4.) Given a complex $K$ and a short exact sequence of abelian groups

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

show the following is a short exact sequence of chain complexes:

$$0 \to C_n(K; G_1) \to C_n(K; G_2) \to C_n(K; G_3) \to 0$$

This induces a long exact sequence in homology. The zig-zag homomorphism, $\beta_* : H_n(K; G_3) \to H_{n-1}(K; G_1)$ is called the **Bockstein homomorphism** associated with the given coefficient sequence.

(a.) Compute $\beta_*$ for the coefficient sequences

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

where $|K|$ equals $\mathbb{R}\mathrm{P}^2$.

(b.) Repeat (a) when $|K|$ = Klein bottle.

5.) State and prove a Mayer-Vietoris Theorem for reduced homoloy. What condition does $A \cap B$ need to satisfy?

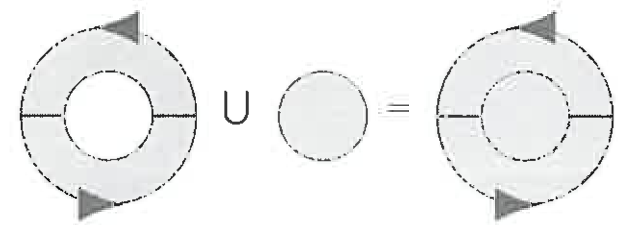

**Figure 1:** Mobius band $\cup$ disk = projective plane = $\mathbb{R}\mathrm{P}^2$

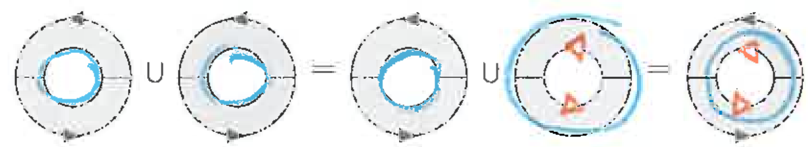

**Figure 2:** $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ = Mobius band $\bigcup$ Mobius band = Klein bottle

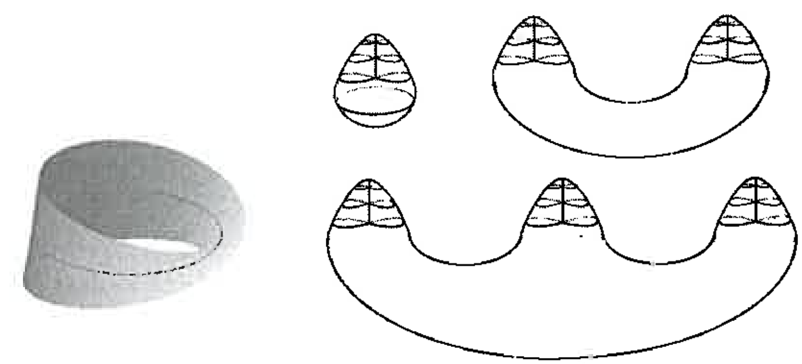

**Figure 3:** Right figures (connected sum of projective planes)
from: people.math.osu.edu/fiedorowicz.1/math655/classification.html

Defn: Let $\mathcal{C} = \{C_p, \partial_C\}$, $\mathcal{D} = \{D_p, \partial_D\}$, $\mathcal{E} = \{E_p, \partial_E\}$ be chain complexes. Let $f : \mathcal{C} \to \mathcal{D}$ and $h : \mathcal{D} \to \mathcal{E}$ be chain maps. Then the sequence

$$0 \to \mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{h} \mathcal{E} \to 0$$

is a **short exact sequence of chain complexes** if in each dimension $n$, the sequence

$$0 \to C_n \xrightarrow{f} D_n \xrightarrow{h} E_n \to 0$$

is an exact sequence of groups.

In other words, the following diagram commutes:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "g_{n+1}"] & D_n \arrow[r, "\delta_n"] \arrow[d, "g_n"] & D_{n-1} \arrow[r] \arrow[d, "g_{n-1}"] & \cdots \\
\cdots \arrow[r] & E_{n+1} \arrow[r, "d_{n+1}"] \arrow[d] & E_n \arrow[r, "d_n"] \arrow[d] & E_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

chain map

chain map

chain complexes

LEMMA. The Zig-Zag Lemma: Given chain complexes, $\mathcal{C} = \{C_n, \partial_C\}$, $\mathcal{D} = \{D_n, \partial_D\}$, $\mathcal{E} = \{E_n, \partial_E\}$ and chain maps $f$ and $g$ such that the following sequence is exact:

$$0 \to \mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{h} \mathcal{E} \to 0$$

Then $\exists$ long exact homology sequence:

$$\cdots \to H_n(\mathcal{C}) \xrightarrow{f_*} H_n(\mathcal{D}) \xrightarrow{h_*} H_n(\mathcal{E}) \xrightarrow{\partial_*} H_{n-1}(\mathcal{C}) \xrightarrow{f_*} H_{n-1}(\mathcal{D}) \to \cdots$$

where $\partial_*$ is induced by $\partial_D$. That is, $\partial_*([e_n]) = [c_{n-1}]$ where $h(d_n) = e_n$ and $f(c_{n-1}) = \partial_D(d_n)$.

*Proof.* By diagram chasing. ☐

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial_C"] & \vdots \arrow[d, "\partial_D"] & \vdots \arrow[d, "\partial_E"] & \\
0 \arrow[r] & C_{n+1} \arrow[r, "1:1", "f_{n+1}"'] \arrow[d, "\partial_C"] & D_{n+1} \arrow[r, "onto", "h_{n+1}"'] \arrow[d, "\partial_D"] & E_{n+1} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & C_n \arrow[r, "f_n"'] \arrow[d, "\partial_C"] & D_n \arrow[r, "h_n"'] \arrow[d, "\partial_D"] & E_n \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r, "f_{n-1}"'] \arrow[d, "\partial_C"] & D_{n-1} \arrow[r, "h_{n-1}"'] \arrow[d, "\partial_D"] & E_{n-1} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$d_n$

$\boxed{\partial d_n}$

$c_{n-1}$

$e_n \in Z_n(E)$

$0$

$h_n$ onto $\Rightarrow \exists\, d_n \in D_n$ st $h_n(d_n) = e_n$

$\mathrm{im}\, f_{n-1} = \ker h_{n-1}$: $h_{n-1}(\partial d_n) \cancel{= h(\partial(d_n))} = \partial(h_n(d_n)) = \partial(e_n) = 0$ since $e_n$ is a cycle

$\Rightarrow \partial d_n \in \ker h_{n-1} = \mathrm{im}\, f_{n-1}$

$$
\begin{tikzcd}
 & \vdots \arrow[d, "\partial_C"'] & \vdots \arrow[d, "\partial_D"'] & \vdots \arrow[d, "\partial_E"'] & \\
0 \arrow[r] & C_{n+1} \arrow[r, "1:1", "f_{n+1}"'] \arrow[d, "\partial_C"'] & D_{n+1} \arrow[r, "onto", "h_{n+1}"'] \arrow[d, "\partial_D"'] & E_{n+1} \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & C_n \arrow[r, "f_n"'] \arrow[d, "\partial_C"'] & D_n \arrow[r, "h_n"'] \arrow[d, "\partial_D"'] & E_n \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r, "f_{n-1}"'] \arrow[d, "\partial_C"'] & D_{n-1} \arrow[r, "h_{n-1}"'] \arrow[d, "\partial_D"'] & E_{n-1} \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$d_{n+1}$ $e_{n+1}$

$\boxed{c_n}$ $\boxed{d_n - d_n' - \partial(d_{n+1})}$ $\boxed{e_n - e_n' - \partial e_{n+1} = 0}$

$\boxed{\partial c_n}$ $c_{n-1} - c_{n-1}'$ $\boxed{\partial d_n - \partial d_n' = 0}$

But $f_{n-1}$ is 1:1

$\Rightarrow c_{n-1} - c_{n-1}' = \partial c_n \Rightarrow [c_{n-1}] = [c_{n-1}']$

Claim: $\partial_*$ is a homomorphism
pf: See HW prob # 6

$\Rightarrow \partial_*$ is well-defined

$$
\begin{tikzcd}
 & \vdots \arrow[d, "\partial_C"'] & \vdots \arrow[d, "\partial_D"'] & \vdots \arrow[d, "\partial_E"'] & \\
0 \arrow[r] & C_{n+1} \arrow[r, "1:1", "f_{n+1}"'] \arrow[d, "\partial_C"'] & D_{n+1} \arrow[r, "onto", "h_{n+1}"'] \arrow[d, "\partial_D"'] & E_{n+1} \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & C_n \arrow[r, "f_n"'] \arrow[d, "\partial_C"'] & D_n \arrow[r, "h_n"'] \arrow[d, "\partial_D"'] & E_n \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r, "f_{n-1}"'] \arrow[d, "\partial_C"'] & D_{n-1} \arrow[r, "h_{n-1}"'] \arrow[d, "\partial_D"'] & E_{n-1} \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Exactness at $H_n(\mathcal{D})$

$$
\begin{tikzcd}
 & \vdots \arrow[d, "\partial_C"] & \vdots \arrow[d, "\partial_D"] & \vdots \arrow[d, "\partial_E"] & \\
0 \arrow[r] & C_{n+1} \arrow[r, "1:1", "f_{n+1}"'] \arrow[d, "\partial_C"] & D_{n+1} \arrow[r, "onto", "h_{n+1}"'] \arrow[d, "\partial_D"] & E_{n+1} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & C_n \arrow[r, "f_n"'] \arrow[d, "\partial_C"] & D_n \arrow[r, "h_n"'] \arrow[d, "\partial_D"] & E_n \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r, "f_{n-1}"'] \arrow[d, "\partial_C"] & D_{n-1} \arrow[r, "h_{n-1}"'] \arrow[d, "\partial_D"] & E_{n-1} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$c_{n-1} \longrightarrow \partial d_n$

$\partial(c_{n-1}) \xrightarrow{f_{n-2}} \partial(\partial d_n) = 0$

$f_{n-2}(\partial(c_{n-1})) = 0 \Rightarrow \partial(c_{n-1}) = 0$ since $f_{n-2}$ is 1:1 $\Rightarrow$ $c_{n-1}$ is a cycle

$$
\begin{tikzcd}
 & \vdots \arrow[d, "\partial_C"] & \vdots \arrow[d, "\partial_D"] & \vdots \arrow[d, "\partial_E"] & \\
0 \arrow[r] & C_{n+1} \arrow[r, "1:1", "f_{n+1}"'] \arrow[d, "\partial_C"] & D_{n+1} \arrow[r, "onto", "h_{n+1}"'] \arrow[d, "\partial_D"] & E_{n+1} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & C_n \arrow[r, "f_n"'] \arrow[d, "\partial_C"] & D_n \arrow[r, "h_n"'] \arrow[d, "\partial_D"] & E_n \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r, "f_{n-1}"'] \arrow[d, "\partial_C"] & D_{n-1} \arrow[r, "h_{n-1}"'] \arrow[d, "\partial_D"] & E_{n-1} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$\boxed{d_{n+1}} \longrightarrow e_{n+1}$

$\partial d_{n+1}$

$d_n \longrightarrow \{e_n\} = \{\partial e_{n+1}\}$

$c_n \longrightarrow \boxed{d_n - \partial d_{n+1}} \longrightarrow \{e_n\} - \{e_n\} = 0$

$$
\begin{tikzcd}
 & \vdots \arrow[d, "\partial_C"] & \vdots \arrow[d, "\partial_D"] & \vdots \arrow[d, "\partial_E"] & \\
0 \arrow[r] & C_{n+1} \arrow[r, "1:1", "f_{n+1}"'] \arrow[d, "\partial_C"] & D_{n+1} \arrow[r, "onto", "h_{n+1}"'] \arrow[d, "\partial_D"] & E_{n+1} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & C_n \arrow[r, "f_n"'] \arrow[d, "\partial_C"] & D_n \arrow[r, "h_n"'] \arrow[d, "\partial_D"] & E_n \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r, "f_{n-1}"'] \arrow[d, "\partial_C"] & D_{n-1} \arrow[r, "h_{n-1}"'] \arrow[d, "\partial_D"] & E_{n-1} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \vdots \arrow[d, "\partial_C"] & \vdots \arrow[d, "\partial_D"] & \vdots \arrow[d, "\partial_E"] & \\
0 \arrow[r] & C_{n+1} \arrow[r, "1:1", "f_{n+1}"'] \arrow[d, "\partial_C"] & D_{n+1} \arrow[r, "onto", "h_{n+1}"'] \arrow[d, "\partial_D"] & E_{n+1} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & C_n \arrow[r, "f_n"'] \arrow[d, "\partial_C"] & D_n \arrow[r, "h_n"'] \arrow[d, "\partial_D"] & E_n \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r, "f_{n-1}"'] \arrow[d, "\partial_C"] & D_{n-1} \arrow[r, "h_{n-1}"'] \arrow[d, "\partial_D"] & E_{n-1} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$