$\boxed{\begin{array}{l}\psi_{e_1} : C_1^\bullet \to \mathbb{Z} \\ \psi_{e_1}(e_1) = 1 \\ \psi_{e_1}(e_i) = 0 \quad i \neq 1\end{array}}$

$H^1(T^2;\mathbb{Z}) \cong \mathbb{Z}\times\mathbb{Z}$ with basis $i = 1,2$, $\phi_i(e_j) = \begin{cases} 1 & j = i, 3 \\ 0 & else \end{cases}$

Note: in terms of basis for C^1 $\phi_i = \psi_{e_1} + \psi_{e_3}$, for $i = 1, 2$.

Note: $(\delta \circ \phi_1)(\sigma_i) = \phi_1(\partial(\sigma_i)) = \phi_1(e_1 + e_2 - e_3)$

$$= \phi(e_1) + \phi(e_2) - \phi(e_3) = 1 + 0 - 1 = 0$$

$(\phi_1 \smile \phi_1)(\sigma_1) = \phi_1(e_1)\cdot\phi_1(e_2) = (1)(0) = 0$

$(\phi_1 \smile \phi_1)(\sigma_2) = \phi_1(e_2)\cdot\phi_1(e_1) = (0)(1) = 0$

Thus $\phi_1 \smile \phi_1 = 0$. Similarly, $\phi_2 \smile \phi_2 = 0$.

$(\phi_1 \smile \phi_2)(\sigma_1) = \phi_1(e_1)\cdot\phi_2(e_2) = (1)(1) = 1$

$(\phi_1 \smile \phi_2)(\sigma_2) = \phi_1(e_2)\cdot\phi_2(e_1) = (0)(0) = 0$

$\Rightarrow \phi_1 \cup \phi_2 = \psi_{\sigma_1}$

$(\phi_2 \smile \phi_1)(\sigma_1) = \phi_2(e_1)\cdot\phi_1(e_2) = (0)(0) = 0$

$(\phi_2 \smile \phi_1)(\sigma_2) = \phi_2(e_2)\cdot\phi_1(e_1) = (1)(1) = 1$

$\Rightarrow \phi_2 \cup \phi_1 = \psi_{\sigma_2}$

Thus $\phi_1 \smile \phi_2 \neq \phi_2 \smile \phi_1$ in C^2.

Multiplication is NOT commutative at chain level

$H_2(T^2;\mathbb{Z}) \cong \mathbb{Z}$ since $C^3 = 0$ implies $ker\delta = C^2$ and

$(\delta \circ \psi_{e_i})(\sigma_j) = \psi_{e_i}(\partial(\sigma_j)) = \psi_{e_i}(e_1 + e_2 - e_3) = \begin{cases} 1 & i = 1,2 \\ -1 & i = 3 \end{cases}$

Thus $\delta \circ \psi_{e_i} = \psi_{\sigma_1} + \psi_{\sigma_2}$ for $i = 1, 2$ and $\delta \circ \psi_{e_3} = -\delta \circ \psi_{e_1}$

Thus $im\delta = < \psi_{\sigma_1} + \psi_{\sigma_2} >$. Thus in H^2, $\psi_{\sigma_1} = -\psi_{\sigma_2}$

$\ker \delta = < \psi_{\sigma_1}, \psi_{\sigma_2} >$

abelian

$H^2_\bullet = < \psi_{\sigma_1}, \psi_{\sigma_2} \mid \psi_{\sigma_1} + \psi_{\sigma_2} = 0 >$

abelian ... presentation

Theorem 1. *When R is commutative, the rings $H^\bullet(X, A; R)$ are graded commutative, that is, the identity*

$$\boxed{\alpha \smile \beta = (-1)^{k\ell}\beta \smile \alpha}$$

holds for all $\alpha \in H^k(X, A; R)$ and $\beta \in H^\ell(X, A; R)$.

$(H^n(S^2 \vee S^4;\mathbb{Z}), +) \cong (H^n(\mathbb{CP}^2;\mathbb{Z}), +) \cong \begin{cases} (\mathbb{Z}, +) & n = 0, 2, 4 \\ 0 & else \end{cases}$ at the group level

But $\cup : H^2(S^2 \vee S^4;\mathbb{Z}) \times H^2(S^2 \vee S^4;\mathbb{Z}) \to H^4(S^2 \vee S^4;\mathbb{Z})$ is the zero map,

multiply 2 elements $\to 0$

while for $\cup : H^2(\mathbb{CP}^2;\mathbb{Z}) \times H^2(\mathbb{CP}^2;\mathbb{Z}) \to H^4(\mathbb{CP}^2;\mathbb{Z})$, $\cup(1, 1) = 1$.

$\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

I.e, the group structure does not distinguish $S^2 \vee S^4$ from $\mathbb{CP}^2$, but the ring structure does. different multiplicat

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Proposition 1. *For a map $f : X \to Y$, the induced maps $f^* : H^n(Y; R) \to H^n(X; R)$ satisfy $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$, and similarly in the relative case.*

Thus $H^\bullet(-; R)$ is a functor from the category of topological spaces to the category of graded R-algebras.

$\phi_1 \cup \phi_2 = -(\phi_2 \cup \phi_1)$ in $H^2_\bullet$