11.6: Curvature

Arclength $s(t) = \int_a^t v(\tau)d\tau$ where $v(t) = |\mathbf{v}(t)|$

$s(t)$ is an increasing function and thus $s^{-1}(t)$ exists. Let $t(s) =^{-1} (t)$.

Arc-length parametrization = reparametrize by replacing t with $t(s)$.

Example: $r(t) = (cos(t), sin(t), t)$

Unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{velocity}{speed} = \frac{\frac{d\mathbf{r}}{dt}}{\frac{ds}{dt}} = \frac{d\mathbf{r}}{ds}$

Thus if $\mathbf{T}$ is parametrized by arclength s, then

$$\mathbf{T}(s) = \frac{\mathbf{v}(s)}{|\mathbf{v}(s)|} = \frac{velocity}{speed} = \frac{\frac{d\mathbf{r}}{ds}}{\frac{ds}{ds}} = \frac{d\mathbf{r}}{ds}$$

Since $\mathbf{T}$ is a unit vector, $\mathbf{T} \cdot \mathbf{T} = 1$.

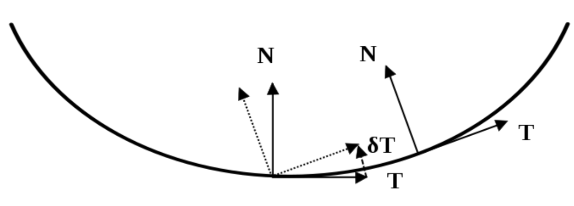
Differentiate with respect to s: $\quad 2\mathbf{T} \cdot \frac{d\mathbf{T}}{ds} = 0$

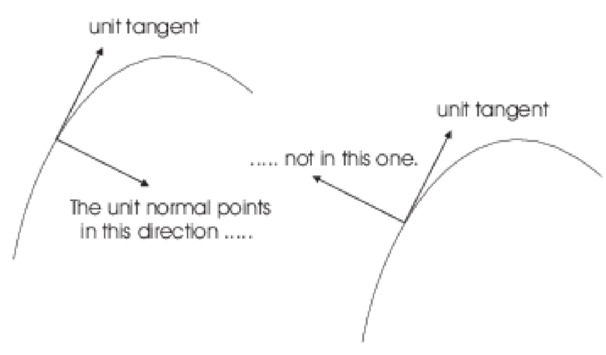
Thus $\mathbf{T}$ is perpendicular to $\frac{d\mathbf{T}}{ds}$

Definition: The **principal unit normal** is $\frac{\frac{d\mathbf{T}}{ds}}{|\frac{d\mathbf{T}}{ds}|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$ where

$$\textbf{curvature } \kappa = |\frac{d\mathbf{T}}{ds}| = |\frac{d\mathbf{T}}{dt}\frac{dt}{ds}| = \frac{1}{\frac{ds}{dt}}|\frac{d\mathbf{T}}{dt}| = \frac{1}{speed}|\frac{d\mathbf{T}}{dt}|$$

The unit normal points in the direction in which the curve is curving:





https://en.wikipedia.org/wiki/Curvature, http://sites.millersville.edu/bikenaga/calculus/tangent-normal-curvature/tangent-normal-curvature.html

Example: Find the unit tangent and normal vectors to the curve $y = x^2$ at (2, 4)

In 2D, if $r(t) = (x(t), y(t))$, let $\phi = tan^{-1}(\frac{y'(t)}{x'(t)})$

Write unit tangent in polar coordinates: $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \mathbf{i}cos\phi + \mathbf{j}sin\phi$

Then $\frac{d\mathbf{T}}{ds} = (-\mathbf{i}sin\phi + \mathbf{j}cos\phi)\frac{d\phi}{ds}$ is obviously perpendicular to $\mathbf{T}$.

curvature $= \kappa = |\frac{d\mathbf{T}}{ds}| = |(-\mathbf{i}sin\phi + \mathbf{j}cos\phi)\frac{d\phi}{ds}| = |\frac{d\phi}{ds}|$

Since $\phi = tan^{-1}(\frac{y'(t)}{x'(t)})$, then

curvature $= \kappa = |\frac{d\phi}{ds}| = \frac{|x'y''-x''y'|}{[(x')^2+(y')^2]^{\frac{3}{2}}}$

Example: Find the point(s) on the curve $y = x^2$ where curvature is maximum.