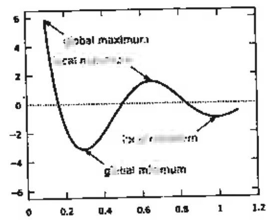
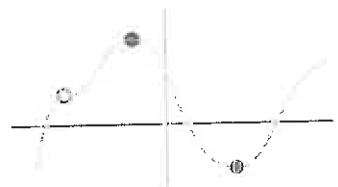
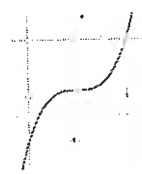
Calc 1 (one independent variables)

If $f(t_0)$ is a local maximum or local minimum, then $f'(t_0) = 0$ or DNE.



en.wikipedia.org/wiki/Maxima_and_minima


mathinsight.org/local_extrema_introduction_two_variables


inflection point

blog.com

Extreme value theorem: If $f : [a, b] \to \mathbb{R}$ is continuous, then f must attain a maximum and a minimum, each at least once.

To find absolute max/min, check all t such that $f'(t) = 0$ or DNE as well as points on the boundary of $[a, b]$ (i.e., also check $f(a)$ and $f(b)$).

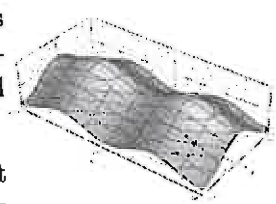
Section 12.5: Math 5 (multiple independent variables)

If $f(\mathbf{t}_0)$ is a local maximum or local minimum, then for all x_i, $\frac{\partial f}{\partial x_i}(\mathbf{t}_0) = 0$ or DNE.

https://mathinsight.org/local_extrema_introduction_two_variables

wolframalpha.com/

Extreme value theorem: If $f : C \to \mathbb{R}$ is continuous where C is a closed and bounded region in $\mathbb{R}^n$, then f must attain a maximum and a minimum, each at least once.


https://en.wikipedia.org/wiki/Surface_(topology)

To find absolute max/min, check all $\mathbf{t}$ such that for all x_i, $\frac{\partial f}{\partial x_i}(\mathbf{t}_0) = 0$ or DNE as well as points on the boundary of C.

Constraint

Example: Find the dimension of an open crate with volume 100 m^3 if material for bottom costs 10 cents/m^2 while the 4 sides cost 5 cents per m^2.

~~Solution~~: $lwh = 100$. Solve for one of the variables: $h = \frac{100}{lw}$

Cost $= 0.1lw + 0.5(2lh + 2wh = 0.1(lw + lh + wh)$ ← constrained optimization

$C(l, w) = 0.1(lw + lh + wh) = 0.1(lw + l(\frac{100}{lw}) + w(\frac{100}{lw}))$

Thus $C(l, w) = 0.1(lw + \frac{100}{w} + \frac{100}{l})$ ← min occurs at critical pt in l or w direction

$\frac{\partial C}{\partial l} = 0.1(w - \frac{100}{l^2}) = 0$ or DNE. $\frac{\partial C}{\partial w} = 0.1(l - \frac{100}{w^2}) = 0$ or DNE.

Note $l \neq 0$ and $w \neq 0$ since volume $\neq 0$. ↑ solve system of nonlinear eqns

$0.1(l - \frac{100}{w^2}) = 0$ implies $l = \frac{100}{w^2}$

$0.1(w - \frac{100}{l^2}) = 0$ implies $0.1(w - \frac{100}{(\frac{100}{w^2})^2}) = 0.1(w - \frac{100w^4}{100^2}) = 0.1(w - \frac{w^4}{100}) = 0$

Thus $w(1 - \frac{w^3}{100}) = 0$. Hence $w = 0$ or $1 - \frac{w^3}{100} = 0$. Thus $w^3 = 100$.

Thus minimum occurs at $w = 100^{\frac{1}{3}} = 10^{\frac{2}{3}}$

$l = \frac{100}{w^2} = \frac{10^2}{10^{\frac{4}{3}}} = 10^{\frac{2}{3}}$ and $h = \frac{100}{lw} = \frac{10^2}{(10^{\frac{2}{3}})(10^{\frac{2}{3}})} = 10^{\frac{2}{3}}$

Thus dimension of box is $10^{\frac{2}{3}} \times 10^{\frac{2}{3}} \times 10^{\frac{2}{3}}$. ← Answer

Cost is $0.1[lw + lh + wh] = 0.1[(10^{\frac{2}{3}})(10^{\frac{2}{3}}) + (10^{\frac{2}{3}})(10^{\frac{2}{3}}) + (10^{\frac{2}{3}})(10^{\frac{2}{3}})]$

Thus cost is \$ $0.3(10^{\frac{4}{3}}) = 0.3(10)(10^{\frac{1}{3}}) = 3(10^{\frac{1}{3}}) \sim 0.3(\frac{64}{3}) = \$6.40$

See 12.6 lecture to see that $\frac{64}{3}$ approximates $10^{\frac{4}{3}}$

Calc 1 (one independent variables)

If $y = f(x)$, recall average rate of change = slope of secant line $= \frac{\Delta y}{\Delta x} = \frac{\Delta f}{\Delta x}$ $= m$

If $y = f(x)$, recall instantaneous rate of change = slope of tangent line:

$$\text{slope} = \frac{dy}{dx} = \frac{df}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

The differential: Useful notation. useful for estimations

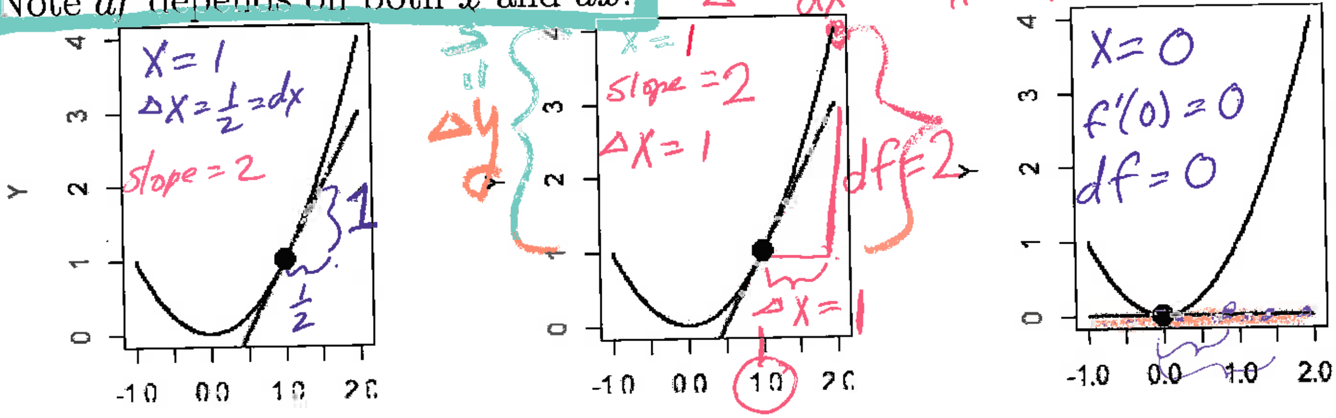
$$\frac{df}{dx} = f'(x). \text{ Thus } df = f'(x)dx = f'(x)\Delta x.$$

Note that x is the independent variable. Thus we can let $dx = \Delta x$.

$\boxed{\text{Note } df \text{ depends on both } x \text{ and } dx.}$ $2 = \frac{df}{dx} = \frac{df}{\Delta x} = \frac{2}{1} \approx \frac{\Delta y}{\Delta x}$

$y = x^2$

$y' = 2x$



Change in the dependent variable y (resulting from change in the independent variable x) can be estimated using the differential.

$$\Delta f = \Delta y \sim dy = df = f'(x)dx = f'(x)\Delta x.$$

or equivalently, $\frac{\Delta f}{\Delta x} \sim f'(x)$. Thus $\Delta f \sim f'(x)\Delta x$. $df(x_1, \underbrace{x_2 - x_1}_{\Delta x})$

Note: depending on f and $dx = \Delta x$, sometimes this is a good estimate and sometimes this is a bad estimate:

$df(x_1, \underbrace{x_2 - x_1}_{\Delta x})$

One can also use the differential to estimate $f(x_2)$ if you know $f(x_1)$

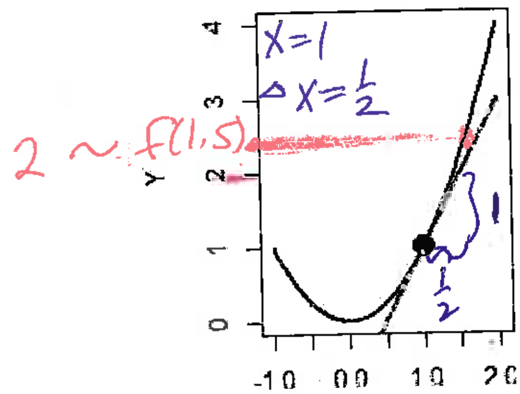
$$f(x_2) = f(x_1) + \underbrace{[f(x_2) - f(x_1)]}_{\Delta f \sim df} = f(x_1) + \Delta f \sim f(x_1) + df = f(x_1) + f'(x_1)\underbrace{dx}_{\Delta x = dx}$$

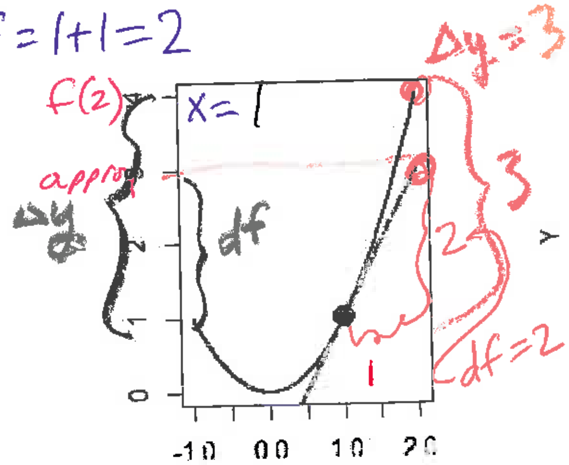
Thus $f(x_2) = f(x_1) + f'(x_1)[x_2 - x_1]$

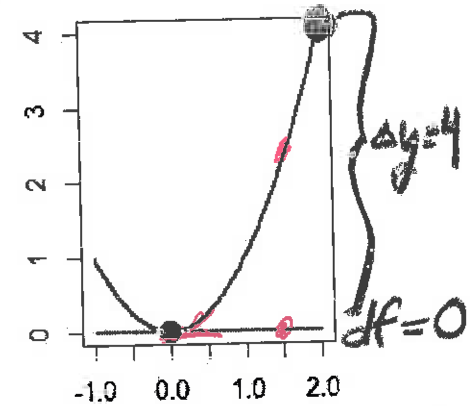
Note equation of tangent line to f at $x = x_1$ is: $\quad y = f(x_1) + f'(x_1)[x - x_1]$

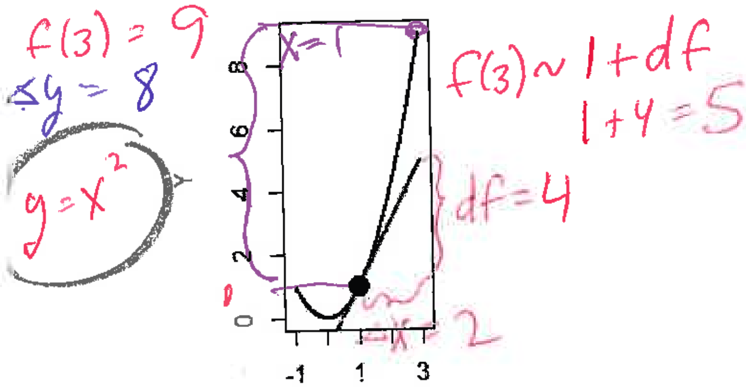
Using tangent line $_{x_1}$ to approx f

$f(1\frac{1}{2}) \sim 1 + df = 1+1 = 2$

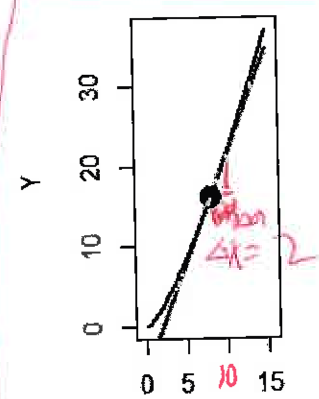








$f(x) = 10^{4/3}$



Approximate $10^{\frac{4}{3}}$:

Let $f(x) = x^{\frac{4}{3}}$. Then $f'(x) = \frac{4}{3}x^{\frac{1}{3}}$

Can calculate $f(8)$ and $f'(8)$, so use $x = 8$ to estimate the value of $f(10)$.

Thus $f(10) = f(8) + \Delta y = f(8) + \Delta f \sim f(8) + f'(8)\Delta x$.

Note $dx = \Delta x = 10 - 8 = 2$.

Note $f(8) = 8^{\frac{4}{3}} = 2^4 = 16$. and $f'(8) = \frac{4}{3}(8)^{\frac{1}{3}} = \frac{4}{3}(2) = \frac{8}{3}$

$$10^{\frac{4}{3}} = f(10) = f(8) + [f(10) - f(8)] = f(8) + \Delta f$$

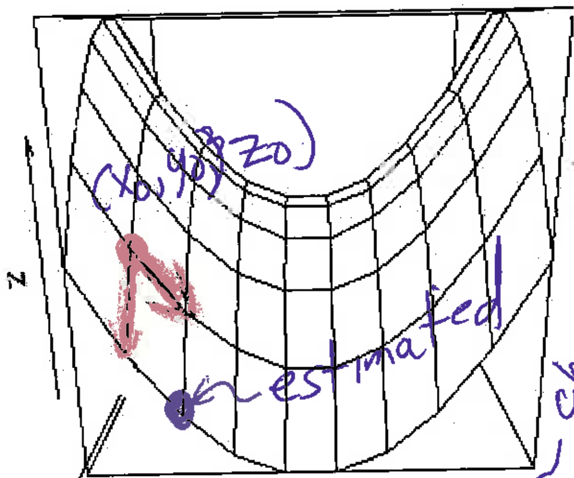
$$\sim f(8) + df = f(8) + f'(8)dx$$

$$= f(8) + f'(8)(10-8) = 16 + (\tfrac{8}{3})(2) = \frac{48}{3} + \frac{16}{3} = \frac{64}{3}$$

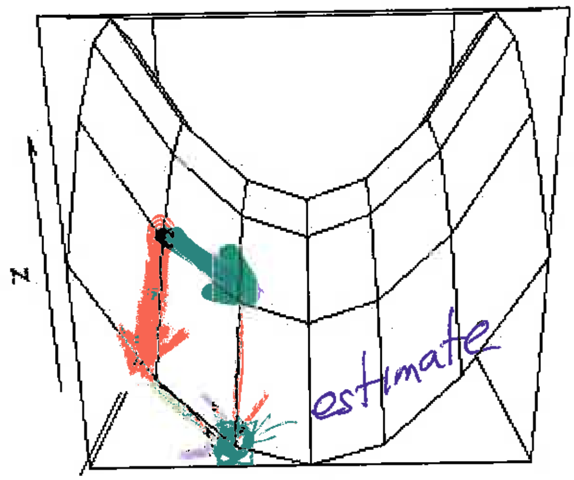
Note: $\frac{64}{3} = 21.333333...$ while $10^{\frac{4}{3}} = 21.544346900318837217592935665...$

Section 12.6: Math 5 (multiple independent variables)

Suppose $z = f(x,y)$. Then the gradient vector of f is: $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$



change in z approximation in y direction (x_0 fixed)



Note $\Delta z \sim df = \underbrace{\frac{\partial f}{\partial x}\Delta x}_{\text{change in z approximation in x direction } (y_0 \text{ fixed})} + \frac{\partial f}{\partial y}\Delta y = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \cdot (\Delta x, \Delta y) = \nabla f \cdot (\Delta x, \Delta y)$

If y fixed $\Delta z \approx \left(\frac{\partial f}{\partial x}\right)(\Delta x)$ by calc 1

$= \underbrace{\left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0,y_0)\right)}_{\nabla f} \cdot (\Delta x, \Delta y)$

(x_0, y_0, z_0)

$(0, \Delta y, \frac{\partial f}{\partial y}\Delta y)$

for x_0 fixed

$(x_0, y_0 + \Delta y, z_0 + df)$

$= (x_0, y_0 + \Delta y, z_0 + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y)$

$(\Delta x, 0, \frac{\partial f}{\partial x}(x_0,y_0)\Delta x)$

$(x_0 + \Delta x, y_0, z_0 + df) = (x_0 + \Delta x, y_0, z_0 + \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)\Delta x)$

for y_0 fixed

$[(x_0 + \Delta x,\ y_0,\ z_0 + \frac{\partial f}{\partial x}\Delta x) - (x_0, y_0, z_0)]$

$+ [(x_0,\ y_0 + \Delta y,\ z_0 + \frac{\partial f}{\partial y}\Delta y) - (x_0, y_0, z_0)]$

direction of tangent line in y direction

$= [(\Delta x,\ 0,\ \frac{\partial f}{\partial x}\Delta x)] + [(0,\ \Delta y,\ \frac{\partial f}{\partial y}\Delta y)] = (\Delta x,\ \Delta y,\ \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y)$

$= (\Delta x,\ \Delta y,\ \nabla f \cdot (\Delta x, \Delta y)) \sim (\Delta x,\ \Delta y,\ \Delta z)$