NOTE: all maps are assumed to be continuous unless otherwise stated.

\[ r : X \to A \text{ is a retraction of } X \text{ onto } A \]

if \( r|_A = \text{identity map on } A \).

\( A \) is a retract of \( X \) if \( \exists \) a retraction of \( X \) onto \( A \).

Ex: The constant map \( c : X \to x_0 \), where \( x_0 \) is a point of \( X \) is a retraction.

\( A \) is a deformation retract of \( X \) if the identity map \( i : X \to X \) is homotopic to a map \( R : X \to X \) where \( R \) is the extension (of the codomain) of a retraction \( r : X \to A \) and where each point of \( A \) remains fixed during the homotopy.

In other words, \( \exists \) homotopy \( H : X \times I \to X \) such that \( H_0 = \text{identity map on } X \), \( H_1(X) \subset A \), and \( H_t|_A = \text{identity map on } A \ \forall \ t \).

Ex: \( X \) is a deformation retract of \( X \times I \).

Ex: \( S^n \) is a deformation retract of \( R^{n+1} - \{0\} \).
Lemma 55.1: If $A$ is a retract of $X$, then 
\[ i_* : \pi_1(A) \to \pi_1(X) \] is injective
where $i_*$ is induced by the inclusion map.

Lemma 58.3: If $A$ is a deformation retract of $X$, then 
\[ i_* : \pi_1(A) \to \pi_1(X) \] is an isomorphism
where $i_*$ is induced by the inclusion map.

Thm 70.1: Seifert-van Kampen Theorem.

Suppose $U$, $V$ and $U \cap V$ are open and path-connected. Let $i : U \cap V \to U$ and $j : U \cap V \to V$ be inclusion maps. If

\[ \pi_1(U) = < a_1, ..., a_i \mid s_1, ..., s_l >, \]

\[ \pi_1(V) = < b_1, ..., b_j \mid t_1, ..., t_m >, \]

\[ \pi_1(U \cap V) = < c_1, ..., c_k \mid r_1, ..., r_n >, \]

then $\pi_1(U \cup V) =$
\[ < a_1, ..., a_i, b_1, ..., b_j \mid s_1, ..., s_l, t_1, ..., t_m, \]
\[ i(c_1) = j(c_1), ..., i(c_n) = j(c_n) > \]