$H_1(T^2;\mathbb{Z}) \cong \mathbb{Z}\times\mathbb{Z}$ with basis $i=1,2$, $\phi_i(e_j) = \begin{cases} 1 & j=i,3 \\ 0 & else \end{cases}$

Note: in terms of basis for C^1, $\quad \phi_i = \psi_{e_1} + \psi_{e_3}, \quad$ for $i=1,2$.

Note: $(\delta\circ\phi_1)(\sigma_i) = \phi_1(\partial(\sigma_i)) = \phi_1(e_1+e_2-e_3)$

$$= \phi(e_1)+\phi(e_2)-\phi_(e_3) = 1+0-1 = 0$$

$(\phi_1 \smile \phi_1)(\sigma_1) = \phi_1(e_1)\cdot\phi_1(e_2) = (1)(0) = 0$

$(\phi_1 \smile \phi_1)(\sigma_2) = \phi_1(e_2)\cdot\phi_1(e_1) = (0)(1) = 0$

Thus $\phi_1\smile\phi_1 = 0$. Similarly, $\phi_2\smile\phi_2 = 0$.

$(\phi_1 \smile \phi_2)(\sigma_1) = \phi_1(e_1)\cdot\phi_2(e_2) = (1)(1) = 1$

$(\phi_1 \smile \phi_2)(\sigma_2) = \phi_1(e_2)\cdot\phi_2(e_1) = (0)(0) = 0$

$(\phi_2 \smile \phi_1)(\sigma_1) = \phi_2(e_1)\cdot\phi_1(e_2) = (0)(0) = 0$

$(\phi_2 \smile \phi_1)(\sigma_2) = \phi_2(e_2)\cdot\phi_1(e_1) = (1)(1) = 1$

Thus $\phi_1\smile\phi_2 \neq \phi_2\smile\phi_1$ in C^2.

$H_2(T^2;\mathbb{Z}) \cong \mathbb{Z}$ since $C^3 = 0$ implies $ker\delta = C^2$ and

$\delta\circ\psi_{e_i}(\sigma_j) = \psi_{e_i}(\partial(\sigma_j)) = \psi_{e_i}(e_1+e_2-e_3) = \begin{cases} 1 & i=1,2 \\ -1 & i=3 \end{cases}$

Thus $\delta\circ\psi_{e_i} = \psi_{\sigma_1}+\psi_{\sigma_2}$ for $i=1,2$ and $\delta\circ\psi_{e_3} = -\delta\circ\psi_{e_1}$

Thus $im\delta =< \psi_{\sigma_1}+\psi_{\sigma_2} >$. Thus in H^2, $\psi_{\sigma_1} = -\psi_{\sigma_2}$

Theorem 1. *When R is commutative, the rings $H^\bullet(X, A; R)$ are graded commutative, that is, the identity*

$$\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha$$

holds for all $\alpha \in H^k(X, A; R)$ and $\beta \in H^\ell(X, A; R)$.

$$H^n(S^2 \vee S^4; \mathbb{Z}) \cong H^n(\mathbb{CP}^2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0, 2, 4 \\ 0 & else \end{cases}$$

But $\cup : H^2(S^2 \vee S^4; \mathbb{Z}) \times H^2(S^2 \vee S^4; \mathbb{Z}) \to H^4(S^2 \vee S^4; \mathbb{Z})$ is the zero map,

while for $\cup : H^2(\mathbb{CP}^2; \mathbb{Z}) \times H^2(\mathbb{CP}^2; \mathbb{Z}) \to H^4(\mathbb{CP}^2; \mathbb{Z})$, $\cup(1, 1) = 1$.

I.e, the group structure does not distinguish $S^2 \vee S^4$ from $\mathbb{CP}^2$, but the ring structure does.

Proposition 1. *For a map $f\colon X \to Y$, the induced maps $f^*\colon H^n(Y; R) \to H^n(X; R)$ satisfy $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$, and similarly in the relative case.*

Thus $H^\bullet(-; R)$ is a functor from the category of topological spaces to the category of graded R-algebras.

Poincaré Duality

Definition 1. *Let X be a space. The cap product is a pairing between certain homology groups and cohomology groups of X. For $k \geq \ell$, we define*

$$\frown : C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$$

$$\sigma \frown \phi = \phi(\sigma|_{[v_0 \cdots v_\ell]})\sigma|_{[v_\ell \cdots v_k]}.$$

Example (using simplicial instead of singular notation):

Suppose $\sigma = [v_0 \cdots v_k] \in C_k$ and $\phi : C_\ell \to \mathbb{Z}$ (i.e, $\phi \in C^\ell(X;\mathbb{Z})$)

If, for example, $\phi[v_0, ..., v_\ell] = 5$, then

$$\sigma \frown \phi = \phi([v_0 \cdots v_\ell])[v_\ell \cdots v_k] = 5[v_\ell \cdots v_k] \in C_{k-\ell}(X;\mathbb{Z}).$$

It is easy to check the following properties:

- $\frown$ is bilinear (by definition)
- $\partial(\sigma \frown \phi) = (-1)^\ell(\partial\sigma \frown \phi - \sigma \frown \delta\phi)$
- $\frown(Z_k \times Z^\ell) \subseteq Z_{k-\ell}$, i.e. cycle $\frown$ cocycle = cycle
- $\frown(B_k \times Z^\ell) \subseteq B_{k-\ell}$, i.e. boundary $\frown$ cocycle = boundary

  $$\delta(\phi) = 0 \text{ implies } \partial(\sigma \frown \phi) = (-1)^\ell(\partial\sigma \frown \phi)$$

- $\frown(Z_k \times B^\ell) \subseteq Z_{k-\ell}$, i.e. cycle $\frown$ coboundary = boundary

  $$\partial(\sigma) = 0 \text{ implies } \partial(\sigma \frown \phi) = (-1)^{\ell+1}(\sigma \frown \delta\phi)$$

These facts imply that the cap product descends to a bilinear map

$$\frown : H_k(X) \times H^\ell(X) \to H_{k-\ell}(X).$$

$$\partial(\sigma \frown \phi) = \partial(\phi([v_0 \cdots v_\ell])[v_\ell \cdots v_k]) = \phi([v_0 \cdots v_\ell])\partial([v_\ell \cdots v_k])$$
$$= \phi([v_0, ..., v_\ell) \sum_{i=\ell}^{k} (-1)^{i-\ell}[v_\ell, ..., \widehat{v_i}, ..., v_k]$$
$$= \sum_{i=\ell}^{k} (-1)^{i-\ell}\phi([v_0, ..., v_\ell)[v_\ell, ..., \widehat{v_i}, ..., v_k]$$

$$\partial\sigma \frown \phi = \sum_{i=0}^{k} (-1)^i [v_0, ..., \widehat{v_i}, ..., v_k] \frown \phi =$$
$$= \sum_{i=0}^{\ell} (-1)^i \phi([v_0, ..., \widehat{v_i}, ..., v_{\ell+1}])[v_{\ell+1}, ..., v_k]$$
$$+ \sum_{i=\ell+1}^{k} (-1)^i \phi([v_0, ..., v_\ell])[v_\ell, ..., \widehat{v_i}, ..., v_k]$$

$$\sigma \cap \delta\phi = \delta\phi([v_0 \cdots v_{\ell+1}])[v_{\ell+1} \cdots v_k] = \phi\partial([v_0 \cdots v_{\ell+1}])[v_{\ell+1} \cdots v_k]$$
$$= \phi(\sum_{i=0}^{\ell+1} (-1)^i [v_0, ..., \widehat{v_i}, ..., v_{\ell+1}])[v_{\ell+1} \cdots v_k]$$
$$= \sum_{i=0}^{\ell+1} (-1)^i \phi([v_0, ..., \widehat{v_i}, ..., v_{\ell+1}])[v_{\ell+1} \cdots v_k]$$

Theorem 2 (Poincaré Duality)**.** *Let M be a closed, R-orientable n-manifold with fundamental class $[M] \in H_n(M; R)$. Then the map $D_M : H^i(M) \to H_{n-i}(M)$ is an isomorphism, where D_M is defined by*

$$D_M([\phi]) = [M] \frown [\phi].$$

Corollary 1. *Let M be a closed, connected, R-orientable n-manifold. The top homology group $H_n(M)$ is isomorphic to $\mathbb{Z}$, and $[M]$ is a generator.*