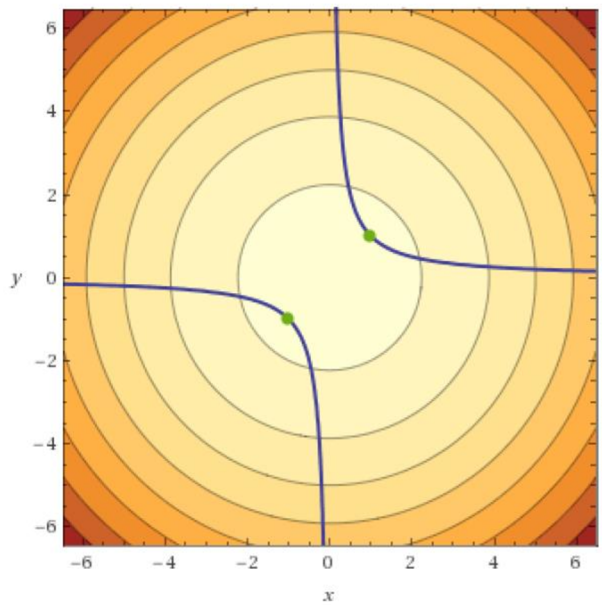
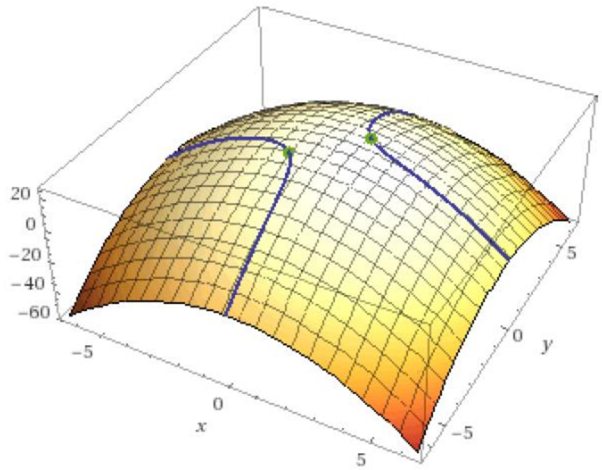
Contour plot:



WolframAlpha

minimize -x^2-y^2 + 25 on xy = 1



Suppose $F(x, y) = (f_1(x, y), f_2(x, y))$

Recall that the Jacobian matrix of F is

$$F'(x, y) = DF = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

12.10: 2nd order derivative test

Suppose $z = f(x, y)$

Recall the derivative matrix of f is $Df = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$

Hessian matrix =

$$D^2 f = \begin{bmatrix} \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) & \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) \\ \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) & \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = Hf$$

Determinant of the Hessian = $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left[\frac{\partial^2 f}{\partial y \partial x}\right]^2 = f_{xx}f_{yy} - [f_{xy}]^2$

Recall $f_{xy} = (f_x)_y = (f_y)_x = f_{yx}$ if 2nd order partials are continuous

Theorem 1: **Two-variable second derivative test**
If 2nd order partials of f are continuous in a neighborhood of a critical point (a, b), then

1. If $det(Hf(a, b)) = \Delta > 0$, then
   if $f_{xx} > 0$, then $f(a, b)$ is a local minimum.
   if $f_{xx} < 0$, then $f(a, b)$ is a local maximum.
2. If $det(Hf(a, b)) = \Delta < 0$, then $f(a, b)$ is neither a local minimum nor a local maximum

Note if $detHf(a, b) = \Delta = 0$, then the 2nd derivative test gives no information.

Example: $f(x, y) = f(x, y) = x^2 + y^2 + pxy = (x + \frac{p}{2}y)^2 - (\frac{p^2}{4} - 1)y^2$

See chalkboard and/or

https://www.khanacademy.org/math/multivariable-calculus/applications-of-multivariable-derivatives/optimizing-multivariable-functions/a/second-partial-derivative-test