Note: If you have more than one independent variable and more than one dependent variable at any stage of the problem, then you must use Jacobian matrices (for example HW 12.7: 5, 13).

Let $g(x, y) = x^2 + y^2$

Let $T(r, \theta) = (rcos\theta, r\theta) = (x(r, \theta), y(r, \theta))$

Then $(g \circ T)(r, \theta) = g(T(r, \theta)) = g(rcos\theta, rsin\theta) = r^2cos^2\theta + r^2sin^2\theta = r^2$.

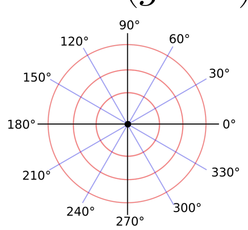
Thus $(g \circ T)'(r, \theta) = [\frac{\partial(g \circ T)}{\partial r} \quad \frac{\partial(g \circ T)}{\partial \theta}] = [\frac{\partial r^2}{\partial r} \quad \frac{\partial r^2}{\partial \theta}] = [2r \quad 0]$

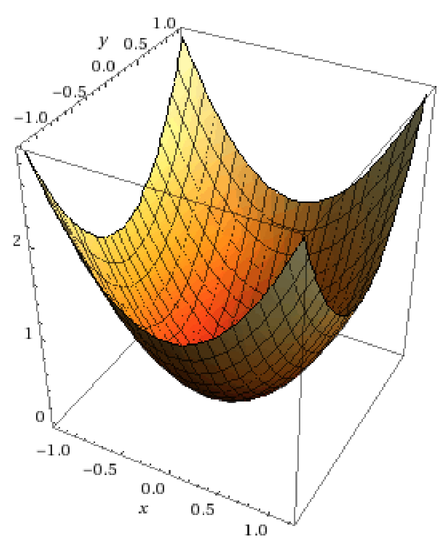
Chain rule: $\quad (g \circ T)'(r, \theta) = g'(T(r, \theta))T'(r, \theta)$

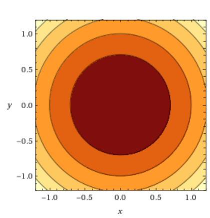
Note that $g'(T(r, \theta)) = [\frac{\partial(x^2+y^2)}{\partial x} \quad \frac{\partial(x^2+y^2)}{\partial x}] = [2x \quad 2y] = [2rcos\theta \quad 2rsin\theta]$

where x and y are functions of r and θ
That is $x = rcos\theta$ and $y = r\theta$
and $g'(x, y)$ is evaluated at $T(r, \theta)$.

$$(g \circ T)'(r, \theta) = g'(T(r, \theta))T'(r, \theta) = [\frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial x}] \begin{bmatrix} \frac{\partial x(r,\theta)}{\partial r} & \frac{\partial x(r,\theta)}{\partial \theta} \\ \frac{\partial y(r,\theta)}{\partial r} & \frac{\partial y(r,\theta)}{\partial \theta} \end{bmatrix}$$



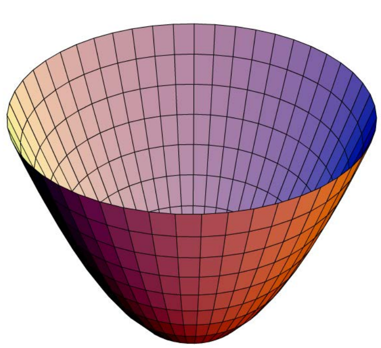




$$= [\frac{\partial(x^2+y^2)}{\partial x} \quad \frac{\partial(x^2+y^2)}{\partial x}] \begin{bmatrix} \frac{\partial x(r,\theta)}{\partial r} & \frac{\partial x(r,\theta)}{\partial \theta} \\ \frac{\partial y(r,\theta)}{\partial r} & \frac{\partial y(r,\theta)}{\partial \theta} \end{bmatrix}$$

$$= [2x \quad 2y] \begin{bmatrix} \frac{\partial(rcos\theta)}{\partial r} & \frac{\partial(rcos\theta)}{\partial \theta} \\ \frac{\partial(rsin\theta)}{\partial r} & \frac{\partial(rsin\theta)}{\partial \theta} \end{bmatrix}$$

$$= [2rcos\theta \quad 2rsin\theta] \begin{bmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{bmatrix}$$



$$= [2rcos^2\theta + 2rsin^2\theta \quad -2r^2cos\theta sin\theta + 2r^2sin\theta cos\theta]$$

$$= [2r \quad 0]$$

If $T(r,\theta) = (x(r,\theta), y(r,\theta))$ and if $z = g(x,y)$,

Then $z = (g \circ T)(r,\theta)$

and $[\frac{\partial z}{\partial r} \;\; \frac{\partial z}{\partial \theta}] = (g \circ T)'(r,\theta) = g'(T(r,\theta))T'(t,\theta) = [\frac{\partial z}{\partial x} \;\; \frac{\partial z}{\partial y}] \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$

$$= [\frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \quad \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}]$$

Thus $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$ and $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$

This matches with 12.6 were we saw $\Delta z \sim dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$.

Application: Suppose that sand is falling at a rate of 40 πm^3/sec, forming a conical pile. If the height of the pile is increasing at a rate of 0.5 m/sec when the height of the pile is 6m and its radius is 12m, find the rate at which the radius is increasing

Note volume of cone is given by $V = \pi r^2(\frac{h}{3})$

Example: If z is a function of x and y and if $xln|z| + ysin(z) = xyz + x$, find $\frac{\partial z}{\partial x}$.

Note z is defined implicitly. We will discuss implicit function theorem more thoroughly later this semester.

Recall the vector $(1, 0, \frac{\partial z}{\partial x})$ is tangent to the surface defined by the above equation.