Study guide for the second exam

Math 2374, Fall 2006

1. Higher order partial derivative (section 3.1)
   (a) Be able to compute all secord-order partial derivatives
   (b) Clairaut's Theorem: mixed partials are equal for twice continuously differentiable functions
   (c) ~~Sample book problems: 3.1 #2, #15(a)~~
2. Parametrized curves, length, and vector fields ~~(Chapter 4)~~
   (a) Paths (parametrized curves)
      i. Key idea: A vector-valued function of one variable (e.g., $\mathbf{c}(t)$) parametrizes a path.
      ii. Find parametrizations of curves such as lines, circles, ellipses, and segments of these (needed especially to compute path and line integrals over curves)
      iii. A parametrization needs both a function $\mathbf{c}(t)$ and a range $a \leq t \leq b$.
      iv. Can parametrize in two directions (orientations). (Could think of unit tangent vector $\mathbf{T} = \mathbf{c}'(t)/\|\mathbf{c}'(t)\|$ as specifying direction.)
   (b) Path length
      i. Key idea: path length element of $\mathbf{c}(t)$ is $ds = \|\mathbf{c}'(t)\|dt$.
      ii. The length of a curve C parametrized by $\mathbf{c}(t)$ for $a \leq t \leq b$ is $L(C) = \int_C ds = \int_a^b \|\mathbf{c}'(t)\|\,dt$.
      iii. Can parametrize a curve in multiple ways, but path length is independent of parametrization.
   (c) Vector fields (section 14.1)
      i. For this class, our main use of vector fields is when we compute line integrals (and later surface integrals) of vector-valued functions (vector fields).
      ii. It's good to know how to sketch vector fields. In particular, it will allow you to estimate values of line integrals to double-check your answers.
   (ci) Divergence and curl
      i. Key idea for divergence: measures outflow per unit volume of fluid flow
      ii. Key idea for curl: measures rotation of fluid flow
      iii. $\cancel{\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F}}$
      iv. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F}$
   (cii) ~~Sample book problems: 4.2 #6, #9, 4.3 #5, 4.4 #11, #14~~

3. Double integrals (sections 5.1 – 5.4)
   (a) Key idea: although defined by Riemann sums over rectangles, these integrals can be computed through iterated integrals.
   (b) Be able to compute bounds for iterated integrals, especially for the di erent orders of integration.
   (c) Remember: outer limits must be constant; inner limits can depend only on variables from the outside integral.
   (d) Finding limits and changing order of integration are easiest if you draw pictures. Inequalities are helpful, too.
   (e) ~~Sample book problems: 5.1 #8, 5.2 #2(b), #7, 5.3 #2(c), #4, 5.4 #2(c), 10, 13~~
4. Triple integrals (section 13.6)
   (a) Key idea: although defined by Riemann sums over boxes, these integrals can be computed through iterated integrals.
   (b) One trick: computing bounds for iterated integrals, especially for the di erent orders of integration.
   (c) Remember: outer limits must be constant; inner limits can depend only on variables from the outside integral(s).
   (d) Finding limits and changing order of integration are easiest if you draw pictures. Inequalities are helpful, too.
   (e) ~~Sample book problems: 5.5 #6, #9, #21, #22~~
5. Path integrals of scalar functions (Section 14.2)
   (a) Key idea: Integrate scalar function $f(\mathbf{x})$ along curve (i.e., $f(\mathbf{c}(t))$) using the ds from path length.
   (b) Formula: $\int_C f\,ds = \int_a^b f(\mathbf{c}(t))\|\mathbf{c}'(t)\|dt$
   (c) If $f(\mathbf{c})$ is density of wire, then $\int_C f\,ds$ is mass of wire.
   (d) If $f(\mathbf{c}) = 1$, then $\int_C f\,ds = \int_C ds$ is length of C.
   (e) $\int_C f\,ds$ is independent of parametrization of C.
   ~~(f) Sample book problems: 7.1 #3(b), #7(a), #10~~
6. Line integrals of vector-valued functions (Section 14.2)
   (a) Key idea: Integrate tangent component of $\mathbf{F}(\mathbf{x})$ along curve (i.e. $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{T}$) using above ds.
   (b) Formula: $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt$.
   (c) If $\mathbf{F}$ is a force field, then $\int_C \mathbf{F} \cdot d\mathbf{s}$ is the work done by the force field on a particle moving along C.

(d) $\int_C \mathbf{F} \cdot d\mathbf{s}$ is independent of parametrization of C, but depends on the direction of C, as $\int_{C^-} \mathbf{F} \cdot d\mathbf{s} = -\int_C \mathbf{F} \cdot d\mathbf{s}$

~~(e) Sample book problems: 7.2 #2(c), #7, #14~~

7. Green's Theorem (section 14.4)

(a) Key idea: If computing a line integral of a vector field $\mathbf{F}$ over a closed curve in 2D, you can convert it to a double integral (if $\mathbf{F}$ is defined in the whole interior of the curve).

(b) Formula: $\displaystyle\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$

(c) Sometimes, we write $\mathbf{F} = (P, Q)$, in which case Green's theorem is written $\displaystyle\int_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$

(d) Important: you need a "positively oriented" boundary $C = \partial D$ correctly. The region D must be on your left as you move along C. (This means inner boundaries will go the opposite direction of outer boundaries.)

(e) Other application: you can use Green's theorem to calculate the area of the region D, which is $\iint_D dA$, by letting, for example, $\mathbf{F} = \frac{1}{2}(-y, x)$ so that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$.

~~(f) Sample book Problems: 8.1 #2, #3(b), #5~~