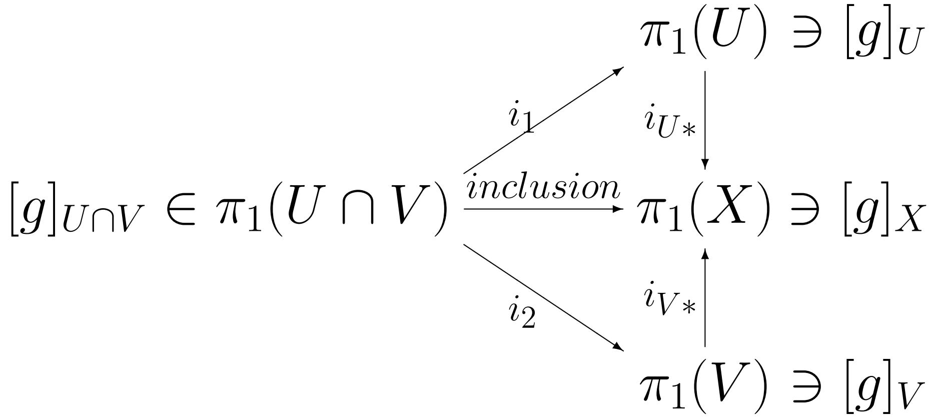
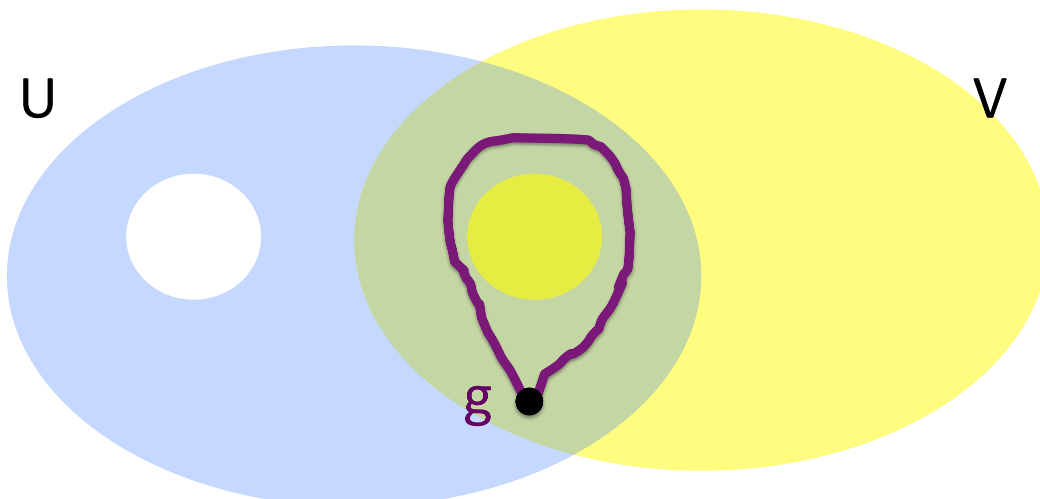


$$i_{U*}(i_1([g]_{U\cap V})) = i_{U*}([g]_U) = [g]_X = i_{U\cap V*}([g]_{U\cap V})$$

$$i_{V*}(i_2([g]_{U\cap V})) = i_{V*}([g]_V) = [g]_X = i_{U\cap V*}([g]_{U\cap V})$$

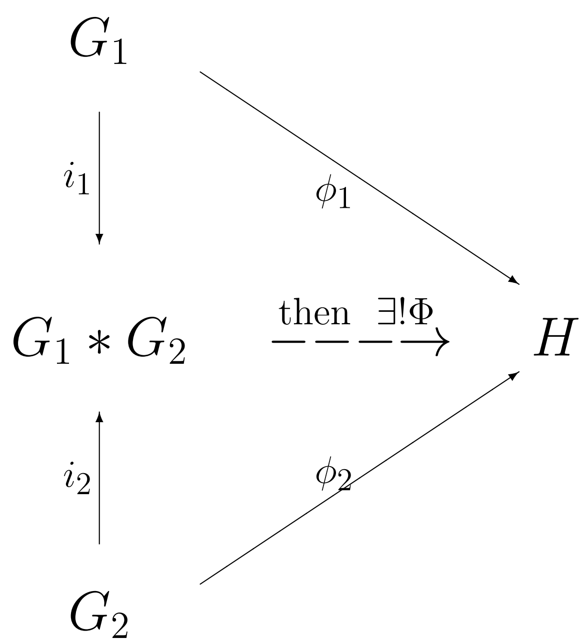


$\pi_1(U) = \langle a, b\rangle$ $[g]_U = b$

$\pi_1(V) = \{e\}$ $[g]_V = e$ $[g]_{U \cup V} = e$

$\pi_1(U \cap V) = \langle b\rangle$ $[g]_{U\cap V} = b$

By group theory, given homomorphism $\phi_1 : G_1 \to H$, then there exists a unique homomorphism $\Phi_i : G_1 * G_2 \to H$ such that the following diagram commutes.



Φ is defined by defining it on its generators:

$$\Phi(g_i) = \phi_i(g_i) \text{ where } g_i \in G_i,\ i = 1 \text{ or } 2.$$

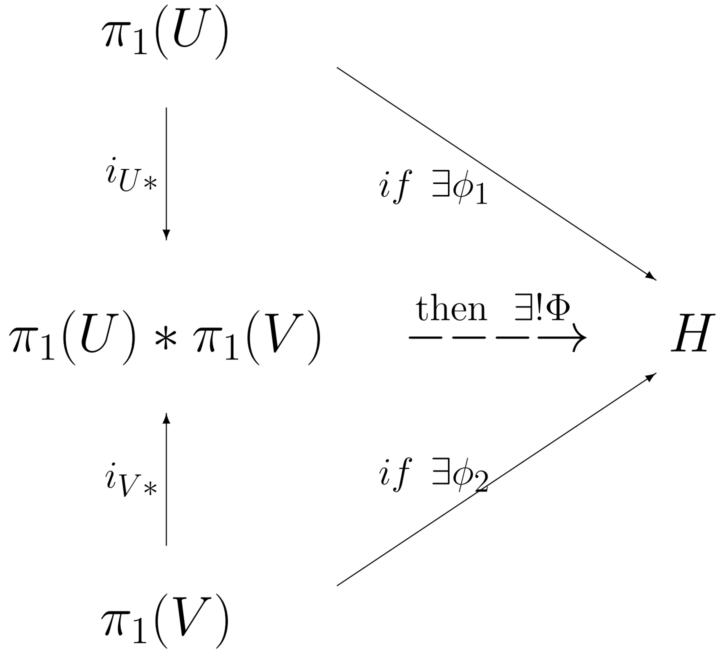
We extend to arbitrary words in $G_1 * G_2$ using the definition of group homomorphism:

$$\Phi(g_1 g_2 \cdots g_n) = \phi_{i_1}(g_1)\phi_{i_2}(g_2) \cdots \phi_{i_n}(g_n) \text{ where } i_k = \begin{cases} 1 & g_k \in G_1 \\ 2 & g_k \in G_2 \end{cases}$$

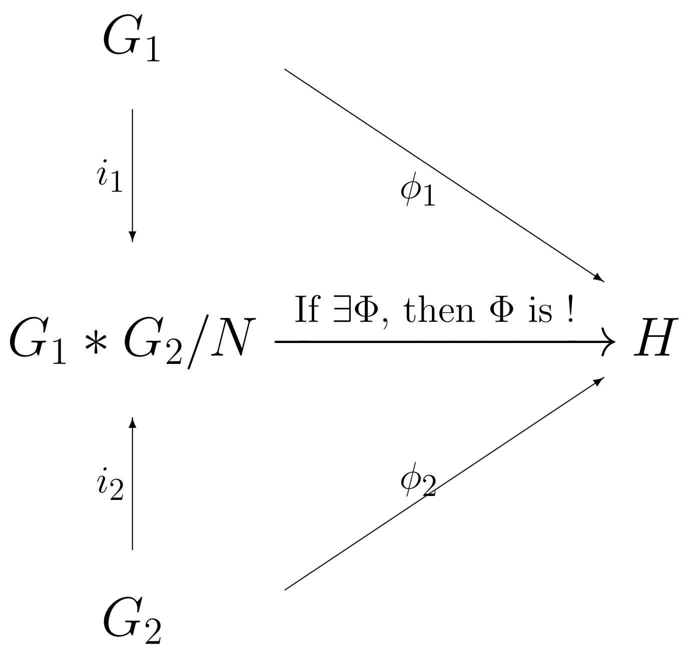
Since ϕ_i are homomorphisms, if $w_1 = w_2$ are two equivalent words in $G_1 * G_2$, then $\Phi(w_1) = \Phi(w_2)$.

Thus Φ is a well-defined homomorphism.

Thus we have the following:

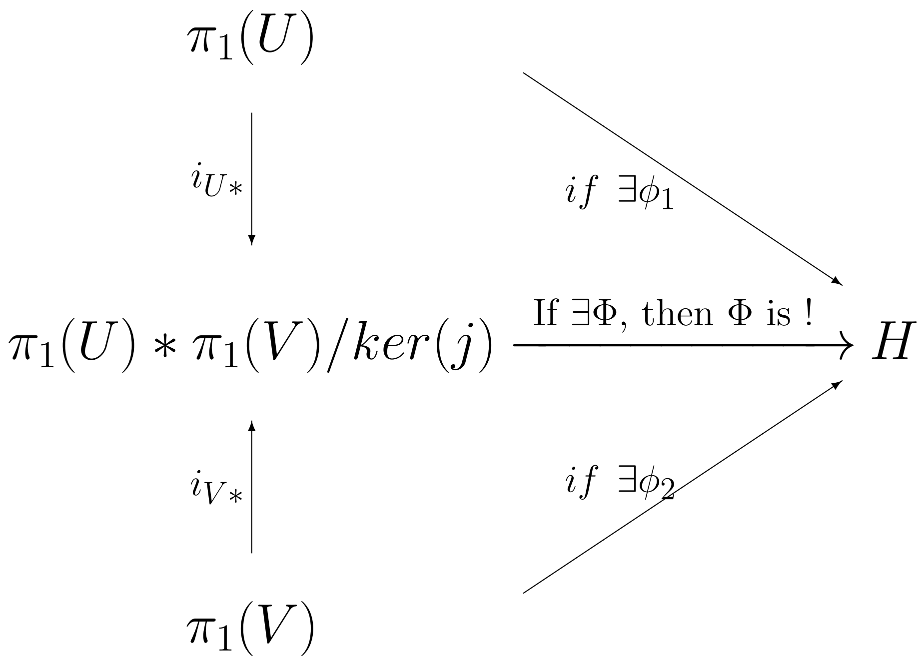


We also know from group theory that if N is any normal subgroup in $G_1 * G_2$, then **IF** Φ exists for the following diagram, then Φ is ! since $\Phi([g_i]) = \phi_i(g_i)$ where $g_i \in G_i$, $i = 1$ or 2



$$\pi_1(X) = \pi_1(U) * \pi_1(V)/ker(j)$$

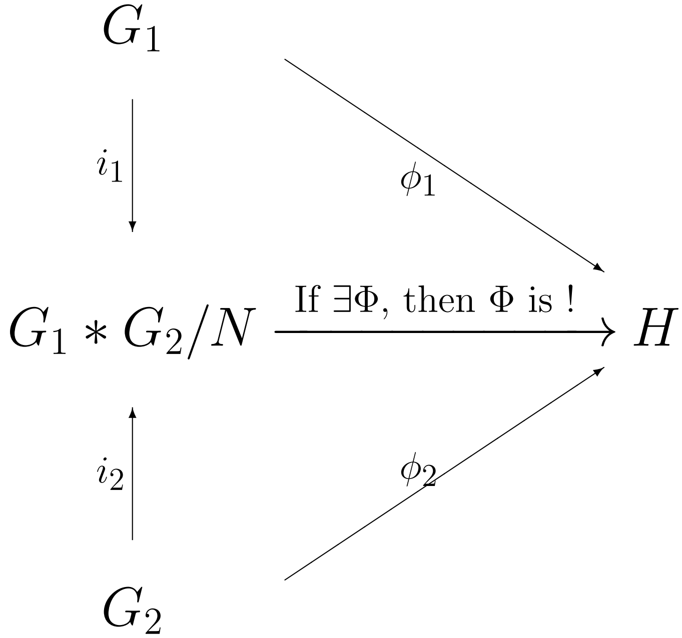
Thus if Φ exists, then Φ is unique.



We know how to define Φ **IF** Φ exists:

$$\Phi([g_i]) = \phi_i(g_i) \text{ where } g_i \in G_i,\ i = 1 \text{ or } 2$$

To determine if this Φ is well-defined, we only need to check if $\Phi(N) = \{e\}$.

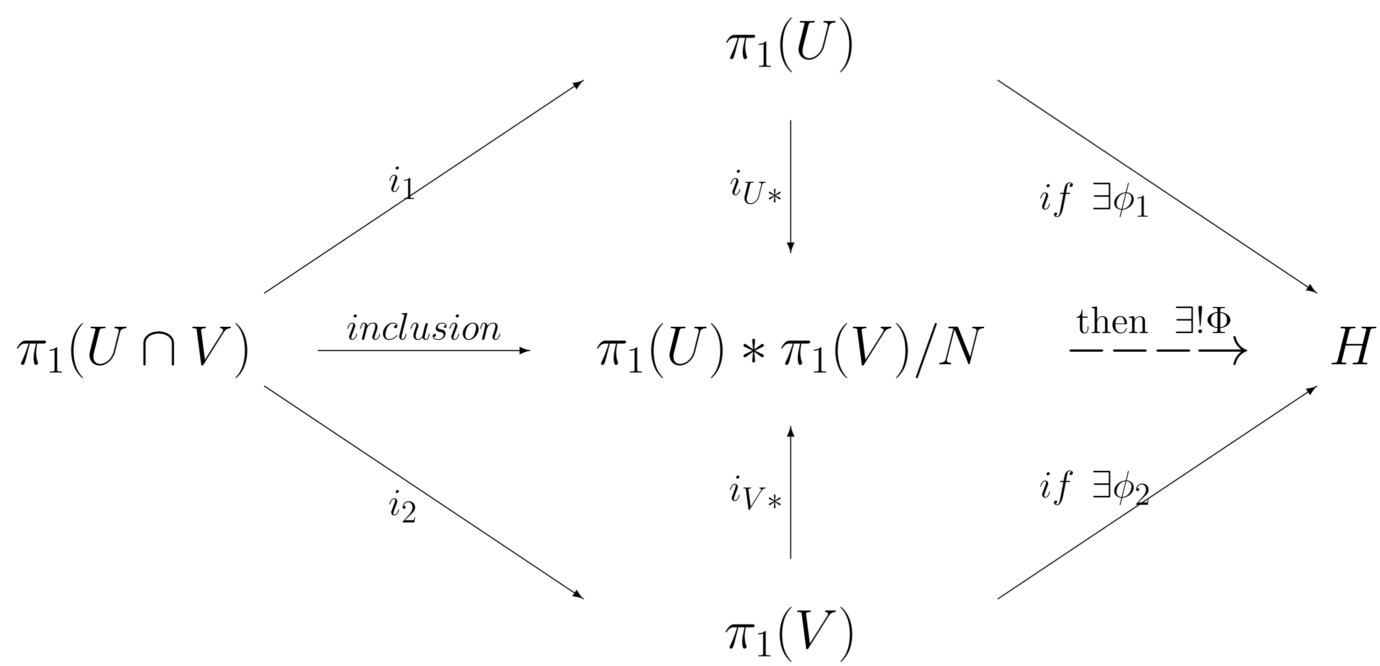


Let N = least normal subgroup generated by

$$\{i_1(c_1)^{-1}i_2(c_1), ..., i_1(c_n)^{-1}i_2(c_n)\}$$

I.e., N is generated by $\{gd_1g^{-1}, ..., gd_ng^{-1} \mid g \in G\}$
where $d_k = i_1(c_k)^{-1}i_2(c_k)$
and where the c_i's are the generators of $\pi(U \cap V)$

So that Φ exists, we need to expand our commutative diagram to include i_1 and i_2 as below:



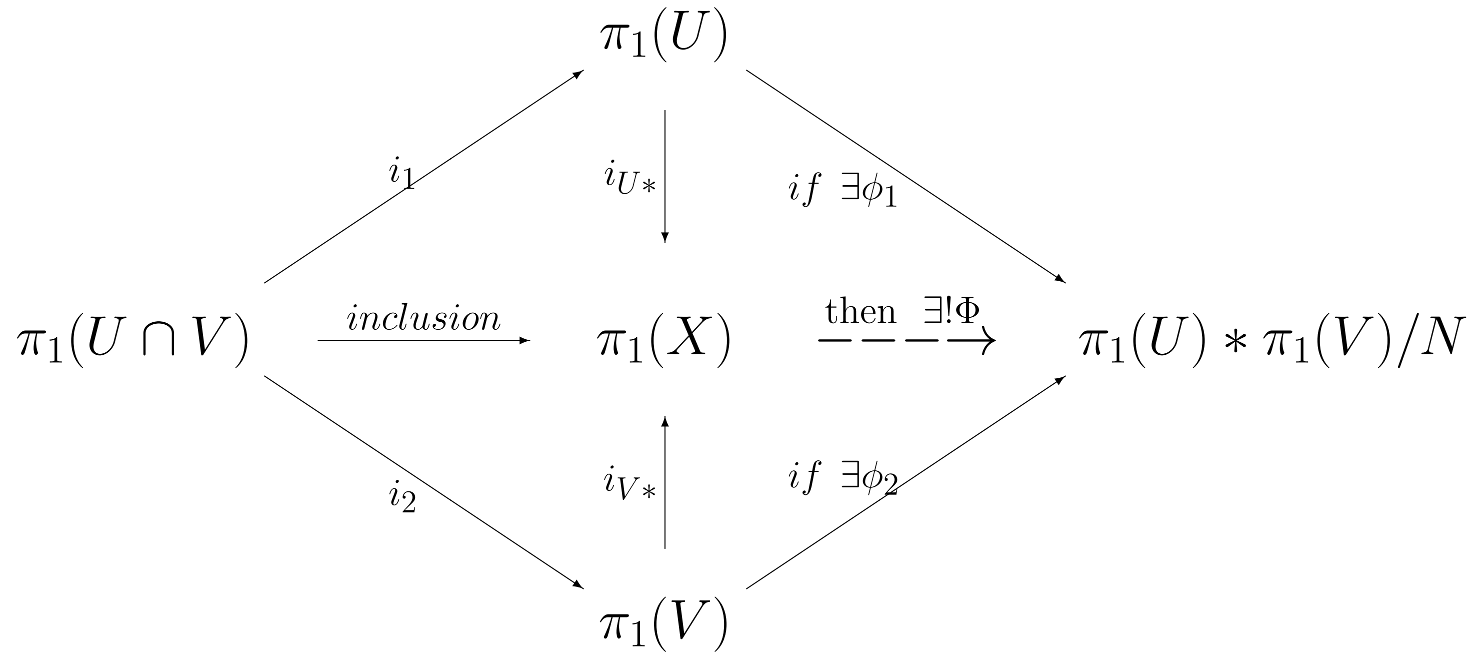
$$\Phi(i_1(c_k)^{-1}i_2(c_k)) = [\Phi(i_1(c_k))]^{-1}\ \Phi(i_2(c_k))$$
$$= [\phi_1(i_1(c_k))]^{-1}\ \phi_2(i_2(c_k)) = [\phi_1(i_1(c_k))]^{-1}\ \phi_1(i_1(c_k)) = e$$

$$\Phi(gd_kg^{-1}) = \Phi(g)\Phi(d_k)\Phi(g^{-1}) = \Phi(g)\Phi(g^{-1}) = e$$

Since Φ sends the generators of N to e, $\Phi(N) = \{e\}$.

Thus Thm 70.2 $[\pi(X) = \pi_1(U) * \pi_1(V)/N]$ implies Thm 70.1.

Thm 70.1 implies Thm 70.2:



$j : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ induced by the two inclusion maps is surjective.

$$\text{I.e., } \pi_1(X) = \pi_1(U) * \pi_1(V)/ker(j)$$

Claim: $N \subset ker(j)$.

$$j(i_1(c_k)^{-1} i_2(c_k)) = [j(i_1(c_k))]^{-1} * j(i_2(c_k)) = [i_{U\cap V}(c_k)]^{-1} i_{U\cap V}(c_k) = e$$

Thus j induces a map

$$k : \pi_1(U) * \pi_1(V)/N \twoheadrightarrow \pi_1(U) * \pi_1(V)/ker(j) = \pi_1(X)$$

since if $N \subset M$ are normal subgroups of G, then

$i : G/N \twoheadrightarrow G/M$, $i(gN) = gM$ is a homomorphism.

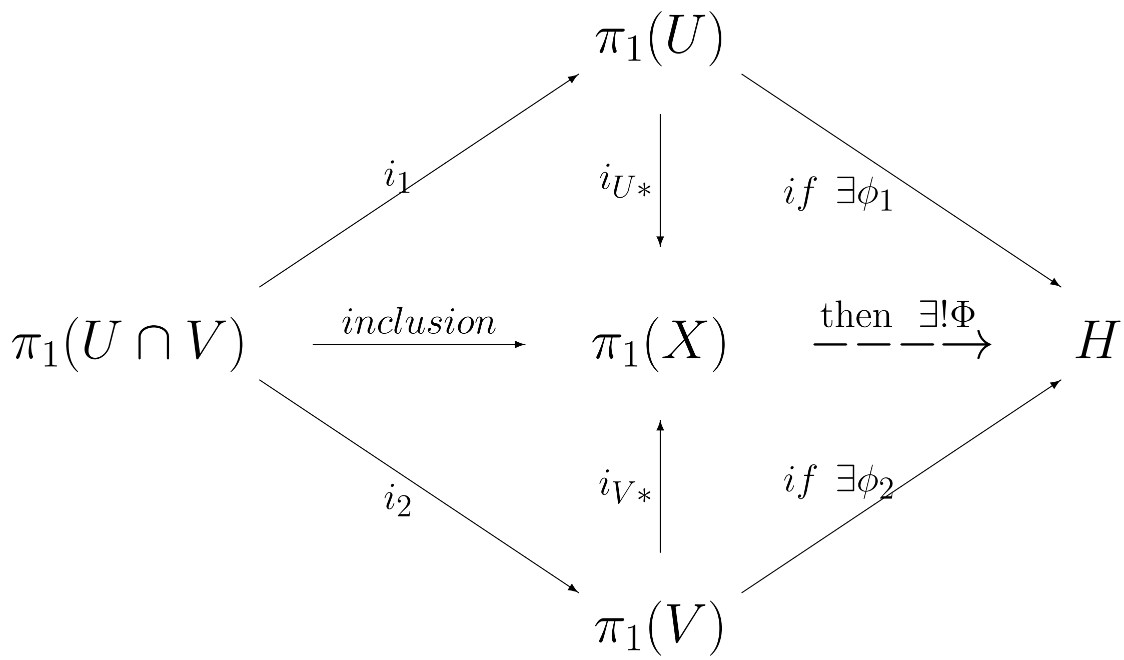
By taking ϕ_i to be inclusion maps in Thm 70.1, $\exists\ \Phi = k^{-1}$.

Thus k is an isomorphism.

$j : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ induced by the two inclusion maps is surjective.

$$\text{Thus } \pi_1(X) = \pi_1(U) * \pi_1(V)/ker(j)$$

Thm 70.1: U, V, $U \cap V$ open and path-connected.



We need to show that Φ is well defined.
Note the definition of Φ is obvious.

Recall that if $g \in \pi_1(X)$,
then $g : I = \cup[t_j, t_{j+1}] \to X$ is a loop in $X = U \cup V$.

We can take $g_j = g|_{[t_{j-1},t_j]}$ to be paths in either U or V.

For each t_j, choose paths α_j from x_0 to $g(t_j)$ such that

$$\circledast \begin{cases} \text{If } g(t_j) \in U \cap V, & \text{choose } \alpha_j \in U \cap V \\ \text{If } g(t_j) \in U - V, & \text{choose } \alpha_j \in U \\ \text{If } g(t_j) \in V - U, & \text{choose } \alpha_j \in V \end{cases}$$

Then $g = (\alpha_0 * g_1 * {\alpha_1}^{-1})(\alpha_1 * g_2 * {\alpha_2}^{-1}) \cdots (\alpha_{n-1} * g_n * {\alpha_n}^{-1})$ where for each j, $(\alpha_{j-1} * g_j * {\alpha_j}^{-1})$ is in $\pi_1(U)$ or in $\pi_1(V)$.

Claim: Given $g = g_1 \cdots g_n \sim f_1 \cdots f_l$, with specified paths α_j from the basepoint x_0 to $g(t_j)$ and paths β_j from x_0 to $f(s_j)$ satisfying conditions $\circledast$, then

$$\Phi(g_1 \cdots g_n) = \Phi(f_1 \cdots f_l).$$

Sub-Claim: If we subdivided the path g_j into h_1 and h_2, then we can replace g_j with $h_1 h_2$.

I.e., $\Phi(g_1 \cdots g_n) = \Phi(g_1 \cdots g_{j-1} h_1 h_2 g_{j+1} \cdots g_n)$

WLOG $g_j : [t_{j-1}, t_j] \to U$. Let $y \in [t_{j-1}, t_j]$ such that

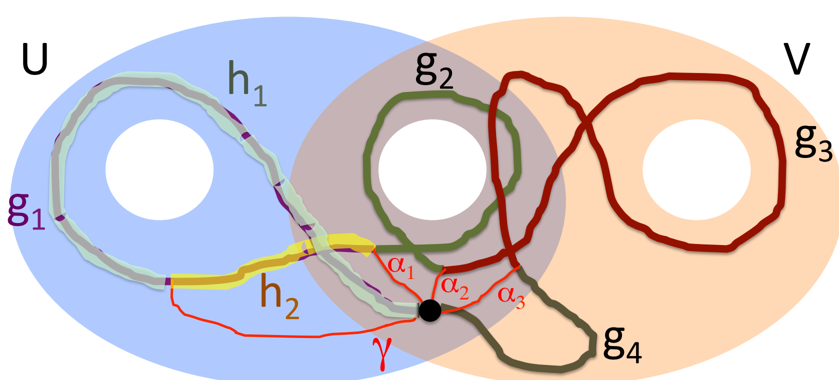
$$h_1 = g_j|_{[t_{j-1}, y]} \text{ and } h_2 = g_j|_{[y, t_j]}.$$

Let γ be any path from x_0 to y satisfying conditions $\circledast$.

Then $\phi_1(\alpha_{j-1} * g_j * {\alpha_j}^{-1}) = \phi_1(\alpha_{j-1} * h_1 h_2 * {\alpha_j}^{-1})$

$= \phi_1(\alpha_{j-1} * h_1 \gamma^{-1} \gamma h_2 * {\alpha_j}^{-1}) = \phi_1(\alpha_{j-1} * h_1 \gamma^{-1}) \phi(\gamma h_2 * {\alpha_j}^{-1})$

Thus $\Phi(g_1 \cdots g_n) = \Phi(g_1 \cdots g_{j-1} h_1 h_2 g_{j+1} \cdots g_n)$ with the associated paths α_i and γ.



$g = (\alpha_0 h_1 h_2 \overline{\alpha_1})(\alpha_1 g_2 \overline{\alpha_2})(\alpha_2 g_3 \overline{\alpha_3})(\alpha_3 g_4 \overline{\alpha_4})$

where α_0 and α_4 are constant maps.

Proof continued from chalkboard:

$h_2 \sim y_1^{-1} z_2 y_2$

$\phi_{i_2}(\alpha_1 h_2 \alpha_2^{-1}) = \phi_{i_2}(\alpha_1 y_1^{-1} z_2 y_2 \alpha_2^{-1})$

$= \phi_{i_2}(\alpha_1 y_1^{-1} \gamma_1^{-1} \cdot \gamma_1 z_2 \gamma_2^{-1} \cdot \gamma_2 y_2 \alpha_2^{-1})$

$= \phi_{i_2}(\alpha_1 y_1^{-1} \gamma_1^{-1}) \phi_{i_2}(\gamma_1 z_2 \gamma_2^{-1}) \phi_{i_2}(\gamma_2 y_2 \alpha_2^{-1})$

Thus $\Phi(h_1 \cdots h_m) = \Phi(z_1 y_1 h_2 \cdots h_m) = \Phi(z_1 y_1 y_1^{-1} z_2 y_2 \cdots h_m)$

$y_1 y_1^{-1} \to \phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1}) \phi_{i_2}(\alpha_1 y_1^{-1} \gamma_1^{-1})$

If $i_1 = i_2$, then

$\phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1}) \phi_{i_2}(\alpha_1 y_1^{-1} \gamma_1^{-1}) = \phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1}) \phi_{i_1}(\alpha_1 y_1^{-1} \gamma_1^{-1})$

$= \phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1} \cdot \alpha_1 y_1^{-1} \gamma_1^{-1}) = \phi_{i_1}(e) = e$

If $i_1 \neq i_2$, then $y \subset U \cap V$. Thus $\alpha_1, \gamma_1 \subset U \cap V$.

Thus $d = \gamma_1 y_1 \alpha_1^{-1} \in \pi_1(U \cap V)$.

$\phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1}) \phi_{i_2}(\alpha_1 y_1^{-1} \gamma_1^{-1}) = \phi_{i_1}(d) \phi_{i_2}(d)^{-1} = e$

$$\begin{aligned} \text{Thus } \Phi(h_1 \cdots h_m) = \Phi(z_1 y_1 h_2 \cdots h_m) &= \Phi(z_1 z_2 y_2 \cdots h_m) \\ &= \cdots = \Phi(f_1 \cdots f_m) \end{aligned}$$

Thus $g \sim f$ in $U \cup V$ implies $\Phi(g) = \Phi(f)$