Directed graph

Idea: extend graphs by letting edges have an explicit direction:

- Representing one-way streets in a street plan
- Expressing asymmetry in social relationships (Alice likes Bob: $A \to B$)
- Expressing asymmetry in communication networks

Definition

A directed graph or digraph D is a tuple (V, A) of vertices V, and a collection of arcs A where each arc $a = \langle \overrightarrow{u, v} \rangle$ joins a vertex (tail) $u \in V$ to another (not necessarily distinct) vertex (head) v.

Basic properties

Definition

For a vertex v of digraph D, the number of arcs with head v is called the indegree $\delta_{in}(v)$ of v. The outdegree $\delta_{out}(v)$ is the number of arcs having v as their tail.

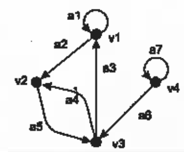
Theorem

$\forall D : \sum_{v \in V(D)} \delta_{in}(v) = \sum_{v \in V(D)} \delta_{out}(v) = |A(D)|$

Proof

- Every arc in D has exactly one head and one tail.
- $\sum_{v \in V(D)} \delta_{in}(v)$ is the same as counting all arc heads
- $\sum_{v \in V(D)} \delta_{out}(v)$ is the same as counting all tails
- Both are equal to the total number of arcs.

Adjacency matrix

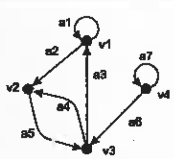


	v_1	v_2	v_3	v_4	$\sum$
v_1	1	1	0	0	2
v_2	0	0	1	0	1
v_3	1	1	0	0	2
v_4	0	0	1	1	2
$\sum$	2	2	2	1	7

Observation

- Adjacency matrix is ***not*** necessarily symmetric: in general, $\mathbf{A}[i,j] \neq \mathbf{A}[j,i]$.
- A digraph D is strict iff $\mathbf{A}[i,j] \leq 1$ and $\mathbf{A}[i,i] = 0$.
- $\forall v_i : \sum_j \mathbf{A}[i,j] = \delta_{out}(v_i)$ and $\sum_j \mathbf{A}[j,i] = \delta_{in}(v_i)$.

Incidence matrix



	a_1	a_2	a_3	a_4	a_5	a_6	a_7
v_1	0	1	-1	0	0	0	0
v_2	0	-1	0	-1	1	0	0
v_3	0	0	1	1	-1	-1	0
v_4	0	0	0	0	0	1	0

$$\mathbf{M}[i,j] = \begin{cases} 1 & \text{if vertex } v_i \text{ is the tail of arc } a_j \\ -1 & \text{if vertex } v_i \text{ is the head of arc } a_j \\ 0 & \textit{otherwise} \end{cases}$$

Observation

Incidence matrices for digraphs cannot capture loops, making these matrices being used less often compared to undirected graphs.

Connectivity

Definition

A directed (v_0, v_k)-walk is an alternating sequence $[v_0, a_0, v_1, a_1, \ldots, v_{k-1}, a_{k-1}, v_k]$ with $a_i = \langle \overrightarrow{v_i, v_{i+1}} \rangle$.

- A directed trail is a directed walk with distinct arcs.
- a directed path is a directed trail with distinct vertices.
- a directed cycle is a directed trail with distinct vertices except for $v_0 = v_k$.

Definition

D is strongly connected if there exists a directed path between every pair of distinct vertices from D. D is weakly connected if its underlying (undirected) graph is connected.

Reachability

Definition

Vertex v is reachable from vertex u if there exists a directed (u, v)-path.

Algorithm (Reachable vertices)

$R_t(u)$ is set of reachable vertices from u found after t steps. $N_{out}(v)$ is out-neighbors of v: $N_{out}(v) = \{w \in V(D) | \exists \langle \overrightarrow{v, w} \rangle \in A(D)\}$.

1. *Set $t \leftarrow 0$ and $R_0(u) \leftarrow \{u\}$.*
2. *Construct the set $R_{t+1}(u) \leftarrow R_t(u) \cup \left(\cup_{v \in R_t(u)} N_{out}(v) \right)$.*
3. *If $R_{t+1}(u) = R_t(u)$, stop: $R(u) \leftarrow R_t(u)$. Otherwise, increment t and repeat the previous step.*