Thus this short exact sequence induces a long exact sequence:

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{\pi_*} H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \longrightarrow \cdots .$$

If $[\sigma] \in \underline{H_n(X,A)}$, then $\sigma \in C_n(X)$ and $\underline{\partial\sigma \in C_{n-1}(A)}$

That is $\partial_*([\sigma]) = \partial_X(\sigma)$ in $C_{n-1}(A)$

$$\begin{tikzcd}
 & \sigma \arrow[r] & {[\sigma]} \\
 & C_n(X) \arrow[r, "\pi_n"'] \arrow[d, "\partial_X"'] & C_n(X,A) \\
C_{n-1}(A) \arrow[r, "i_{n-1}"'] & C_{n-1}(X) & \\
\partial\sigma \arrow[r] & \partial\sigma &
\end{tikzcd}$$

Ex:

$$\cdots \longrightarrow H_1(\{0,1\}) \xrightarrow{i_*} H_1(\mathbb{R}) \xrightarrow{\pi_*} H_1(\mathbb{R},\{0,1\}) \xrightarrow{\partial_*} \widetilde{H}_0(\{0,1\}) \xrightarrow{i_*} \widetilde{H}_0(\mathbb{R}) \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \xrightarrow{i_*} 0 \xrightarrow{\pi_*} H_1(\mathbb{R},\{0,1\}) \xrightarrow{\partial_*} \mathbb{Z} \xrightarrow{i_*} 0 \longrightarrow \cdots$$

Thus $H_1(\mathbb{R},\{0,1\}) = \mathbb{Z}$

The following diagram does NOT commute:

$$
\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[dl, "K_n"', "\text{chain homotopy}"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[r] \arrow[dl, "K_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\delta_{n+1}"] & D_n \arrow[r, "\delta_n"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}
$$

The chain maps $f_n, g_n : C_n \to D_n$ are chain homotopic if $\exists K_n : C_n \to D_n$ such that

$$\boxed{\delta_{n+1}K_n + K_{n-1}\partial_n = f_n - g_n}$$

Claim: $f_* = g_* : H_\bullet^C \to H_\bullet^D$

Proof: If $\alpha \in Z_n^C$, then $\partial\alpha = 0$.

$$\delta_{n+1}K_n(\alpha) + K_{n-1}\partial_n(\alpha) = f_n(\alpha) - g_n(\alpha)$$

$$f_n(\alpha) = \delta_{n+1}K_n(\alpha) + g_n(\alpha)$$

$$[f_n(\alpha)] = [\delta_{n+1}K_n(\alpha) + g_n(\alpha)] = [g_n(\alpha)]$$

Thus $f_*(\alpha) = g_*(\alpha)$