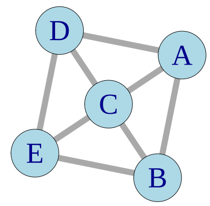
Math 4060 Mini-exam 1   Feb 13, 2019   SHOW ALL WORK

1.) Calculate the following for the graph below:



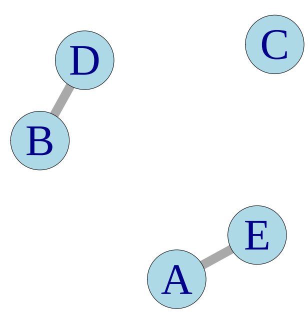
[10] 1a) $\delta(A) =$ __3__ $N(A) =$ __$\{B, C, D\}$__ $\kappa(G) =$ __3__ $\lambda(G) =$ __3__

[4] 1b) The degree sequence for G is __$[4, 3, 3, 3, 3]$__

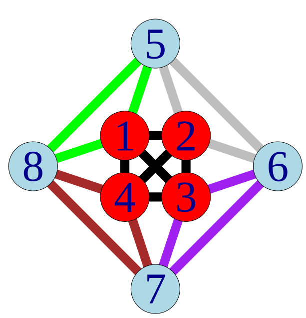
[4] 1c) The adjacency matrix of G is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

[4] 1d) Draw, $\overline{G}$ = the complement of G:



[4] 1e) Draw, $L(G)$ = the line graph of G:



[14] 2.) Choose **2** from the following 3 problems. **Clearly indicate your choices**. You may attempt all problems for additional partial credit as discussed in class.

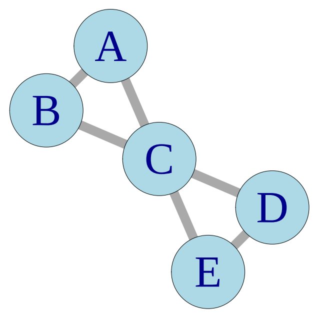
2a.) Give an example of a planar graph, G, with 5 vertices that contains an Eulerian circuit where $\kappa(G) = 1$ and $\lambda(G) = 2$.

What is the Eulerian circuit? *ABCDECA or BCDECAB or ...*

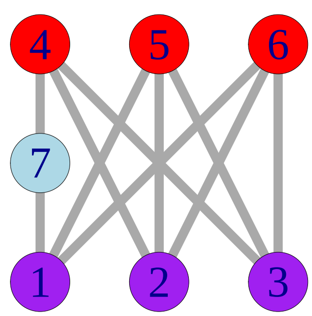
A minimal vertex cut for G is $\{C\}$

A minimal edge cut for G is $\{<A, B>, <A, C>\}$ or $\{<A, B>, <B, C>\}$

or $\{<D, E>, <D, C>\}$ or $\{<E, D>, <E, C>\}$

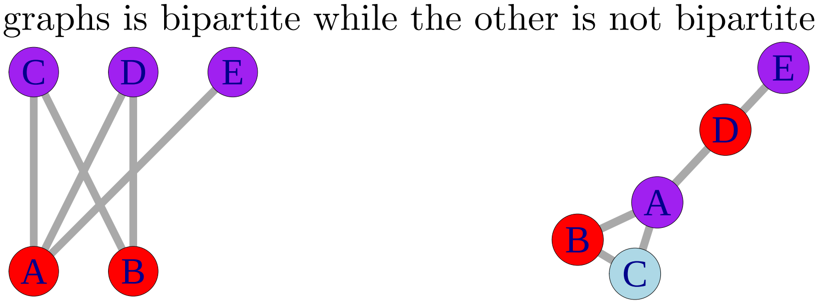


2b.) Give an example of non-planar graph with 7 vertices.



Or subdivide K_5, adding 2 vertices (splitting 2 edges or 1 edge twice).

2c.) Give an example of two non-isomorphic graphs with degree sequence $[3, 2, 2, 2, 1]$ where one of the graphs is bipartite while the other is not bipartite.



Note the 2nd graph is obtained using the Havel-Hakimi algorithm.

[10] 3.) Choose **1** from the following 2 problems. **Clearly indicate your choice**. You may attempt both problems for additional partial credit as discussed in class.

3a.) Prove that the following 2 graphs are isomorphic. Hint: Start the proof by labeling the vertices.

Proof: Two graphs G_1 and G_2 are isomorphic if $\exists$ a bijection $f : V(G_1) \to V(G_2)$ such that f induces a bijection $f : E(G_1) \to E(G_2)$, $f(< u, v >) = < f(u), f(v) >$

Alternatively, two graphs G_1 and G_2 are isomorphic if $|V(G_1)| = |V(G_2)|$, $|E(G_1)| = |E(G_2)|$, and $\exists$ an injective map $f : V(G_1) \to V(G_2)$ such that f induces a map $f : E(G_1) \to E(G_2)$, $f(< u, v >) = < f(u), f(v) >$

Define $f : G_1 \to G_2$ by $f(a_i) = c_i$ and $f(< u, v >) = < f(u), f(v) >$

Note that $f(< a_1, a_i >) =< c_1, c_i >$ for $i = 2, 3, 5$

Also, $f(< a_4, a_i >) =< c_4, c_i >$ for $i = 2, 3$

Moreover, $f(< a_5, a_i >) =< c_5, c_i >$ for $i = 2, 3$

Alternate proof: Note the adjacency matrix of G_1 is the same as the adjacency matrix of G_2.

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

3b.) Prove that a graph $G = (V, E)$ where $V = \{v_1, ..., v_n\}$ is bipartite if and only if the vertices can be ordered so that the adjacency matrix is of the form $\begin{pmatrix} 0_{k\times k} & A \\ B & 0_{\ell\times\ell} \end{pmatrix}$ where $0_{m\times m}$ is an $m \times m$ matrix whose entries are all 0.

Proof: ($\Rightarrow$)

Suppose G is bipartite. Then $V(G) = U \cup W$ where $U \cap W = \emptyset$ and
$E(G) \subset \{< u, w > \ | \ u \in U, w \in W\}$

Let $k = |U|$

Order $V(G) = \{v_1, ...v_k, v_{k+1}, ..., v_n\}$ where $v_i \in U$ if $i \leq k$ and $v_i \in W$ if $i > k$.

Let A be the adjacency matrix of G where $a_{ij} = \begin{cases} 1 & \text{if } < v_i, v_j > \in \ E(G) \\ 0 & \text{else} \end{cases}$

Note $a_{ij} = 0$ for $i, j \leq k$ since $v_i, v_j \in U$ when $i, j \leq k$

Note $a_{ij} = 0$ for $i, j > k$ since $v_i, v_j \in W$ when $i, j > k$

Thus $A = \begin{pmatrix} 0_{k\times k} & A \\ B & 0_{\ell\times\ell} \end{pmatrix}$

($\Leftarrow$) Suppose the adjacency matrix of G is $A = \begin{pmatrix} 0_{k\times k} & A \\ B & 0_{\ell\times\ell} \end{pmatrix}$.

Let $V(G) = \{v_1, ...v_k, v_{k+1}, ..., v_n\}$, so that the subscripts correspond to the rows (or equivalently columns) of A.

Recall G is bipartite if $\exists U, V$ such that $V(G) = U \cup W$ where $U \cap W = \emptyset$ and
$E(G) \subset \{< u, w > \ | \ u \in U, w \in W\}$

Let $U = \{v_1, ...v_k\}$ and $W = \{v_{k+1}, ..., v_n\}$

Let $< x, y > \in E(G)$. Then since $a_{ij} = 0$ for $i, j \leq k$, $a_{ij} = 0$ for $i, j > k$, then either $x \in U$ and $y \in W$ or $y \in U$ and $x \in W$.

Since $< x, y > = < y, x >$, WLOG assume $x \in U$ and $y \in W$.

Thus $E(G) \subset \{< u, w > \ | \ u \in U, w \in W\}$.

Therefore G is bipartite.