Recall, assume $X, \widetilde{X}$ path connected & locally path connected.

$$F : \left\{ \begin{array}{c} (\widetilde{X}, \widetilde{x_0}) \\ \downarrow p \\ (X, x_0) \end{array} \;\middle|\; p \text{ is a covering map} \right\} \to \{H \mid H < \pi_1(X, x_0)\}$$

$$F(p) = p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$$

Note since $p_* : \pi_1(\widetilde{X}, \widetilde{x_0}) \to \pi_1(X, x_0)$ is a homomorphism, $p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$ is a subgroup of $\pi_1(X, x_0)$. Thus F is well-defined.

Prop (82.1). Suppose X is semilocally simply-connected. Then for every subgroup $H \subset \pi_1(X, x_0)$ there is a covering space $p \colon \widetilde{X_H} \to X$ such that $p_*(\pi_1(\widetilde{X_H}, \widetilde{x_0})) = H$ for a suitably chosen basepoint $\widetilde{x_0} \in \widetilde{X_H}$.

Cor: If X is semilocally simply connected, then F is onto.

Definition 0.1. X is *semilocally simply connected* if $\forall x \in X, \exists U$ open in X such that $x \in U$ and the homomorphism induced by inclusion is trivial:

$$i_* : \pi_1(U, x) \to \pi_1(X, x), \qquad i([\alpha]) = [\alpha] = [e].$$

Prop (79.2).

$$\begin{array}{ccc} (\widetilde{X_1}, \widetilde{x_1}) & \overset{\text{then } \exists! h}{- - \rightarrow} & (\widetilde{X_2}, \widetilde{x_2}) \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & (X, x_0) & \end{array}$$

$p_1\colon \widetilde{X_1} \to X$ and $p_2\colon \widetilde{X_2} \to X$ are equivalent
via a pointed homeomorphism $h\colon (\widetilde{X_1}, \widetilde{x_1}) \to (\widetilde{X_2}, \widetilde{x_2})$

if and only if

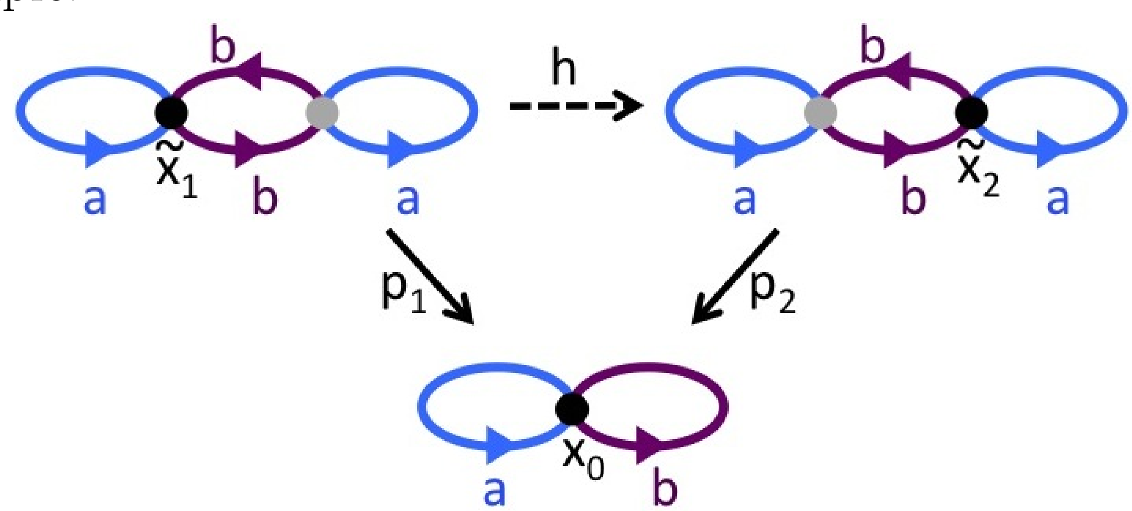
$$(p_1)_*(\pi_1(\widetilde{X_1}, \widetilde{x_1})) = (p_2)_*(\pi_1(\widetilde{X_2}, \widetilde{x_2})).$$

Cor: If $F(p) = p_*(\pi_1(\widetilde{X}, \widetilde{x_0})) = H$, then

$$F^{-1}(H) = \{\pi_1(\widetilde{X_1}, \widetilde{x_1}) \mid \exists \text{homeomorphism } h\colon (\widetilde{X}, \widetilde{x_0}) \to (\widetilde{X_2}, \widetilde{x_2}) \text{ such that above diagram commutes } \}$$

Cor: F is 1:1 if we mod out by pointed equivalence.

Example:



Prop (79.3a). Given covering map $\begin{array}{c}\widetilde{X}\\ \downarrow p\\ X\end{array}$ and $\widetilde{x_1}, \widetilde{x_2} \in p^{-1}(x_0)$,

$p_*(\pi_1(\widetilde{X}, \widetilde{x_1}))$ and $p_*(\pi_1(\widetilde{X}, \widetilde{x_2}))$are conjugate in $\pi_1(X, x_0)$.

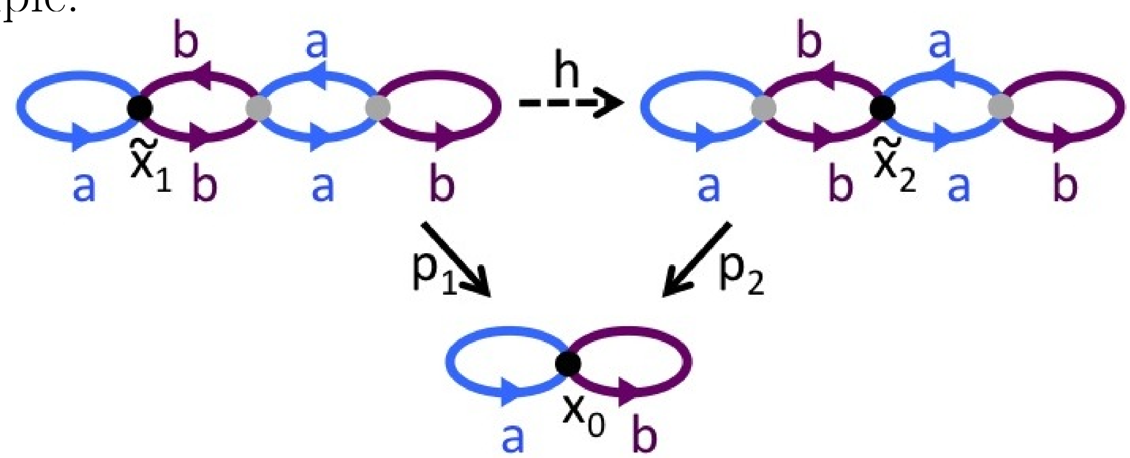
Moreover, let $H_1 = p_*(\pi_1(\widetilde{X}, \widetilde{x_1}))$ and $H_2 = p_*(\pi_1(\widetilde{X}, \widetilde{x_2}))$, let γ be a path in $\widetilde{X}$ from $\widetilde{x_1}$ to $\widetilde{x_2}$, and let $\alpha = p \circ \gamma \in \pi_1(X, x_0)$ then $H_1 = \alpha H_2 \alpha^{-1}$

Prop (79.3b). Given covering map $\begin{array}{c}(\widetilde{X}, \widetilde{x_0})\\ \downarrow p\\ (X, x_0)\end{array}$, $H_0 = p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$.

If H is a subgroup of $\pi_1(X, x_0)$, such that $H_0 = \alpha H \alpha^{-1}$, then

$\exists \widetilde{x_1} \in p^{-1}(x_0)$ such that $H = p_*(\pi_1(\widetilde{X_1}, \widetilde{x_1}))$.

Example:



PROP (79.4).

$$\begin{array}{ccc} \widetilde{X_1} & \xrightarrow{\text{then } \exists h} & \widetilde{X_2} \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & X & \end{array}$$

Suppose $p_1(\widetilde{x_1}) = p_2(\widetilde{x_2}) = x_0$ The covering maps p_1, and p_2 are equivalent iff the subgroups $(p_1)_*(\pi_1(\widetilde{X_1}, \widetilde{x_1}))$ and $(p_2)_*(\pi_1(\widetilde{X_2}, \widetilde{x_2}))$ are conjugate in $\pi_1(X, x_0)$.

Note if $h(\widetilde{x_1}) = \widetilde{x_2}$, then $(p_1)_*(\pi_1(\widetilde{X_1}, \widetilde{x_1})) = (p_2)_*(\pi_1(\widetilde{X_2}, \widetilde{x_2}))$

If $h(\widetilde{x_1}) \neq \widetilde{x_2}$, then
$(p_1)_*(\pi_1(\widetilde{X_1}, \widetilde{x_1})) = \alpha[(p_2)_*(\pi_1(\widetilde{X_2}, \widetilde{x_2}))]\alpha^{-1}$ for some $\alpha^{-1} \neq e$

$$F : \left\{ \begin{array}{c} (\widetilde{X}, \widetilde{x_0}) \\ {\scriptstyle p} \big\downarrow \\ (X, x_0) \end{array} \;\middle|\; p \text{ is a covering map} \right\} \to \{H \mid H < \pi_1(X, x_0)\}$$

$F(p) = p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$

Let $[H]$ denote the conjugacy class of H in $\pi_1(X, x_0)$:

$$[H] = \{K \mid K = gHg^{-1}\}$$

Let $[p]$ denote the set of covering maps equivalent to p. Then

$F : \{[p]\} \to \{[H]\}$ is a bijection if X semilocally simply connected.