Proof of thm 59.1:

Let $g : I \to X$ be a loop in $X = U \cup V$.

Since U and V are open, $I \subset g^{-1}(U) \cup g^{-1}(V)$.

By the Lebesgue number lemma, $\exists\ \delta > 0$ such that if $A \subset X$ with $\text{diam}(A) < \delta$, then $A \subset g^{-1}(U)$ or $A \subset g^{-1}(V)$.

Thus if $I = \cup[a_i, a_{i+1}]$ where $\text{diam}([a_i, a_{i+1}]) < \delta$,
then $[a_i, a_{i+1}] \subset g^{-1}(U)$ or $[a_i, a_{i+1}] \subset g^{-1}(V)$.

Thus

$$g([a_i, a_{i+1}]) \subset g(g^{-1}(U)) \subset U \text{ or } g([a_i, a_{i+1}]) \subset g(g^{-1}(V)) \subset V.$$

If there exists a_k such that $g(a_k) \notin U \cap V$,
replace the intervals $[a_{k-1}, a_k]$ and $[a_k, a_{k+1}]$ with $[a_{k-1}, a_{k+1}]$.

If $g(a_k) \in U$, then $g(a_k) \notin V$. Hence

$$g([a_{k-1}, a_k]) \subset U \ \&\ g([a_k, a_{k+1}]) \subset U. \text{ Thus } g([a_{k-1}, a_{k+1}]) \subset U.$$

Similary, $g(a_k) \in V$ implies $g([a_{k-1}, a_{k+1}]) \subset V$.

Thus we can write $I = \cup[c_i, c_{i+1}]$ where
$c_i \in U \cap V\ \forall i$ and $g([c_i, c_{i+1}]) \subset U$ or $g([c_i, c_{i+1}]) \subset V$

Since $U \cap V$ is path connected, $\exists$ a path α_i between x_0 and c_i.

Thus $g = (\alpha_0 * g_1 * \overline{\alpha_1})(\alpha_1 * g_2 * \overline{\alpha_2}) \cdots (\alpha_{n-1} * g_n * \overline{\alpha_n})$

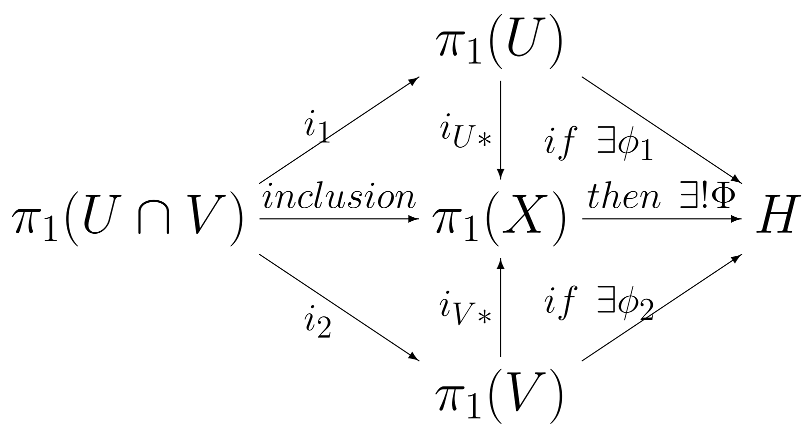
where for each i, $(\alpha_{i-1} * g_i * \overline{\alpha_i})$ is in $\pi_1(U)$ or in $\pi_1(V)$.

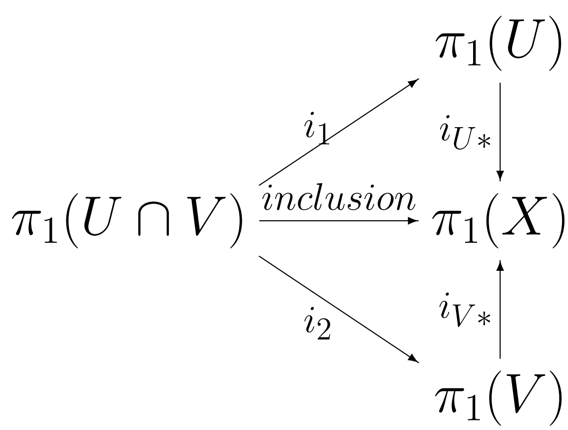
I.e., $j : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ induced by the two inclusion maps is surjective.

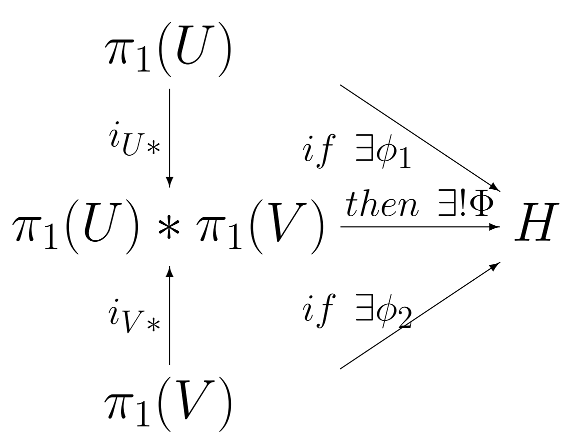
$j : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ induced by the two inclusion maps is surjective.

$$\text{Thus } \pi_1(X) = \pi_1(U) * \pi_1(V)/ker(j)$$

Thm 70.1: U, V, $U \cap V$ open and path-connected.

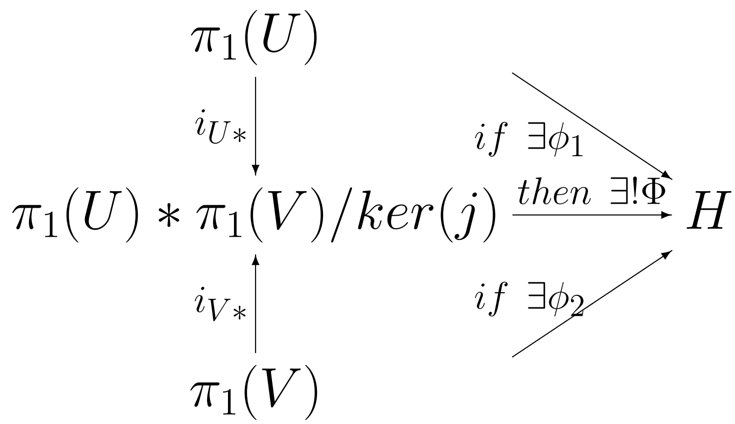






$$\pi_1(X) = \pi_1(U) * \pi_1(V)/ker(j)$$

If Φ exists, then Φ is unique.

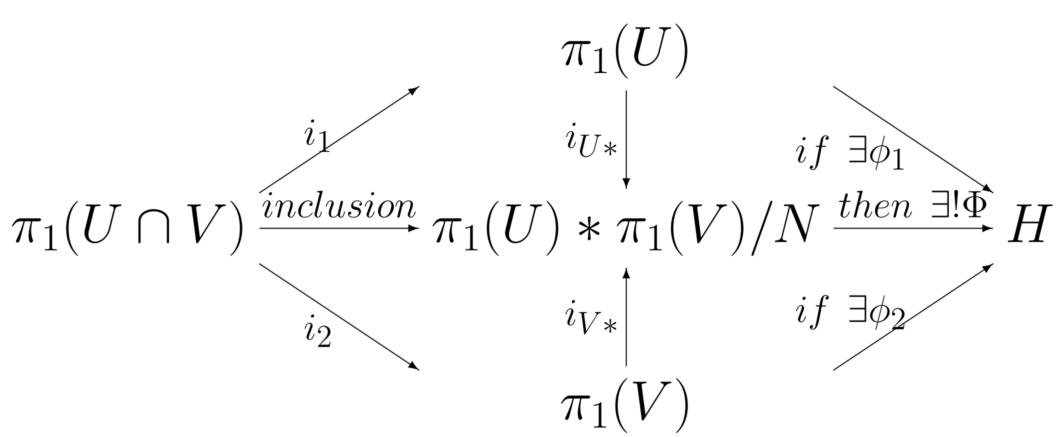


Let N = least normal subgroup generated by

$$\{i_1(c_1)^{-1}i_2(c_1), ..., i_1(c_n)^{-1}i_2(c_n)\}$$

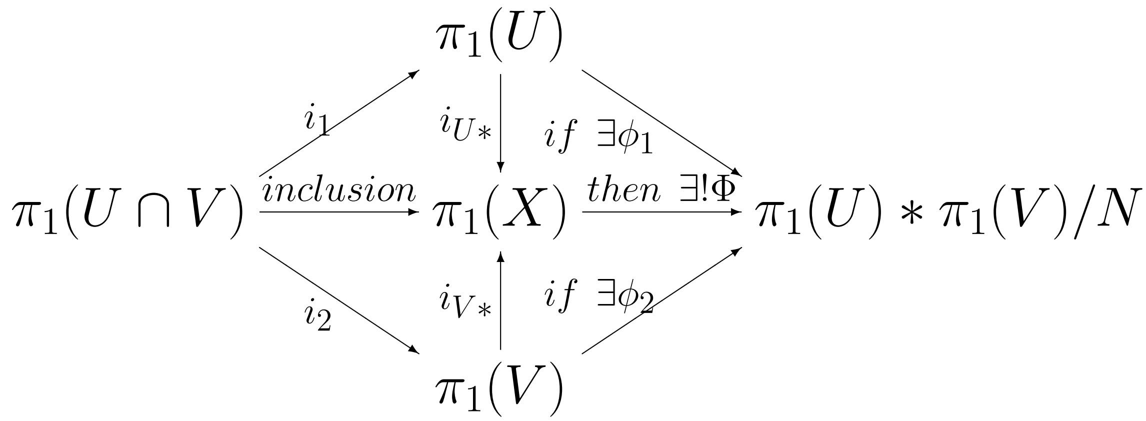
I.e., N is generated by $\{gd_1g^{-1}, ..., gd_ng^{-1} \mid g \in G\}$
where $d_k = i_1(c_k)^{-1}i_2(c_k)$

and where the c_i's are the generators of $\pi(U \cap V)$



Thus Thm 70.2 $[\pi(X) = \pi_1(U) * \pi_1(V)/N]$ implies Thm 70.1.

Thm 70.1 implies Thm 70.2:



$j : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ induced by the two inclusion maps is surjective.

$$\text{I.e., } \pi_1(X) = \pi_1(U) * \pi_1(V)/ker(j)$$

Claim: $N \subset ker(j)$.

$$j(i_1(c_k)^{-1} i_2(c_k)) = [j(i_1(c_k))]^{-1} * j(i_2(c_k)) = [i_{U\cap V}(c_k)]^{-1} i_{U\cap V}(c_k) = e$$

Thus j induces a map

$$k : \pi_1(U) * \pi_1(V)/N \twoheadrightarrow \pi_1(U) * \pi_1(V)/ker(j) = \pi_1(X)$$

.

By taking ϕ_i to be inclusion maps in Thm 70.1, $\exists\ \Phi = k^{-1}$.

Proof continued from chalkboard:

$h_2 \sim y_1^{-1} z_2 y_2$

$\phi_{i_2}(\alpha_1 h_2 \alpha_2^{-1}) = \phi_{i_2}(\alpha_1 y_1^{-1} z_2 y_2 \alpha_2^{-1})$

$= \phi_{i_2}(\alpha_1 y_1^{-1} \gamma_1^{-1} \cdot \gamma_1 z_2 \gamma_2^{-1} \cdot \gamma_2 y_2 \alpha_2^{-1})$

$= \phi_{i_2}(\alpha_1 y_1^{-1} \gamma_1^{-1}) \phi_{i_2}(\gamma_1 z_2 \gamma_2^{-1}) \phi_{i_2}(\gamma_2 y_2 \alpha_2^{-1})$

Thus $\Phi(h_1 \cdots h_m) = \Phi(z_1 y_1 h_2 \cdots h_m) = \Phi(z_1 y_1 y_1^{-1} z_2 y_2 \cdots h_m)$

$y_1 y_1^{-1} \to \phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1}) \phi_{i_2}(\alpha_1 y_1^{-1} \gamma_1^{-1})$

If $i_1 = i_2$, then

$\phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1}) \phi_{i_2}(\alpha_1 y_1^{-1} \gamma_1^{-1}) = \phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1}) \phi_{i_1}(\alpha_1 y_1^{-1} \gamma_1^{-1})$

$= \phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1} \cdot \alpha_1 y_1^{-1} \gamma_1^{-1}) = \phi_{i_1}(e) = e$

If $i_1 \neq i_2$, then $y \subset U \cap V$. Thus $\alpha_1, \gamma_1 \subset U \cap V$.

Thus $d = \gamma_1 y_1 \alpha_1^{-1} \in \pi_1(U \cap V)$.

$\phi_{i_1}(\gamma_1 y_1 \alpha_1^{-1}) \phi_{i_2}(\alpha_1 y_1^{-1} \gamma_1^{-1}) = \phi_{i_1}(d) \phi_{i_2}(d)^{-1} = e$

$$\begin{aligned} \text{Thus } \Phi(h_1 \cdots h_m) = \Phi(z_1 y_1 h_2 \cdots h_m) &= \Phi(z_1 z_2 y_2 \cdots h_m) \\ &= \cdots = \Phi(f_1 \cdots f_m) \end{aligned}$$

Thus $g \sim f$ in $U \cup V$ implies $\Phi(g) = \Phi(f)$