just need 1 pair

Thm 4.6: Let G be a graph with n vertices. Suppose also that $\exists$ non-adjacent vertices u and w such that $\delta(u) + \delta(w) \geq n$. and for this pair

Then G is Hamiltonian $\Longleftrightarrow$ $G + < u, w >$ is Hamiltonian.

Proof ($\Rightarrow$) A Hamiltonian cycle in G is also a Hamiltonian cycle in $G + < u, w >$.

($\Leftarrow$) Suppose $G + < u, w >$ is Hamiltonian.

Proof by contradiction: Suppose G is not Hamiltonian.

Thus the Hamiltonian cycle in $G + < u, w >$ must contain the edge $< u, w >$ (else this cycle would be a Hamiltonian cycle in G).

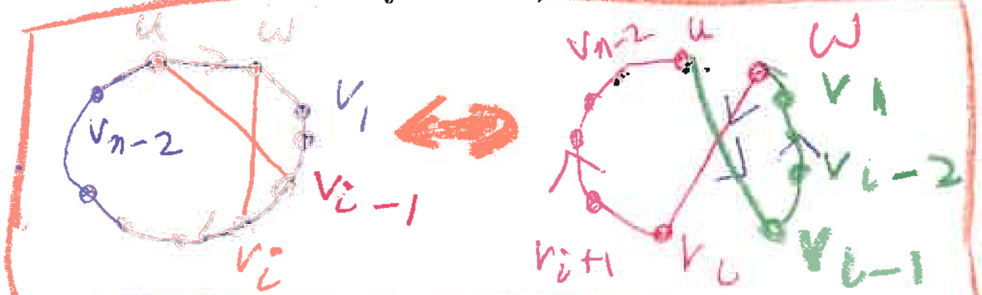
Thus we can write the Hamiltonian cycle as

$$\boxed{u, w, v_1, v_2, \ldots, v_{n-2}, u}$$ ← specific

Suppose $< u, v_{i-1} >$ and $< w, v_i >$ are both edges in G. Then

$$w, v_i, \ldots, v_{n-2}, u, v_{i-1}, v_{i-2}, \ldots, v_1, w$$

is a Hamiltonian cycle in G, a contradiction.



$\Rightarrow$ $v_{j_i+1} \notin N_G(w)$ for $i = 1, \ldots k-1$

$N_G(u) = \{v_{n-2}\} \cup \{v_{j_1}, \ldots v_{j_{k-1}}\}$

Thus $< u, v_{i-1} > \in E(G)$ implies $< w, v_i > \notin E(G)$.

Suppose in G, $\boxed{\delta(u) = k}$, then $\delta_G(w) \leq n - 1 - 1 - (k-1) = n-k-1$

no loops

Thus in G, $\delta_G(u) + \delta_G(w) \leq$

$\delta_G(u) + \delta_G(w) \leq k + n - k - 1 \leq n-1 < n$ not neighbor of w in G

Defn 4.3: If $V(G) = n$, the **closure** of G is the graph obtained from G by iteratively adding edges to G joining non-adjacent vertices u and w where $\delta(u) + \delta(w) \geq n$.

To create the closure of G, create the sequence

$$G_0 = G,\ G_1 = G_0 + < u_0, w_0 >, \ldots, \quad G_k = G_{k-1} + < u_{k-1}, w_{k-1} >$$

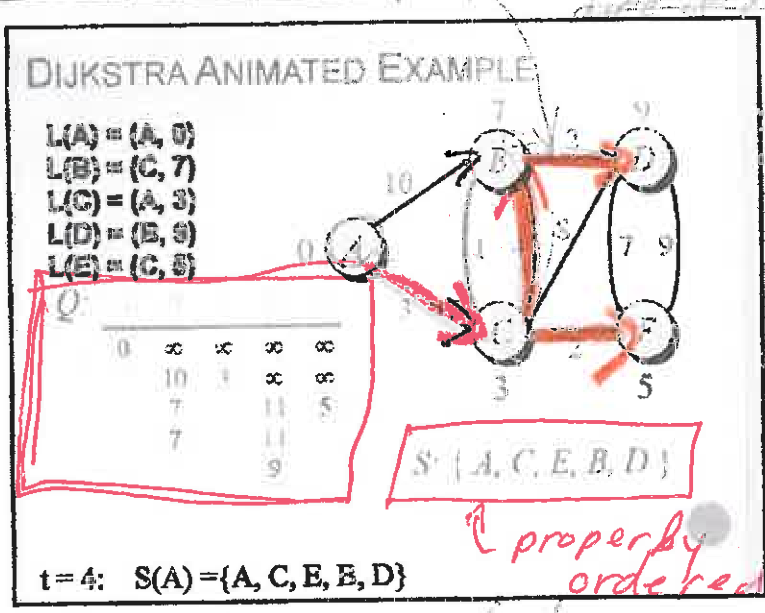
where $< u_i, w_i > \notin G_i$ and $\delta(u_i) + \delta(w_i) \geq n$ in G_i.

Moreover if $\delta(u) + \delta(w) \geq n$ in G_k, then $< u, w > \in G_k$.

Thm 4.7: A simple graph is Hamiltonian if and only if its closure is Hamiltonian.

Found a spanning tree ~~for undirected graph at least for directed graph~~

starts from A



Spanning tree includes all vertices

Given (graph G) Connected

Given vertex $A \in G$

Find all paths P

$P = A, v_1, \ldots v_k = v_0, v_1, \ldots v_k$ where $A = v_0$

st $\sum_{i=0}^{k-1} w(\langle v_i, v_{i+1} \rangle)$ (directed or undirected)

is ~~min~~ smallest among all paths from $v_0 = A$ to v_k

If unweighted graph, let $w(\langle v, w \rangle) = 1$ $\forall$ edges

Note for shortest path problem, greedy algorithm works

We don't need to look at all paths