Thm 3.1.1: Pigeonhole Principle (weak form): If you have $n+1$ objects placed in n boxes, then at least one box will be occupied by 2 or more objects.

Thm 3.1.1: Pigeonhole Principle (weak form): If you have $n+1$ pigeons in n pigeonholes, then at least one pigeonhole will be occupied by 2 or more pigeons.

Thm 3.1.1: If $f : A \to B$ is a function and $|A| = n+1$, and $|B| = n$, then f is not 1:1.

Thm 3.2.1 Pigeonhole Principle (strong form): Let $q_1, q_2, ..., q_n$ be positive integers. If $q_1 + q_2 + ... + q_n - n + 1$ objects are put into n boxes, then for some i the ith box contains at least q_i objects

Proof Outline:

Cor: Pigeonhole Principle (weak form):

Proof. Let $q_i = 2$ for all i.

Cor: If $n(r-1)+1$ objects are put into n boxes, then there exists a box containing at least r objects.

Proof: Let $q_i = r$ for all i. Note $nr - n + 1 = n(r-1)+1$.

Cor A: If $m_i, r \in \mathcal{Z}_+$ and if $\frac{m_1+\ldots+m_n}{n} > r-1$, then there exists an i such that $m_i \geq r$.

Cor A: If $m_i \in \mathcal{Z}_+$ and if $\frac{m_1+\ldots+m_n}{n} \geq r$, then there exists an i such that $m_i \geq r$.

Lemma B: If $\frac{m_1+\ldots+m_n}{n} < r$, then there exists an i s. t. $m_i < r$.

Appl 7: If you have an arbitrary number of apples, bananas and oranges, what is the smallest number of these fruits that one needs to put in a basket in order to guarantee there are at least 8 apples or at least 6 bananas or at least 9 oranges in the basket.

Appl: Suppose you have 20 pairs of socks. If 7 are black and 13 are white, and if you grab n socks at random, what should n be so that you are guaranteed to have a pair of socks of the same color.

Appl: Suppose you have 20 pairs of different shoes in your closet. If you grab n shoes at random, what should n be so that you are guaranteed to have a matching pair of shoes.

Appl 9: Show that every sequence $a_1, a_2, ..., a_{n^2+1}$ contains either an increasing or decreasing subsequence of length $n+1$.

Example ($n = 2$):

$a_1 =$, $a_2 =$, $a_3 =$, $a_4 =$, $a_5 =$

Let m_k = length of largest increasing subsequence beginning with a_k.

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a_1: $m_1 =$

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a_2: $m_2 =$

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a_3: $m_3 =$

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a_4: $m_4 =$

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a_5: $m_5 =$

Example ($n = 2$):

$a_1 = 8,\ a_2 = 4,\ a_3 = 10,\ a_4 = 6,\ a_5 = 4$

Need $n + 1$ objects in our subsequence. Suppose $r = n + 1$.

Hence might need $n(r - 1) + 1 = n(n + 1 - 1) + 1 = n^2 + 1$ objects in n boxes in order to obtain at least $r = n + 1$ objects in one of the boxes.

Let m_k = length of largest increasing subsequence beginning with a_k.

8 8, 10 $m_1 = 2$

4 4, 10 4, 6 4, 4 $m_2 = 2$

10 $m_3 = 1$ **6** $m_4 = 1$ **4** $m_5 = 1$

Proof: Let m_k = length of largest increasing subsequence beginning with a_k, $k = 1, ..., n^2 + 1$.

Suppose there exists an $m_k \geq n + 1$. Then there exists an increasing subsequence of length $m_k \geq n + 1$. Hence there exists an increasing subsequence of length $n+1$.

Suppose $m_k < n+1$. Then $m_k = 1, 2, ...,$ or n.

Hence there exists an i such that $m_k = i$ for $n+1$ a_k's.

There exists $a_{k_1}, a_{k_2}, ..., a_{k_{n+1}}$ such that

$$m_{k_1} = m_{k_2} = ... = m_{k_{n+1}} = i$$

Show $a_{k_1}, a_{k_2}, ..., a_{k_{n+1}}$ is a decreasing sequence.

Suppose not. Then there exists a j such that $a_{k_j} > a_{k_{j+1}}$.

$\exists$ an increasing subsequence of length i starting at a_{k_j}

There does not exist an increasing subsequence of length $i+1$ starting at a_{k_j}

$\exists$ an increasing subsequence of length i starting at $a_{k_{j+1}}$

There does not exist an increasing subsequence of length $i+1$ starting at $a_{k_{j+1}}$

Suppose $a_{k_{j+1}}, a_{h_2}, a_{h_3}, ..., a_{h_i}$ is an increasing subsequence of length i.

Then $a_{k_j}, a_{k_{j+1}}, a_{h_2}, a_{h_3}, ..., a_{h_i}$ is an increasing subsequence of length $i+1$, a contradiction.

Application 6: Chinese remainder theorem:
Suppose $m, n, a, b \in \mathcal{Z}$, $(m, n) = 1$, $0 \leq a \leq m-1$, $0 \leq b \leq n-1$, then $\exists x \geq 0$ such that $x = pm + a = qn + b$ for $p, q \in \mathcal{Z}$.

Moreover can take $p \in \{0, ..., n-1\}$.

Scratch work:

a is the remainder when x is divided by m.
b is the remainder when x is divided by n.

$x = a \bmod m, \quad x = b \bmod n.$

Proof plus thoughts:

We need to use the Pigeonhole principle (or related theorem). Thus we need to create objects. We are interested in $pm + a$ for some unknown $p \in \mathcal{Z}$. Thus one idea is to create the following objects:

$\mathcal{O} = \{a, m + a, 2m + a, ..., (n - 1)m + a\}$.

Note $\mathcal{O}$ has ________ distinct objects.

We need to create boxes. What else are we interested in? How about remainders?

Let $r_k =$ the remainder of $km + a$ when divided by n.

Properties of r_k: