

$$\text{Hom}(A, G) = \{h : A \rightarrow G \mid h \text{ homomorphism}\}$$

$\text{Hom}(A, G)$ is a group under function addition.

The **dual homomorphism to** $f : A \rightarrow B$ is the homomorphism $f^* : \text{Hom}(A, G) \leftarrow \text{Hom}(B, G)$ defined by $f^*(\psi) = \psi \circ f : A \rightarrow B \rightarrow G$

That is the assignment

$$A \rightarrow \text{Hom}(A, G) \quad \text{and} \quad f \rightarrow f^*$$

is a **contravariant functor** from the category of abelian groups and homomorphisms to itself since

If $i : A \rightarrow A$ is the identity map on A , then

$$i_*(\psi) = \psi \circ i = \psi \text{ is the identity map on } \text{Hom}(A, G).$$

And if $f : A \rightarrow B$, $g : B \rightarrow C$, $\psi : C \rightarrow G$

$$(f^* \circ g^*)(\psi) = f^*(g^*(\psi)) = f^*(\psi \circ g) = \psi \circ g \circ f = (g \circ f)^*(\psi)$$

In other words, if the diagram on the left commutes, so does the one on the right:

$$\begin{array}{ccc} A & \xrightarrow{k} & C \\ f \downarrow & \nearrow g & \\ B & & \end{array} \qquad \begin{array}{ccc} \text{Hom}(A, G) & \xleftarrow{k^*} & \text{Hom}(C, G) \\ f^* \uparrow & \nwarrow g^* & \\ \text{Hom}(B, G) & & \end{array}$$

- Hence f isomorphism implies f^* is an isomorphism.

- The constant fn $f = 0$ implies $f^* = 0$ since $f^*(\psi) = \psi \circ f = \psi \circ 0$.

Given a chain complex:

$$\dots \rightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \dots$$

Its dual is also a chain complex:

$$\dots \leftarrow \text{Hom}(C_{n+1}, G) \xleftarrow{\partial_{n+1}^*} \text{Hom}(C_n, G) \xleftarrow{\partial_n^*} \text{Hom}(C_{n-1}, G) \leftarrow \dots$$

Cohomology

$$\text{Cochains: } \Delta^n(X; G) = \text{Hom}(C_n, G) = \prod_{\sigma_\alpha} G$$

$$\text{Coboundary map: } \delta^1 = \partial_1^* : \Delta^0(X; G) \rightarrow \Delta^1(X; G)$$

$$\text{Cohomology: } H^n(X; G) = Z^n(X; G) / B^n(X; G) = \ker(\delta_{n+1}) / \text{im}(\delta_n)$$

$n = 0$:

$$\text{The dual of } C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \text{ is } \Delta^1(X; G) \xleftarrow{\delta_1} \Delta^0(X; G) \xleftarrow{\delta_0} 0$$

$$\text{im}(\delta_0) = 0. \text{ Thus } H^0(X; G) = \ker(\delta_1) / \text{im}(\delta_0) = \ker(\delta_1)$$

$$\psi : C_0 = \langle V \rangle \longrightarrow G, \text{ defined by } \psi(v_\alpha) = g_\alpha$$

$$\delta_1(\psi) : C_1 = \langle E \rangle \longrightarrow G,$$

$$\delta_1(\psi)([v_1, v_2]) = \psi \circ \delta([v_1, v_2]) = \psi(v_2 - v_1) = \psi(v_2) - \psi(v_1).$$

Application: ψ = elevation, $\delta_1(\psi)$ = change in elevation.

Application:

ψ = voltage at connection points, $\delta_1(\psi)$ = voltage across components.

$$\delta_1(\psi) = 0 \text{ iff}$$

$$\delta_1(\psi)([v_1, v_2]) = \psi \circ \delta_1([v_1, v_2]) = \psi(v_2 - v_1) = \psi(v_2) - \psi(v_1) = 0.$$

Thus

$$\ker(\delta_1) = \{\psi : C_0 \rightarrow G \mid \psi \text{ is constant on the components of } X\}$$

$$\text{Hence } H^0(X; G) = \prod_{\text{components of } X} G.$$

$$\text{Recall } H_0(X; G) = \sum_{\text{components of } X} G.$$

$n = 1$:

$$\text{Dual of } C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \text{ is } \Delta^2(X; G) \xleftarrow{\delta_2} \Delta^1(X; G) \xleftarrow{\delta_1} \Delta^0(X; G)$$

$$H^1(X; G) = Z^1(X; G)/B^1(X; G) = \ker(\delta_2)/\text{im}(\delta_1)$$

$$\text{im}(\delta_1) = ?$$

$$\text{Suppose } \delta_1(\psi) = \sigma : \Delta^1 \rightarrow G$$

Then σ is determined by trees in the 1-skeleton of $X = X^1$.

Let $T =$ a set of maximal trees for X^1 & let $A = \{e_a \in \Delta^1 \mid e_a \notin T\}$.

$$\text{If } \Delta^2 = 0, H^1(X; G) = \ker(\delta_2)/\text{im}(\delta_1) = \Delta^1/\text{im}(\delta_1) = \prod_{e_a \in A} G$$

$$\text{Recall if } \Delta^2 = 0, H^1(X; G) = \sum_{e_a \in A} G$$

Lemma: If $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is exact, then

$$\text{Hom}(A, G) \xleftarrow{f^*} \text{Hom}(B, G) \xleftarrow{g^*} \text{Hom}(C, G) \leftarrow 0 \text{ is exact.}$$

Proof:

Claim: g onto implies g^* is 1:1.

Suppose $g^*(\psi) = \psi \circ g = 0$. Since g is onto, $\psi(x) = 0$ for all $x \in C$. Thus $\psi = 0$ and g^* is 1:1.

Thus we have exactness at $\text{Hom}(C, G)$.

Claim: $\text{im}(f) = \ker(g)$ implies $\text{im}(g^*) = \ker(f^*)$.

$\text{im}(f) \subset \ker(g)$ implies $g \circ f = 0$

implies $f^* \circ g^* = (g \circ f)^* = 0^* = 0$ and thus $\text{im}(g^*) \subset \ker(f^*)$.

Suppose $\psi \in \ker(f^*)$, $\psi : B \rightarrow G$. Then $f^*(\psi) = \psi \circ f = 0$. Thus $\psi(f(A)) = 0$ and ψ induces homomorphism $\psi' : B/f(A) \rightarrow G$

g induces an isomorphism $g' : B/\ker(g) = B/f(A) \rightarrow C$.

$$\begin{array}{ccccc} G & \xleftarrow{\psi} & B & \xrightarrow{g} & C \\ & \searrow \psi' & \downarrow & \nearrow \cong g' & \\ & & B/f(A) & & \end{array}$$

$$g^*(\psi' \circ (g')^{-1}) = \psi' \circ (g')^{-1} \circ g = \psi$$

Lemma: If $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is split exact, then $0 \rightarrow \text{Hom}(A, G) \xleftarrow{f^*} \text{Hom}(B, G) \xleftarrow{g^*} \text{Hom}(C, G) \leftarrow 0$ is split exact.

Proof: $\exists \pi : B \rightarrow A$ such that $\pi \circ f = id_A$.

Thus $(\pi \circ f)^* = f^* \circ \pi^* = \text{identity on } \text{Hom}(A, G)$.

Thus f^* is surjective and the dual sequence splits.

Note also that $\text{Hom}(\bigoplus A_\alpha, G) = \prod \text{Hom}(A_\alpha, G)$,

and thus $\text{Hom}(A \oplus C, G) = \text{Hom}(A, G) \oplus \text{Hom}(C, G)$

Example: The dual of the exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \rightarrow 0$

$$0 \leftarrow \text{Hom}(\mathbb{Z}, G) \xleftarrow{t^*} \text{Hom}(\mathbb{Z}, G) \xleftarrow{\pi^*} \text{Hom}(\mathbb{Z}_2, G) \leftarrow 0$$

$\pi^*(\psi) = \psi \circ \pi : \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \xrightarrow{\psi} G$, defined by $(\psi \circ \pi)(1) = \psi(1)$.

$$t^*(\psi) = \psi \circ t : \mathbb{Z} \xrightarrow{t} \mathbb{Z} \xrightarrow{\psi} G,$$

defined by $(\psi \circ t)(1) = \psi(2) = \psi(1) + \psi(1) = 2\psi(1)$.

Defn: A **free resolution** of an abelian group H is an exact sequence of abelian groups,

$$\dots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \rightarrow 0$$

where each F_i is free.

Recall an exact sequence is a chain complex, and the dual of a chain complex is a chain complex.

Thus the dualization of this free resolution is a chain complex:

$$\dots \xleftarrow{f_2^*} \text{Hom}(F_1, G) \xleftarrow{f_1^*} \text{Hom}(F_0, G) \xleftarrow{f_0^*} \text{Hom}(H, G) \leftarrow 0$$

Let $H^n(F; G) = \text{Ker}(f_{n+1}^*) / \text{im}(f_n^*)$

Lemma 3.1: a.) Given two free resolutions F and F' of H and H' , respectively, every homomorphism $\alpha : H \rightarrow H'$ can be extended to a chain map from F to F' :

$$\begin{array}{ccccccccc} \dots & \longrightarrow & F_2 & \longrightarrow & F_1 & \xrightarrow{\phi} & F_0 & \xrightarrow{\psi} & H & \longrightarrow & 0 \\ & & \exists \alpha_2 \downarrow & & \exists \alpha_1 \downarrow & & \exists \alpha_0 \downarrow & & \downarrow \alpha & & \\ \dots & \longrightarrow & F'_2 & \longrightarrow & F'_1 & \xrightarrow{\phi} & F'_0 & \xrightarrow{\psi} & H' & \longrightarrow & 0 \end{array}$$

Furthermore, any two such chain maps extending α are chain homotopic.

b.) For any two free resolutions F and F' of H , \exists canonical isomorphism $H^n(F; G) = H^n(F', G)$ for all n .

Example: A short exact sequence of abelian groups,

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow H \rightarrow 0$$

where F_i are free is called a **free resolution of H** .

Example: $0 \rightarrow B_p(X) \hookrightarrow Z_p(X) \rightarrow H_p(X) \rightarrow 0$

Example:

Let $F_0 =$ the free abelian group generated by the generators of H .

Let $F_1 =$ kernel of projection map $F_0 \rightarrow H$.

Dual of the exact seq $0 \rightarrow F_1 \rightarrow F_0 \rightarrow H \rightarrow 0$ is the chain complex:

$$0 \xleftarrow{f_2^*} \text{Hom}(F_1, G) \xleftarrow{f_1^*} \text{Hom}(F_0, G) \xleftarrow{f_0^*} \text{Hom}(H, G) \leftarrow 0$$

Recall $F_1 \rightarrow F_0 \rightarrow H \rightarrow 0$ exact implies its dual is also exact:

$$\text{Hom}(F_1, G) \xleftarrow{f_1^*} \text{Hom}(F_0, G) \xleftarrow{f_0^*} \text{Hom}(H, G) \leftarrow 0$$

Note $H^n(F; G) = \text{Ker}(f_{n+1}^*)/\text{im}(f_n^*) = 0$ for $n > 1$.

And $H^0(F; G) = \text{Ker}(f_1^*)/\text{im}(f_0^*) = 0$.

But $H^1(F; G) = \text{Ker}(f_2^*)/\text{im}(f_1^*) = ?$.

Definition: $\text{Ext}(H, G) = H^1(F; G)$ (the extension of G by H).

For computational purposes, the following properties are useful.

(a) $\text{Ext}(H \oplus H', G) \cong \text{Ext}(H, G) \oplus \text{Ext}(H', G)$ since the direct sum of free resolutions is the free resolution of the direct sum.

(b) $\text{Ext}(H, G) = 0$ if H is free
since $0 \rightarrow H \rightarrow H \rightarrow 0$ is a free resolution of H .

(c) $\text{Ext}(\mathbb{Z}/n, G) \cong G/nG$
by dualizing the free resolution $0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow \mathbb{Z}_n \rightarrow 0$.

to produce the exact sequence:

$$0 \leftarrow \text{Ext}(\mathbb{Z}_n, G) \leftarrow \text{Hom}(\mathbb{Z}, G) \xleftarrow{n} \text{Hom}(\mathbb{Z}, G) \leftarrow \text{Hom}(\mathbb{Z}_n, G) \leftarrow 0$$

Theorem 1. *If a chain complex C_\bullet of free abelian groups has homology groups $H_\bullet(C)$, then the cohomology groups $H^\bullet(C; G)$ of the cochain complex $\text{Hom}(C_\bullet, G)$ are determined by the split exact sequences*

$$0 \rightarrow \text{Ext}(H_{n-1}(C), G) \rightarrow H^n(C; G) \xrightarrow{h} \text{Hom}(H_n(C), G) \rightarrow 0.$$

Corollary 1. *If the homology groups H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n$ and $T_{n-1} \subset H_n$, then*

$$H^n(C; \mathbb{Z}) \cong (H_n/T_n) \oplus T_{n-1}.$$

Corollary 2. *If a chain map between chain complexes of free abelian groups induces an isomorphism on homology groups, then it induces an isomorphism on cohomology groups with any coefficient group G .*