Midterm 2 **Show all steps** name: e-mail:

FYI: $\int cos^2(x)dx = \frac{1}{2}(x + sin(x)cos(x)) + C,$ $\int sin^2(x)dx = \frac{1}{2}(x - sin(x)cos(x)) + C$

[15] 1.) Evaluate the given integral by making an appropriate change of variables where R is the parallelogram enclosed by the lines $y = 2x$, $y = \frac{1}{3}x$, $y = 2x - 10$, and $y = \frac{1}{3}x + 5$.

$$\int\int_R (x + 2y)dxdy$$

2.) Consider the force field $\mathbf{F}(x, y) =< y, 0 >$ and a circle with center the origin and radius r.

[5] (a) Graph the force field and circle on the same graph. Note your graph need not be to scale (since you don't know the radius of the circle, but the relative lengths of your vectors should be correct. In other words, you don't need to include units on your axes).

Find the work done by the force field on a particle that moves once around the circle oriented in the clockwise direction. Solve this problem 2 different ways:

[8] (b) Use a line integral

[8] (c) Use Green's theorem.

[2] (d) What is the work done by a force field $\mathbf{F}(x, y) =< x, y >$ on a particle that moves once around the circle oriented in the clockwise direction. Briefly explain your answer without using integration or Green's theorem.

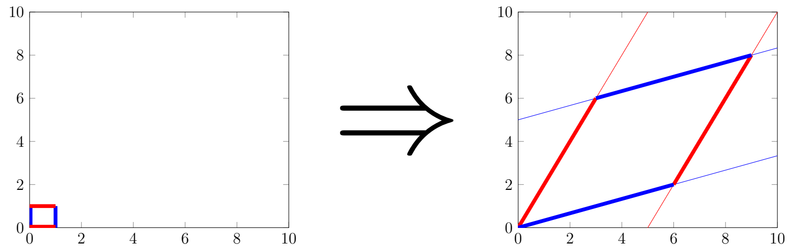
[2] (e) What is the work done by force field $\mathbf{F}(x, y) =< x, 0 >$ on a particle that moves once around the circle oriented in the clockwise direction. Briefly explain your answer without using integration or Green's theorem.

[15] 1.) Evaluate the given integral by making an appropriate change of variables where R is the parallelogram enclosed by the lines $y = 2x$, $y = \frac{1}{3}x$, $y = 2x - 10$, and $y = \frac{1}{3}x + 5$.

$$\int\int_R (x + 2y)dxdy$$

Answer: The lines $y = \frac{1}{3}x$ and $y = 2x - 10$ intersect at $(6, 2)$ since $\frac{1}{3}x = 2x - 10$ implies $x = 6x - 30$ and hence $5x = 30$ and thus $x = 6$, $y = \frac{1}{3}(6) = 2$

The lines $y = \frac{1}{3}x + 5$ and $y = 2x$ intersect at $(3, 6)$ since $\frac{1}{3}x + 5 = 2x$ implies $x + 15 = 6x$ and hence $5x = 15$ and thus $x = 3$, $y = 2(3) = 6$



Thus the parallelogram, R, has sides $< 6, 2 >$ and $< 3, 6 >$ since the origin is a vertex of the parallelogram.

If we let $< x, y >= T(u, v) = u < 6, 2 > +v < 3, 6 >$, then $T(D) = R$ where D is the unit square $[0, 1] \times [0, 1]$

Thus we can let $x = 6u + 3v$ and $y = 2u + 6v$. 5

Thus $\frac{\partial(x,y)}{\partial(u,v)} = det \begin{bmatrix} 6 & 3 \\ 2 & 6 \end{bmatrix} = 36 - 6 = 30.$ 5

Note the absolute value of our Jacobian is > 1. This makes sense as our change of variable changes the larger parallelogram region R into the smaller square region $[0, 1] \times [0, 1]$. Thus we need to scale our integral over the smaller square region $[0, 1] \times [0, 1]$ by

$$abs(\frac{\partial(x, y)}{\partial(u, v)}) = abs(det \begin{bmatrix} 6 & 3 \\ 2 & 6 \end{bmatrix}) = |36 - 6| = 30 > 1$$ 5

.

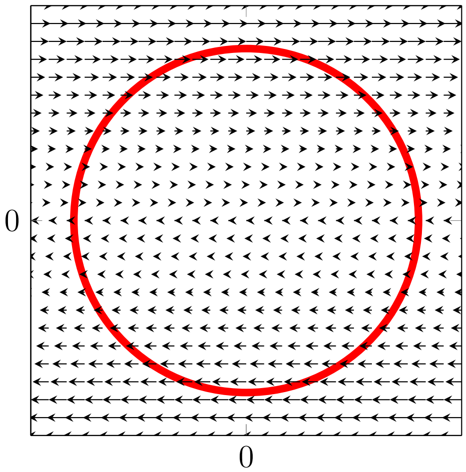
$$\int\int_R (x + 2y)dxdy = \int_0^1 \int_0^1 [(6u + 3v) + 2(2u + 6v)] \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv = \int_0^1 \int_0^1 [10u + 15v](30)dudv$$

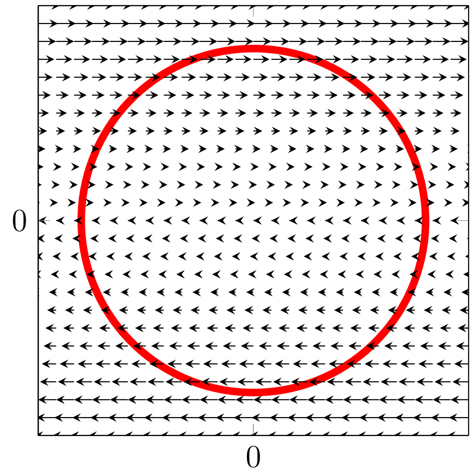
$= 150 \int_0^1 \int_0^1 [2u + 3v]dudv = 150 \int_0^1 [u^2 + 3uv]|_0^1 dv = 150 \int_0^1 [1 + 3v]|_0^1 dv$

$= 150[v + \frac{3}{2}v^2]|_0^1 = 150[1 + \frac{3}{2}] = 150[\frac{5}{2}] = 75(5) = 375$

2.) Consider the force field $\mathbf{F}(x,y) =< y, 0 >$ and a circle with center the origin and radius r.

[5] (a) Graph the force field and circle on the same graph. Note your graph need not be to scale (since you don't know the radius of the circle, but the relative lengths of your vectors should be correct. In other words, you don't need to include units on your axes).





Find the work done by the force field on a particle that moves once around the circle oriented in the clockwise direction. Solve this problem 2 different ways:

[8] (b) Use a line integral

Let $\mathbf{r}(\theta) = (-rcos\theta, rsin\theta)$ be a parametrization of the circle where the circle is oriented in the clockwise direction where $0 \leq \theta \leq 2\pi$.

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Alternatively, can use normal counter-clockwise orientation of the circle and use that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_{C} \mathbf{F} \cdot d\mathbf{r}$$

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Using the parametrization $\boxed{\mathbf{r}(\theta) = (-rcos\theta, rsin\theta)}$, then $\mathbf{F}(-rcos\theta, rsin\theta) =< rsin\theta, 0 >$

$\mathbf{r}'(t) =< rsin\theta, rcos\theta >$ and hence $d\mathbf{r} =< rsin\theta, rcos\theta > d\theta$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C < y, 0 > \cdot < dx, dy > = \boxed{\int_0^{2\pi} < rsin\theta, 0 > \cdot < rsin\theta, rcos\theta > d\theta}$ 3

$= \int_0^{2\pi} r^2 sin^2\theta d\theta = r^2 \int_0^{2\pi} sin^2\theta d\theta = r^2(\frac{\theta - sin\theta cos\theta}{2}|_0^{2\pi}) = \pi r^2$ 1

[8] (c) Use Green's theorem.

$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA = -\int\int_D (\frac{\partial 0}{\partial x} - \frac{\partial y}{\partial y}) dA = -\int\int_D (-1) dA = \int\int_D dA = \pi r^2$

since the area of a circle with radius r is πr^2.

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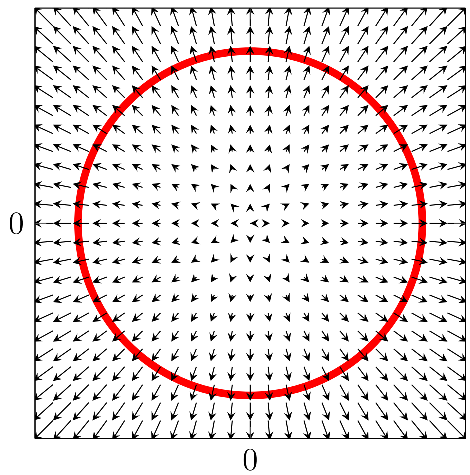
[2] (d) What is the work done by a force field $\mathbf{F}(x, y) = < x, y >$ on a particle that moves once around the circle oriented in the clockwise direction. Briefly explain your answer without using integration or Green's theorem.

Work is 0 since the vector field is perpendicular to the circle centered at the origin.



[2] (e) What is the work done by force field $\mathbf{F}(x, y) = < x, 0 >$ on a particle that moves once around the circle oriented in the clockwise direction. Briefly explain your answer without using integration or Green's theorem.

Work is 0 since the amount of positive work cancels out with the amount of negative work due to the symmetry of the vector field with respect to a circle centered at the origin.