Existence and Uniqueness

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a,b) \to R$ and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

$$y' + p(t)y = g(t),$$
$$y(t_0) = y_0$$

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a,b) \to R$, $q : (a,b) \to R$, and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0,$$
$$y'(t_0) = y_0'$$

Thm 3.2.2: If ϕ_1 and ϕ_2 are two solutions to a homogeneous linear differential equation, the $c_1\phi_1 + c_2\phi_2$ is also a solution to this linear differential equation.

Definition: The Wronskian of two differential functions, f and g is

$$W(f,g) = fg' - f'g = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Thm 3.2.3: Suppose that ϕ_1 and ϕ_2 are two solutions to $y'' + p(t)y' + q(t)y = 0$. If $W(\phi_1,\phi_2)(t_0) = \phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0) \neq 0$, then there is a unique choice of constants c_1 and c_2 such that $c_1\phi_1 + c_2\phi_2$ satisfies this homogeneous linear differential equation and initial conditions, $y(t_0) = y_0$, $y'(t_0) = y_0'$.

Thm 3.2.4: Given the hypothesis of thm 3.2.1 Suppose that ϕ_1 and ϕ_2 are two solutions to $y'' + p(t)y' + q(t)y = 0$. If $W(\phi_1,\phi_2)(t_0) \neq 0$, for some $t_0 \in (a,b)$, then any solution to this homogeneous linear differential equation can be written as $y = c_1\phi_1 + c_2\phi_2$ for some c_1 and c_2.

Defn If ϕ_1 and ϕ_2 satisfy the conditions in thm 3.2.4, then ϕ_1 and ϕ_2 form a fundamental set of solutions to $y'' + p(t)y' + q(t)y = 0$.

Thm 3.2.5: Given any second order homogeneous linear differential equation, there exist a pair of functions which form a fundamental set of solutions.

3.3: Linear Independence and the Wronskian

Defn: f and g are linearly dependent if there exists constants c_1, c_2 such that $c_1 \neq 0$ or $c_2 \neq 0$ and $c_1 f(t) + c_2 g(t) = 0$ for all $t \in (a, b)$

Thm 3.3.1: If $f : (a, b) \to R$ and $g(a, b) \to R$ are differentiable functions on (a, b) and if $W(f, g)(t_0) \neq 0$ for some $t_0 \in (a, b)$, then f and g are linearly independent on (a, b). Moreover, if f and g are linearly dependent on (a, b), then $W(f, g)(t) = 0$ for all $t \in (a, b)$

If $c_1 f(t) + c_2 g(t) = 0$ for all t, then $c_1 f'(t) + c_2 g'(t) = 0$

Solve the following linear system of equations for c_1, c_2

$$\begin{aligned} c_1 f(t_0) + c_2 g(t_0) = 0 \\ c_1 f'(t_0) + c_2 g'(t_0) = 0 \end{aligned}$$

$$\begin{bmatrix} f(t_0) & g(t_0) \\ f'(t_0) & g'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$