

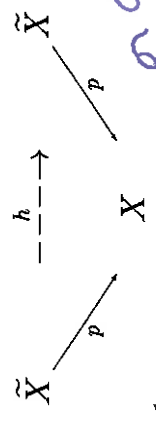
PROP. Let $H_0 = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

$\varphi : \pi_1(X, x_0) \rightarrow p^{-1}(x_0), \varphi(\alpha) = \tilde{\alpha}(1)$ induces

$$\Phi : \pi_1(X, x_0)/H_0 \rightarrow p^{-1}(x_0),$$

$$\Phi(H_0[\alpha]) = \tilde{h}\alpha(1) = \tilde{\alpha}(1)$$

If \tilde{X} path connected, then Φ is bijective.



DEFINITION 0.1.

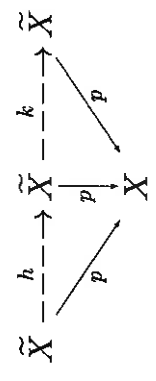
$C(\tilde{X}, p, X) = \{h : \tilde{X} \rightarrow \tilde{X} \mid h \text{ is a covering transformation for } p\}$
 $= \{h : \tilde{X} \rightarrow \tilde{X} \mid h \text{ is a homeomorphism and } p = h \circ p\}$

Examples:

$$C(\mathbb{R}, p = e^{2\pi i \theta}, S^1) = \{h_j : \mathbb{R} \rightarrow \mathbb{R}, h_j(x) = x + j \mid j \in \mathbb{Z}\} \simeq \mathbb{Z}$$

$$C(S^1, p = z^k, S^1) = \{h_j : S^1 \rightarrow S^1, h_j(e^x) = e^{x+2\pi i j/k} \mid j \in \{0, 1, \dots, k-1\}\} \simeq \mathbb{Z}_k$$

Note: $C(\tilde{X}, p, X)$ is a group under composition.



$$H = H_0$$

DEFINITION 0.2. Suppose $H < G$. The normalizer of H in G is

$$N(H) = \{g \in G \mid gHg^{-1} = H\}$$

Note: $N(H) < G$ and $H < N(H)$

$N(H)$ is the largest subgp of G containing H as a normal subgp.

Example 0.3. $N(\{e\}) = G$

Example 0.4. $N(H) = G$ if G is abelian.

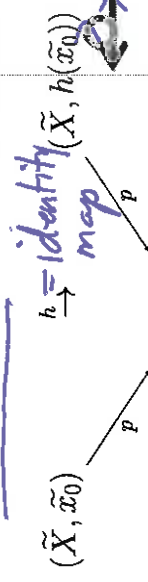
Claim: Let $H_0 = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Then $C(\tilde{X}, p, X) \simeq N(H_0)/H_0$.

Examples:

$$C(\mathbb{R}, p = e^{2\pi i \theta}, S^1) \simeq \mathbb{Z} \simeq \mathbb{Z}/\{e\} = N(\{e\})/\{e\}$$

$$C(S^1, p = z^k, S^1) \simeq \mathbb{Z}_k \simeq \mathbb{Z}/k\mathbb{Z} = N(k\mathbb{Z})/k\mathbb{Z}$$

$$C(\tilde{X}, p, S^1 \vee S^1) = \text{identity} \simeq \{e\}$$



$$H_0 = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) =$$

$$\langle a^n b a^{-n}, b^k a b^{-k} \mid n, k \in \mathbb{Z} \rangle$$

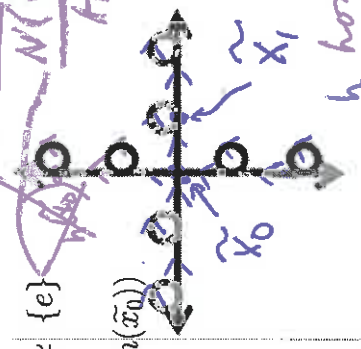
$$H = p_*(\pi_1(\tilde{X}, \tilde{x}_1)) =$$

$$\langle b, a^n b a^{-n}, a^{-1} b^k a b^{-k} a \mid n, k \in \mathbb{Z}, n \neq -1 \rangle$$

conjugate

but not equal

$$H_0 = \frac{N(H_0)}{H_0}$$



$\tilde{\alpha}(1)$

$h = e$

$N(H_0)$
||

Recall, $\Phi : \pi_1(X, x_0)/H_0 \rightarrow p^{-1}(x_0)$, $\Phi(H[\alpha]) = \tilde{\alpha}(1)$

Define: $\Psi : \mathcal{C}(\tilde{X}, p, X) \rightarrow p^{-1}(x_0)$, $\Psi(h) = h(\tilde{x}_0)$

Claim: $\Psi(\mathcal{C}(\tilde{X}, p, X)) = \Phi(N(H_0)/H_0)$
 $\Psi(\mathcal{C}(\tilde{X}, p, X)) = \{h(\tilde{x}_0) \mid h \in \mathcal{C}(\tilde{X}, p, X)\}$

$$= \{h(\tilde{x}_0) \mid p_*(\pi_1(\tilde{X}, h(\tilde{x}))) = H_0\}$$

\tilde{X} path connected implies $\exists \tilde{\alpha}$, a path in \tilde{X} from \tilde{x}_0 to $h(\tilde{x})$

$$[\alpha] \in \pi_1(X, x_0)/H_0 \text{ and } \tilde{\alpha}(1) = h(\tilde{x}_0). = \Phi(\alpha)$$

I.e., α is a loop in X and $\tilde{\alpha}$ is a path in \tilde{X} from \tilde{x}_0 to $\tilde{\alpha}(1) = h(\tilde{x}_0)$

Thus by Lemma 79.3a, $\alpha p_*(\pi_1(\tilde{X}, h(\tilde{x}))) \alpha^{-1} = H_0$.

But $h \in \mathcal{C}(\tilde{X}, p, X)$ iff $p_*(\pi_1(\tilde{X}, h(\tilde{x}_0))) = H_0$

$$H_0 = \alpha p_*(\pi_1(\tilde{X}, h(\tilde{x}))) \alpha^{-1} = \alpha H_0 \alpha^{-1} \text{ iff } \alpha \in N(H_0).$$

Thus $\Psi(\mathcal{C}(\tilde{X}, p, X)) = \Phi(N(H_0)/H_0)$

THM (81.2). The bijection $\Phi^{-1} \circ \Psi : \mathcal{C}(\tilde{X}, p, X) \rightarrow N(H_0)/H_0$ is an isomorphism of groups.

normal

COR (81.3). $H_0 \triangleleft \pi_1(X, x_0)$ iff $\forall \tilde{x} \in p^{-1}(x_0)$, \exists covering transformation h such that $h(\tilde{x}_0) = \tilde{x}$.

Note that in this case $N(H_0)/H_0 = \pi_1(X, x_0)/H_0$

COR (81.4). \tilde{X} simply connected implies $\mathcal{C}(\tilde{X}, p, X) \simeq \pi_1(X, x_0)$

$$H_0 = e \Rightarrow N(H_0) = \pi_1(X, x_0)$$

The covering map $p : \tilde{X} \rightarrow X$ is **regular** if $H_0 \triangleleft \pi_1(X, x_0)$

I.e., the covering map is regular if $N(H_0) = \pi_1(X, x_0)$

A (left) **group action** A of a group G on a set X is a function $A : G \times X \rightarrow X$, $A(g, x) = g(x)$ that satisfies the following two axioms:

Identity: $e(x) = x \quad \forall x \in X$.

Compatibility: $(gh)x = g(h(x)) \quad \forall g, h \in G \text{ and } \forall x \in X$.

The **orbit** of a point $x \in X = G(x) = \{g(x) \mid g \in G\}$.

The **orbit space** $X/G = X/\sim$ where $x \sim y$ iff $y \in G(x)$.

If G acts on a topological space X , the action is **properly discontinuous** if $\forall x \in X \exists$ an open neighborhood U of x in X , such that $g(U) \cap U \neq \emptyset$ implies $g = e$.

THM (81.5). Let \tilde{X} be pc and lpc and let G be a group of homeomorphisms of \tilde{X} . The quotient map $\pi : \tilde{X} \rightarrow \tilde{X}/G$ is a covering map iff the action of G is properly discontinuous. In this case π is regular and $\mathcal{C}(\tilde{X}, \pi, \tilde{X}/G) \simeq G$

THM (81.5). If $p : \tilde{X} \rightarrow X$ is a regular covering map and $G = \mathcal{C}(\tilde{X}, p, X)$ then \exists homeomorphism $k : X/G \rightarrow X$ such that $p = k \circ \pi$

$$\begin{array}{ccc} \tilde{X} & = & \tilde{X} \\ \pi \downarrow & & \downarrow p \\ \tilde{X}/G & \xrightarrow{k} & X \end{array}$$