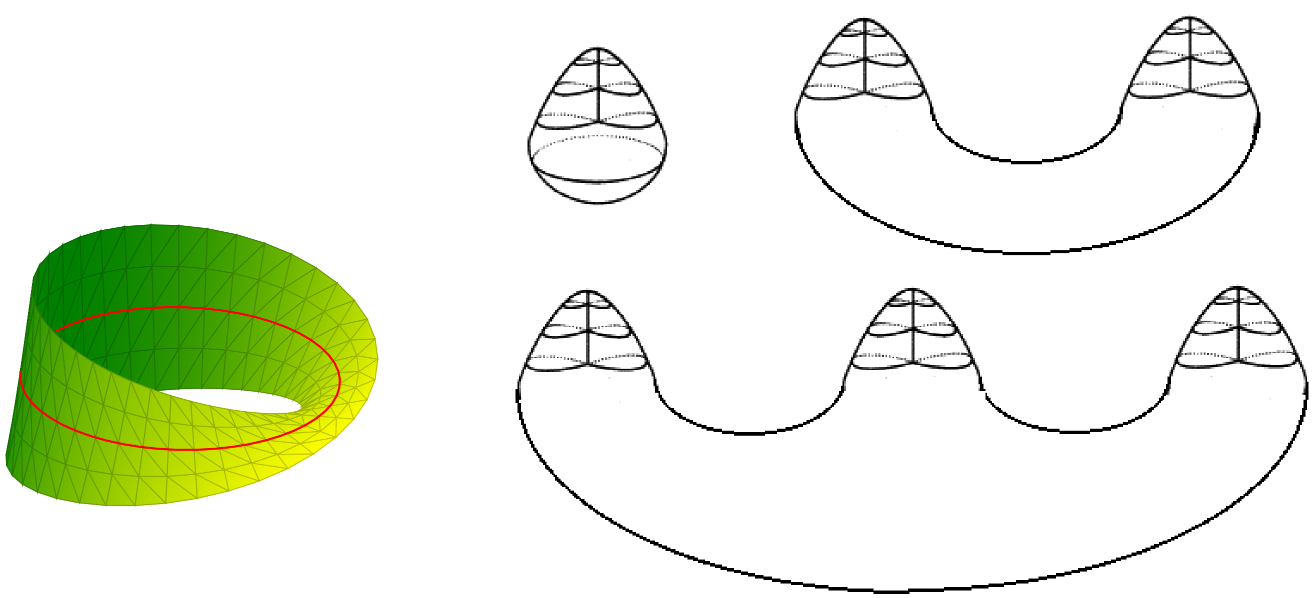

Figure 3: Right figures (connected sum of projective planes)
from: people.math.osu.edu/fiedorowicz.1/math655/classification.html