11.6: Curvature

Arclength $s(t) = \int_a^t v(\tau)d\tau$ where $v(t) = |\mathbf{v}(t)|$

$s(t)$ is an increasing function and thus $s^{-1}(t)$ exists. Let $t(s) =^{-1} (t)$.

Arc-length parametrization = reparametrize by replacing t with $t(s)$.

Example: $r(t) = (cos(t), sin(t), t)$

Unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{velocity}{speed} = \frac{\frac{d\mathbf{r}}{dt}}{\frac{ds}{dt}} = \frac{d\mathbf{r}}{ds}$

Thus if $\mathbf{T}$ is parametrized by arclength s, then

$$\mathbf{T}(s) = \frac{\mathbf{v}(s)}{|\mathbf{v}(s)|} = \frac{velocity}{speed} = \frac{\frac{d\mathbf{r}}{ds}}{\frac{ds}{ds}} = \frac{d\mathbf{r}}{ds}$$

Since $\mathbf{T}$ is a unit vector, $\mathbf{T} \cdot \mathbf{T} = 1$.

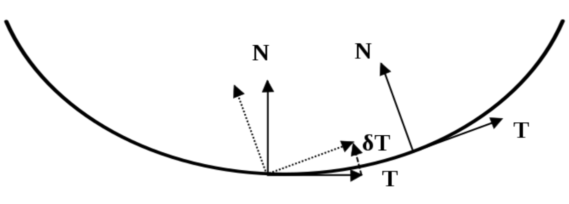
Differentiate with respect to s: $2\mathbf{T} \cdot \frac{d\mathbf{T}}{ds} = 0$

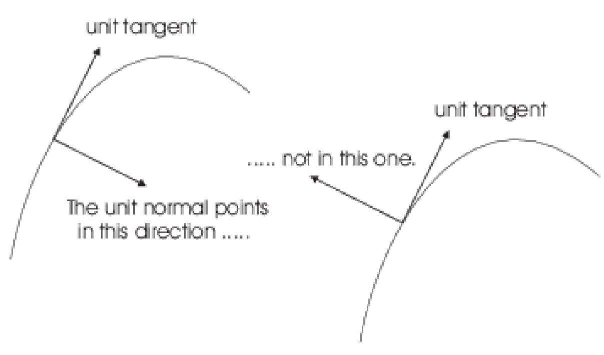
Thus $\mathbf{T}$ is perpendicular to $\frac{d\mathbf{T}}{ds}$

Definition: The **principal unit normal** is $\frac{\frac{d\mathbf{T}}{ds}}{|\frac{d\mathbf{T}}{ds}|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$ where

$$\textbf{curvature } \kappa = |\frac{d\mathbf{T}}{ds}| = |\frac{d\mathbf{T}}{dt}\frac{dt}{ds}| = \frac{1}{\frac{ds}{dt}}|\frac{d\mathbf{T}}{dt}| = \frac{1}{speed}|\frac{d\mathbf{T}}{dt}|$$

The unit normal points in the direction in which the curve is curving:





https://en.wikipedia.org/wiki/Curvature, http://sites.millersville.edu/bikenaga/calculus/tangent-normal-curvature/tangent-normal-curvature.html

Example: Find the unit tangent and normal vectors to the curve $y = x^2$ at (2, 4)

In parametric form: $\mathbf{r}(t) = (t, t^2)$. Thus $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{(1,2t)}{\sqrt{1+4t^2}}$

At $t = 2$, $\mathbf{T}(2) = \frac{(1,4)}{\sqrt{17}} = (\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}})$

Alternate method: one can use the slope of the tangent line to find $\mathbf{T}$:

slope of line at $x = 2$ is 4.

(2, 4) is a point on the tangent line and

the direction of the line with slope $\frac{4}{+1}$ is $(\Delta x, \Delta y) = (1, 4)$.

Thus the equation of the tangent line to $y = x^2$ at (2, 4) is

$$(x, y) = (2, 4) + t(1, 4)$$

Note we only needed the direction of the tangent line which is given by the vector (1, 4).

Thus $\mathbf{T}(2) = \frac{(1,4)}{\sqrt{17}} = (\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}})$

The unit normal to $y = x^2$ at (2, 4) is either $(-\frac{4}{\sqrt{17}}\frac{1}{\sqrt{17}}, (\frac{4}{\sqrt{17}}, -\frac{1}{\sqrt{17}})$ as these are the only two vectors in R^2 that have length one and are perpendicular to $\mathbf{T}$ (check dot product).

Since the unit normal points in the direction in which the curve is curving, $\mathbf{N}(2) = (-\frac{4}{\sqrt{17}}\frac{1}{\sqrt{17}})$ (draw picture)

In 2D, if $r(t) = (x(t), y(t))$, let $\phi = tan^{-1}(\frac{y'(t)}{x'(t)})$

Write unit tangent in polar coordinates: $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \mathbf{i}cos\phi + \mathbf{j}sin\phi$

Then $\frac{d\mathbf{T}}{ds} = (-\mathbf{i}sin\phi + \mathbf{j}cos\phi)\frac{d\phi}{ds}$ is obviously perpendicular to $\mathbf{T}$.

curvature $= \kappa = |\frac{d\mathbf{T}}{ds}| = |(-\mathbf{i}sin\phi + \mathbf{j}cos\phi)\frac{d\phi}{ds}| = |\frac{d\phi}{ds}|$

Since $\phi = tan^{-1}(\frac{y'(t)}{x'(t)})$, then

curvature $= \kappa = |\frac{d\phi}{ds}| = \frac{|x'y''-x''y'|}{[(x')^2+(y')^2]^{\frac{3}{2}}}$

Note if $r(x) = (x, f(x)) = (x, y)$, then $x' = 1$ and $x'' = 0$. Thus

$$\kappa = \frac{|y''|}{[(1+(y')^2]^{\frac{3}{2}}} = \frac{|y''|}{|(1,y')|^3}$$

Example: Find the point(s) on the curve $y = x^2$ where curvature is maximum.

$r(x) = (x, x^2)$, $\kappa = \frac{|x'y''-x''y'|}{[(x')^2+(y')^2]^{\frac{3}{2}}} = \frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}} = \frac{2}{[1+4x^2]^{\frac{3}{2}}} = 2[1 + 4x^2]^{-\frac{3}{2}}$

$\kappa'(x) = -3[1 + 4x^2]^{-\frac{5}{2}}(8x) = 0$ iff $x = 0$.

Note $\kappa'(x) > 0$ when $x < 0$, and $\kappa'(x) < 0$ when $x > 0$.

Thus $\kappa(x)$ has a maximum at $x = 0$.

When $x = 0, y = 0$. Thus maximum curvature occurs at $(x, y) = (0, 0)$ (as expected).