

Directed graph

Idea: extend graphs by letting edges have an explicit direction:

- Representing one-way streets in a street plan
- Expressing asymmetry in social relationships (Alice likes Bob: $A \rightarrow B$)
- Expressing asymmetry in communication networks

Definition

A **directed graph** or **digraph** D is a tuple (V, A) of **vertices** V , and a collection of **arcs** A where each arc $a = (\overrightarrow{u, v})$ joins a vertex (tail) $u \in V$ to another (not necessarily distinct) vertex (head) v .

Basic properties

Definition

For a vertex v of digraph D , the number of arcs with head v is called the **indegree** $\delta_{in}(v)$ of v . The **outdegree** $\delta_{out}(v)$ is the number of arcs having v as their tail.

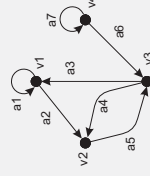
Theorem

$$\forall D : \sum_{v \in V(D)} \delta_{in}(v) = \sum_{v \in V(D)} \delta_{out}(v) = |A(D)|$$

Proof

- Every arc in D has exactly one head and one tail.
- $\sum_{v \in V(D)} \delta_{in}(v)$ is the same as counting all arc heads
- $\sum_{v \in V(D)} \delta_{out}(v)$ is the same as counting all tails
- Both are equal to the total number of arcs.

Adjacency matrix

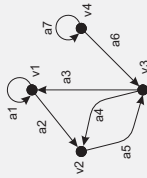


	V_1	V_2	V_3	V_4	Σ
V_1	1	1	0	0	2
V_2	0	0	1	0	1
V_3	1	1	0	0	2
V_4	0	0	1	1	2
Σ	2	2	2	1	7

Observations

- Adjacency matrix is *not* necessarily symmetric: in general, $A[i, j] \neq A[j, i]$.
- A digraph D is **strict** iff $A[i, j] \leq 1$ and $A[i, j] = 0$.
- $\forall v_i : \sum_j A[i, j] = \delta_{out}(v_i)$ and $\sum_j A[j, i] = \delta_{in}(v_i)$.

Incidence matrix



$$M[i, j] = \begin{cases} 1 & \text{if vertex } v_i \text{ is the tail of arc } a_j \\ -1 & \text{if vertex } v_i \text{ is the head of arc } a_j \\ 0 & \text{otherwise} \end{cases}$$

Observation

Incidence matrices for digraphs cannot capture loops, making these matrices being used less often compared to undirected graphs.

Connectivity

Definition

A **directed** (v_0, v_k) -**walk** is an alternating sequence $[v_0, a_0, v_1, a_1, \dots, v_{k-1}, a_{k-1}, v_k]$ with $a_i = (v_i, v_{i+1})$.

- A **directed trail** is a directed walk with distinct arcs.
- A **directed path** is a directed trail with distinct vertices.
- A **directed cycle** is a directed trail with distinct vertices except for $v_0 = v_k$.

Definition

D is **strongly connected** if there exists a directed path between every pair of distinct vertices from D . D is **weakly connected** if its underlying (undirected) graph is connected.

Reachability

Definition

Vertex v is **reachable** from vertex u if there exists a directed (u, v) -path.

Algorithm (Reachable vertices)

$R_t(u)$ is set of **reachable vertices** from u found after t steps.
 $N_{out}(v)$ is **out-neighbors** of v : $N_{out}(v) = \{w \in V(D) \mid \exists (v, w) \in A(D)\}$.

1. Set $t \leftarrow 0$ and $R_0(u) \leftarrow \{u\}$.
2. Construct the set $R_{t+1}(u) \leftarrow R_t(u) \cup \left(\bigcup_{v \in R_t(u)} N_{out}(v) \right)$.
3. If $R_{t+1}(u) = R_t(u)$, stop. $R(u) \leftarrow R_t(u)$. Otherwise, increment t and repeat the previous step.