Let $\mathcal{U} = \{\mathcal{U}_\alpha\}$ such that $X \subset \cup U_\alpha^o$.

Then $C_n^{\mathcal{U}}(X) = \{\sum r_i \sigma_i \mid \sigma_i \subset U_\alpha \text{ for some } \alpha\}$ is a subgroup of $C_n(X)$.

$\partial(C_n^{\mathcal{U}}(X)) \subset C_{n-1}^{\mathcal{U}}(X)$ and $\partial^2 = 0$. Thus $\exists H_n^{\mathcal{U}}(X)$

Prop 2.2.1: The inclusion map $i : C_n^{\mathcal{U}}(X) \to C_n(X)$ is a chain homotopy equivalence.

I.e., $\exists \rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ wuthc that $i\rho$ and ρi are chain homotopic to the identity.

Hence i induces an isomorphism $H_n^{\mathcal{U}}(X) \cong H_n(X)$.

(1) Barycentric subdivision of of (ideal) simplices.

Simplex $[v_0, ..., v_n] = \{\sum t_i v_i \mid \sum t_i = 1, t_i \geq 0\}$

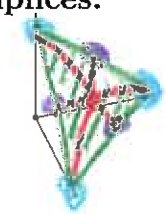


Figure 1: http://www.wikiwand.com/en/Simplex

The barycenter = center of mass ~~gravity~~ $= b = \sum_{i=0}^{n} \frac{1}{n+1} v_i$

Barycentric subdivision: decompose $[v_0, ..., v_n]$ into the $n-$simplices $[b, w_0, ..., w_{n-1}]$, inductively.

Divide each edge $[v_1, v_2]$ in half, forming 2 new edges $[b, v_1]$, $[b, v_2]$.

Note: $diam[b, v_i] = ||v_i - b|| = \frac{1}{2}||v_2 - v_1|| = \frac{1}{2}(diam[v_1, v_2])$

$[b, b_e, v]$

http://drorbn.net/AcademicPensieve/2010-06/

Claim:
If b is a barycenter of $[v_0, ..., v_{k-1}]$, then $||b - v_i|| \leq \left(\frac{k-1}{k}\right) ||v_j - v_k||$.

Thus diam $[b, w_0, ..., w_{k-1}] \leq \left(\frac{k-1}{k}\right) diam[v_0, ..., v_n]$

Note: Claim is true for $k = 2$. Suppose claim is true for $k = n - 1$.

Suppose all the faces of $[v_0, ..., v_n]$ have been subdivided. For all $n-1$-simplices $[w_0, ..., w_{n-1}]$ in this subdivision, form the $n-$simplices $[b, w_0, ..., w_{n-1}]$, where b is the barycenter of $[v_0, ..., v_n]$

By induction $||w_i - w_j|| \leq \left(\frac{n-1}{n}\right) ||v_l - v_k||$.

Let b_i be the barycenter of $[v_0, ..., \widehat{v_i}, ..., v_n]$

$$b = \sum_{j=0}^{n} \frac{1}{n+1} v_j = \left(\frac{1}{n+1}\right) v_i + \sum_{j \neq i} \left(\frac{1}{n+1}\right) v_j = \left(\frac{1}{n+1}\right) v_i + \left(\frac{n}{n+1}\right) \sum_{j \neq i} \left(\frac{1}{n}\right) v_j$$

$$= \boxed{\left(\frac{1}{n+1}\right) v_i + \left(\frac{n}{n+1}\right) b_i}$$

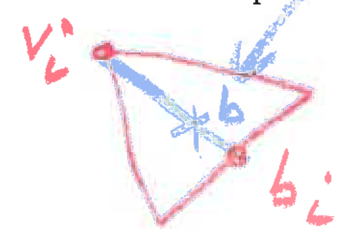
$\frac{1}{n+1} + \frac{n}{n+1} = 1$

Thus $||b - v_i|| = \left(\frac{n}{n+1}\right) ||b_i - v_i|| \leq \left(\frac{n}{n+1}\right) ||v_j - v_i||$

Thus $diam[b, w_0, ..., w_{n-1}] \leq \left(\frac{n}{n+1}\right) diam[v_0, ..., v_n]$

Thus repeated barycentric subdivision leads to simplices whose diameter is arbitrarily small.

b lies on this line



2. Barycentric subdivision of Linear Chains

For Y convex, define $LC_n(Y) = \{\lambda : \Delta^n \to Y \mid \lambda \text{ is linear }\}$

$\partial(LC_n(Y)) \subset LC_{n-1}(Y)$.

For convenience, define $LC_{-1}(Y) = \mathbb{Z} =< [\emptyset] >$ where $\partial[v] = [\emptyset]$

If $b \in Y$, define homomorphism $b : LC_n(Y) \to LC_{n+1}(Y)$, $b([w_0, ..., w_n]) = [b, w_0, ..., w_n]$, the cone operator.

$\partial b([w_0, ..., w_n]) = \partial[b, w_0, ..., w_n] = [w_0, ..., w_n] - b\partial[w_0, ..., w_n]$.

Thus if $\alpha = \sum_{i=1}^{n} r_i\lambda_i$, then $(\partial \circ b)(\alpha) = \alpha - (b \circ \partial)(\alpha)$, $\forall \alpha \in LC_n(Y)$.

$(\partial \circ b)(\alpha) + (b \circ \partial)(\alpha) = \alpha$

That is $\partial \circ b + b \circ \partial = id - 0$, where id = the identity homomorphism and 0 = the constant zero homomorphism on $LC_n(Y)$.

Thus b is a chain homotopy between the identity map and the zero homomorphism on the augmented chain complex $LC(Y)$.

Define subdivision homomorphism $S : LC_n(Y) \to LC_n(Y)$ by induction on n.

Let $\lambda : \Delta^n \to Y$ be a generator of $LC_n(Y)$.

Let $b_\lambda = \lambda(b)$ where b is the barycenter of Δ^n.

S is defined inductively

Define $S([\emptyset]) = [\emptyset]$ and $S(\lambda) = b_\lambda(S(\partial(\lambda)))$

Ex: If $\lambda = [v]$, then $b_\lambda = v$ and

$$S([v]) = b_\lambda(S(\partial([v]))) = v(S([\emptyset])) = v([\emptyset]) = [v].$$

Thus S is the identity on $LC_{-1}(Y)$ and $LC_0(Y)$.

Ex: If $\lambda = [v, w]$, $S([v, w]) = b_\lambda(S(\partial([v, w])))$
$= b_\lambda(S([w]) - S([v])) = b_\lambda([w] - [v]) = [b_\lambda, w] - [b_\lambda, v]$.

Ex: If $\lambda = [u, v, w]$, $S(u, [v, w]) = b_\lambda(S(\partial([u, v, w])))$ $= [v, b_\lambda] + [b_\lambda, u]$

$= b_\lambda(S([v, w]) - S([u, w]) + S([u, v]))$

$= b_\lambda([b_{v,w}, w] - [b_{v,w}, v] - ([b_{u,w}, w] - [b_{u,w}, u]) + [b_{u,v}, v] - [b_{u,v}, u])$

$= [b_\lambda, b_{v,w}, w] - [b_\lambda, b_{v,w}, v] - [b_\lambda, b_{u,w}, w] + [b_\lambda, b_{u,w}, u] + [b_\lambda, b_{u,v}, v] - [b_\lambda, b_{u,v}, u]$

If λ is an embedding, $S(\lambda)$ is the alternating sum of the simplices in the barycentric subdivision of λ.

Claim: S is a chain ~~homotopy~~ map between $LC_n(Y)$ and itself.

That is $\boxed{\partial S = S\partial.}$

$LC_n(Y) \Rightarrow LC_n(Y)$

Proof by induction on n:

True for $n = -1, 0$ since $S = id$.

$\partial(S(\lambda)) = \partial(b_\lambda(S(\partial(\lambda)))) = (1 - b_\lambda\partial)(S(\partial(\lambda)))$

$[v] \times I$

$$= S(\partial(\lambda)) - b_\lambda(\partial(S(\partial(\lambda)))) = S(\partial(\lambda)) - b_\lambda(S(\partial(\partial(\lambda))))$$

$$= S(\partial(\lambda)) - b_\lambda(S(0)) = S(\partial(\lambda))$$

Define a chain homotopy between S and id,

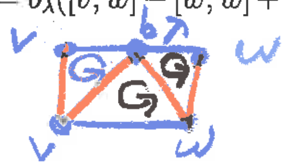
$$T : LC_n(Y) \to LC_{n+1}(Y) \text{ inductively:}$$

$\boxed{T = 0 \text{ for } n = -1, \text{ and } T(\lambda) = b_\lambda(\lambda - T\partial\lambda).}$

Thus $T([v]) = v([v] - T\partial[v]) = v([v] - T[\emptyset]) = v([v]) = [v, v]$.

$$T([v, w]) = b_\lambda([v, w] - T\partial[v, w]) = b_\lambda([v, w] - T([w] - [v]))$$

$$= b_\lambda([v, w] - [w, w] + [v, v]) = [b_\lambda, v, w] - [b_\lambda, w, w] + [b_\lambda, v, v]$$



barycenter of w

barycenter of v

$$T([u, v, w]) = b_\lambda([u, v, w] - T\partial[u, v, w])$$

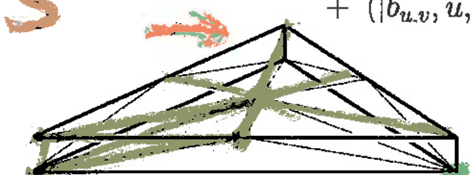
$$= b_\lambda([u, v, w] - T([v, w] - [u, w] + [u, v])$$

$$= b_\lambda([u, v, w] - ([b_{v,w}, v, w] - [b_{v,w}, w, w] + [b_{v,w}, v, v]$$

$$- ([b_{u,w}, u, w] - [b_{u,w}, w, w] + [b_{u,w}, u, u])$$

$$+ ([b_{u,v}, u, v] - [b_{u,v}, v, v] - [b_{u,v}, u, u])$$

S



$\leftarrow \Delta \times I$

id

from: Hatcher

$\partial b + b\partial = id$

$\partial b = id - b\partial$

$$\partial(T(\lambda)) = \partial b_\lambda(\lambda - T\partial\lambda) \text{ by defn of } T$$

$$= \lambda - T\partial\lambda - b_\lambda\partial(\lambda - T\partial\lambda) \qquad \text{since } \partial b_\lambda = id - b_\lambda\partial$$

$$= \lambda - T\partial\lambda - b_\lambda[\partial\lambda - \partial T(\partial\lambda)] \qquad \text{since } \partial \text{ is a homomorphism.}$$

$$= \lambda - T\partial\lambda - b_\lambda[S(\partial\lambda) - T\partial(\partial\lambda)] \text{ by } id - \partial T = S - T\partial \text{ for dim(n-1).}$$

$$= \lambda - T\partial\lambda - b_\lambda[S(\partial\lambda)] \qquad \text{since } \partial^2 = 0$$

$$= \lambda - T\partial\lambda - S(\lambda) \qquad \text{since } S(\lambda) = b_\lambda(S(\partial(\lambda)))$$

Thus $\partial T(\lambda) = \lambda - T\partial(\lambda) - S(\lambda)$. I.e., $\boxed{\partial T + T\partial = id - S.}$

In other words, T is a chain homotopy between id and S.

3. Barycentric subdivision of general chains:

Currently S is only defined on convex subsets Y.

For example: $S : C_n(\Delta^n) \to C_n(\Delta^n)$.

For example if $n = 1$, $\Delta^n = [v, w]$ with barycenter b_λ, then

$$S(id_{[v,w]}) = id_{[b_\lambda,w]} - id_{[b_l,v]}$$

We can extend S to $C_n(X)$ as follows:

$$S : C_n(X) \to C_n(X) \text{ by } S(\sigma) = \sigma_\# S(\Delta^n).$$

For example, if $\sigma : [v, w] \to X \in C_n(X)$ with barycenter b_λ,

$$S(\sigma) = \sigma_\# S(\Delta^n) = \sigma \circ (id_{[b_\lambda,w]} - id_{[b_\lambda,v]}) = \sigma_{[b_\lambda,w]} - \sigma_{[b_\lambda,v]}.$$