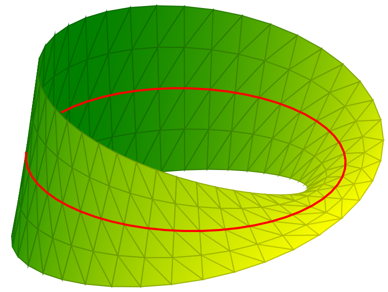
Mobius band = Cross cap



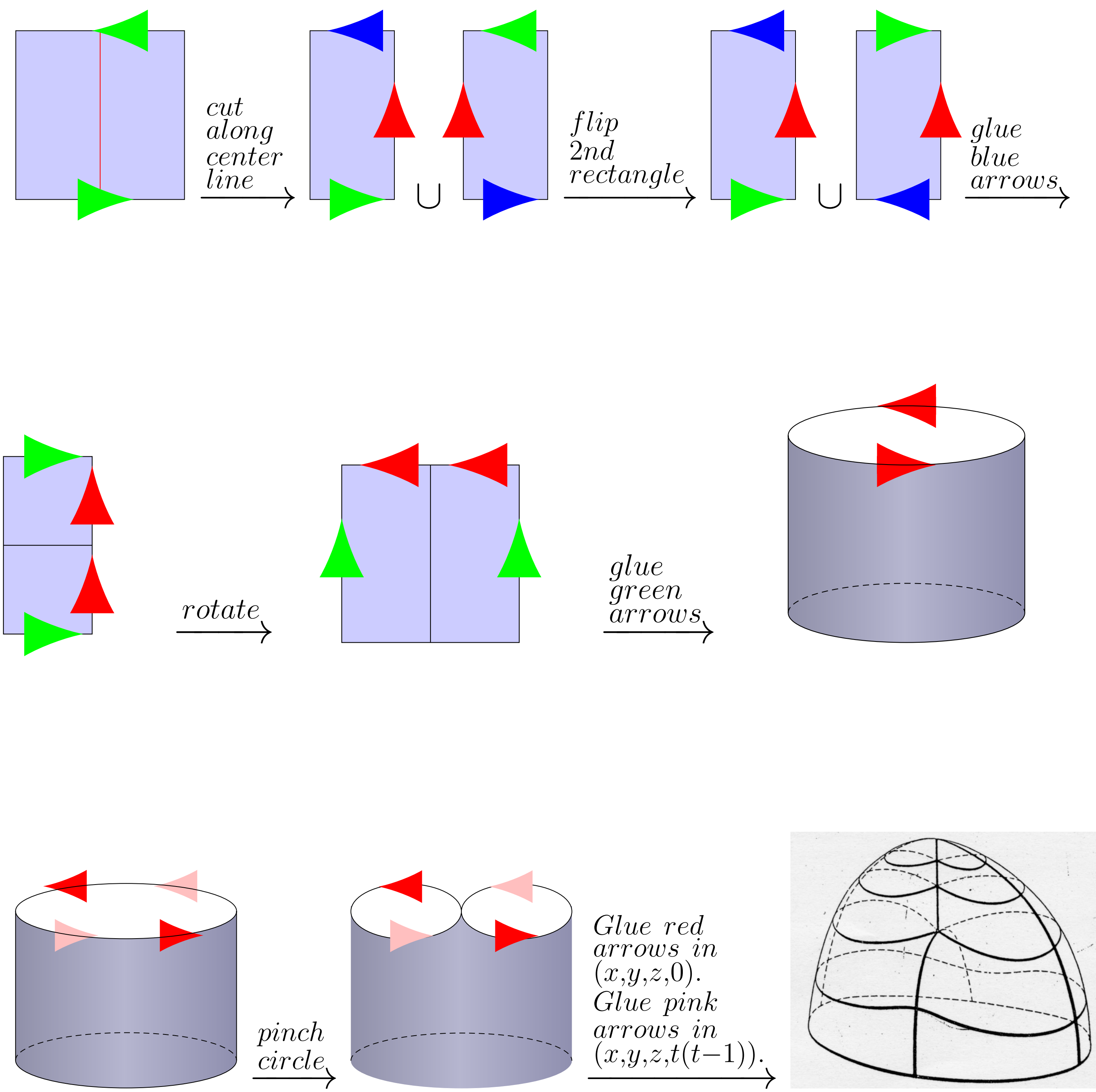


Figure 1: Cross cap in $\mathbb{R}^4$. Last figure from http://www.freud-lacan.com/freud/Champs_specialises/Langues_etrangeres/Anglais/Le_cross_cap_de_Lacan_ou_asphere

A chain complex is a sequence of homomorphisms of abelian groups such that $\partial_n \partial_{n+1} = 0$ for all n:

$$... \xrightarrow{\partial_{n+2}} G_{n+1} \xrightarrow{\partial_{n+1}} G_n \xrightarrow{\partial_n} G_{n-1} \xrightarrow{\partial_{n-1}} ....$$

Given a chain complex, define homology $H_n = Ker(\partial_n)/Im(\partial_{n+1})$.

A *chain map* $\phi : (C_\bullet, \partial_\bullet) \to \left(\widetilde{C}_\bullet, \widetilde{\partial}_\bullet\right)$ is a collection of homomorphisms $\phi_n : C_n \to \widetilde{C}_n$ such that the following diagram commutes.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle \phi_{n+1}} & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_{n-1}} & & \\
\cdots & \longrightarrow & \widetilde{C}_{n+1} & \xrightarrow{\widetilde{\partial}_{n+1}} & \widetilde{C}_n & \xrightarrow{\widetilde{\partial}_n} & \widetilde{C}_{n-1} & \xrightarrow{\widetilde{\partial}_{n-1}} & \cdots
\end{array}$$

A chain map $\phi : (C_\bullet, \partial_\bullet) \to \left(\widetilde{C}_\bullet, \widetilde{\partial}_\bullet\right)$ induces a map on homology $\phi_* : H_n \to \widetilde{H}_n$.

Suppose $f : X \to Y$ is continuous.

f induces the the chain map $f_\# : (C_\bullet(X), \partial_\bullet) \to (C_\bullet(Y), \partial_\bullet)$.

$f_\#(\sigma : \Delta \to X) = f \circ \sigma : \Delta \to Y$ and extend linearly.

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4 methods for calculating homology:

1.) via definition 2.) via matrices 3.) hand-waving

4.) $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ for X path connected. That is H_1 is the abelianization of π_1 (See Appendix 2A Hatcher).