Proof of thm 59.1:

Let \( g : I \to X \) be a loop in \( X = U \cup V \).

Since \( U \) and \( V \) are open, \( I \subset g^{-1}(U) \cup g^{-1}(V) \).

By the Lebesgue number lemma, \( \exists \delta > 0 \) such that if \( A \subset X \) with \( \text{diam}(A) < \delta \), then \( A \subset g^{-1}(U) \) or \( A \subset g^{-1}(V) \).

Thus if \( I = \cup [a_i, a_{i+1}] \) where \( \text{diam}([a_i, a_{i+1}]) < \delta \), then \( [a_i, a_{i+1}] \subset g^{-1}(U) \) or \( [a_i, a_{i+1}] \subset g^{-1}(V) \).

Thus
\[
g([a_i, a_{i+1}]) \subset g(g^{-1}(U)) \subset U \text{ or } g([a_i, a_{i+1}]) \subset g(g^{-1}(V)) \subset V.
\]

If there exists \( a_k \) such that \( g(a_k) \notin U \cap V \), replace the intervals \( [a_{k-1}, a_k] \) and \( [a_k, a_{k+1}] \) with \( [a_{k-1}, a_{k+1}] \).

If \( g(a_k) \in U \), then \( g(a_k) \notin V \). Hence \( g([a_{k-1}, a_k]) \subset U \& g([a_k, a_{k+1}]) \subset U \). Thus \( g([a_{k-1}, a_{k+1}]) \subset U \).

Similary, \( g(a_k) \in V \) implies \( g([a_{k-1}, a_{k+1}]) \subset V \).

Thus we can write \( I = \cup [c_i, c_{i+1}] \) where
\[
c_i \in U \cap V \forall i \text{ and } g([c_i, c_{i+1}]) \subset U \text{ or } g([c_i, c_{i+1}]) \subset V
\]

Since \( U \cap V \) is path connected, \( \exists \) a path \( \alpha_i \) between \( x_0 \) and \( c_i \).

Thus
\[
g = (\alpha_0 \ast g_1 \ast \overline{\alpha_1})(\alpha_1 \ast g_2 \ast \overline{\alpha_2}) \cdots (\alpha_{n-1} \ast g_n \ast \overline{\alpha_n})
\]
where for each \( i \), \( (\alpha_{i-1} \ast g_i \ast \overline{\alpha_i}) \) is in \( \pi_1(U) \) or in \( \pi_1(V) \).

I.e., \( j : \pi_1(U) \ast \pi_1(V) \to \pi_1(X) \) induced by the two inclusion maps is surjective.
\[ j : \pi_1(U) \ast \pi_1(V) \to \pi_1(X) \] induced by the two inclusion maps is surjective.

Thus \( \pi_1(X) = \pi_1(U) \ast \pi_1(V)/\ker(j) \)

---

Thm 70.1: \( U, V, U \cap V \) open and path-connected.
\[
\pi_1(X) = \pi_1(U) \ast \pi_1(V)/\ker(j)
\]

If \( \Phi \) exists, then \( \Phi \) is unique.

\[
\begin{array}{cccc}
\pi_1(U) & \xrightarrow{i_U^*} & \pi_1(U) \ast \pi_1(V)/\ker(j) & \xrightarrow{\text{if } \exists \phi_1} H \\
\pi_1(V) & \xrightarrow{i_V^*} & \pi_1(U) \ast \pi_1(V)/\ker(j) & \xrightarrow{\text{if } \exists \phi_2}
\end{array}
\]

Let \( N = \) least normal subgroup generated by

\[
\{i_1(c_1)^{-1}i_2(c_1), \ldots, i_1(c_n)^{-1}i_2(c_n)\}
\]

I.e., \( N \) is generated by \( \{gd_1g^{-1}, \ldots, gd_ng^{-1} \mid g \in G\} \)

where \( d_k = i_1(c_k)^{-1}i_2(c_k) \)

and where the \( c_i \)'s are the generators of \( \pi(U \cap V) \)

\[
\begin{array}{cccc}
\pi_1(U) & \xrightarrow{i_1} & \pi_1(U) \ast \pi_1(V)/N & \xrightarrow{\text{then } \exists !\Phi} H \\
\pi_1(V) & \xrightarrow{i_2} & \pi_1(U) \ast \pi_1(V)/N & \xrightarrow{\text{then } \exists !\Phi}
\end{array}
\]

Thus Thm 70.2 \([\pi(X) = \pi_1(U) \ast \pi_1(V)/N]\) implies Thm 70.1.
Thm 70.1 implies Thm 70.2:

\[ \pi_1(U \cap V) \xrightarrow{\text{inclusion}} \pi_1(U) \xrightarrow{i_*} \pi_1(X) \xrightarrow{\text{then } \exists ! \Phi} \pi_1(U) * \pi_1(V)/N \]

\[ \pi_1(V) \xrightarrow{i_*} \pi_1(U) \xrightarrow{\text{if } \exists \phi_2} \]

\[ j : \pi_1(U) * \pi_1(V) \to \pi_1(X) \text{ induced by the two inclusion maps is surjective.} \]

I.e., \( \pi_1(X) = \pi_1(U) * \pi_1(V)/\ker(j) \)

Claim: \( N \subset \ker(j) \).

\[ j(i_1(c_k)^{-1}i_2(c_k)) = [j(i_1(c_k))]^{-1} * j(i_2(c_k)) = [i_{U \cap V}(c_k)]^{-1}i_{U \cap V}(c_k) = e \]

Thus \( j \) induces a map

\[ k : \pi_1(U) * \pi_1(V)/N \to \pi_1(U) * \pi_1(V)/\ker(j) = \pi_1(X) \]

By taking \( \phi_i \) to be inclusion maps in Thm 70.1, \( \exists \Phi = k^{-1} \).
Proof continued from chalkboard:

\[ h_2 \sim y_1^{-1}z_2y_2 \]

\[
\phi_{i_2}(\alpha_1 h_2 \alpha_2^{-1}) = \phi_{i_2}(\alpha_1 y_1^{-1}z_2y_2 \alpha_2^{-1}) \\
= \phi_{i_2}(\alpha_1 y_1^{-1}\gamma_1^{-1} \cdot \gamma_1 z_2 \gamma_2^{-1} \cdot \gamma_2 y_2 \alpha_2^{-1}) \\
= \phi_{i_2}(\alpha_1 y_1^{-1}\gamma_1^{-1}) \phi_{i_2}(\gamma_1 z_2 \gamma_2^{-1}) \phi_{i_2}(\gamma_2 y_2 \alpha_2^{-1})
\]

Thus \( \Phi(h_1 \cdots h_m) = \Phi(z_1 y_1 h_2 \cdots h_m) = \Phi(z_1 y_1^{-1} z_2 y_2 \cdots h_m) \)

\[ y_1 y_1^{-1} \to \phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1}) \phi_{i_2}(\alpha_1 y_1^{-1} \gamma_1^{-1}) \]

If \( i_1 = i_2 \), then

\[
\phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1}) \phi_{i_2}(\alpha_1 y_1^{-1} \gamma_1^{-1}) = \phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1}) \phi_{i_1}(\alpha_1 y_1^{-1} \gamma_1^{-1})
\]

\[ = \phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1} \cdot \alpha_1 y_1^{-1} \gamma_1^{-1}) = \phi_{i_1}(e) = e \]

If \( i_1 \neq i_2 \), then \( y \subset U \cap V \). Thus \( \alpha_1, \gamma_1 \subset U \cap V \).

Thus \( d = \gamma_1 y_1 \alpha_1^{-1} \in \pi_1(U \cap V) \).

\[
\phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1}) \phi_{i_2}(\alpha_1 y_1^{-1} \gamma_1^{-1}) = \phi_{i_1}(d) \phi_{i_2}(d)^{-1} = e
\]

Thus \( \Phi(h_1 \cdots h_m) = \Phi(z_1 y_1 h_2 \cdots h_m) = \Phi(z_1 z_2 y_2 \cdots h_m) = \cdots = \Phi(f_1 \cdots f_m) \)

Thus \( g \sim f \) in \( U \cup V \) implies \( \Phi(g) = \Phi(f) \)