Second order differential equation:

Linear equation with constant coefficients:

If the second order differential equation is

\[ ay'' + by' + cy = 0, \]

then \( y = e^{rt} \) is a solution

Need to have two independent solutions. Solve the following IVPs:

1.) \( y'' - 6y' + 9y = 0 \)

\( y(0) = 1, \quad y'(0) = 2 \)

2.) \( 4y'' - y' + 2y = 0 \)

\( y(0) = 3, \quad y'(0) = 4 \)

3.) \( 4y'' + 4y' + y = 0 \)

\( y(0) = 6, \quad y'(0) = 7 \)

4.) \( 2y'' - 2y = 0 \)

\( y(0) = 5, \quad y'(0) = 9 \)

\[ ay'' + by' + cy = 0, \quad y = e^{rt}, \text{ then} \]

\[ ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \text{ implies } ar^2 + br + c = 0, \]

Suppose \( r = r_1, r_2 \) are solutions to \( ar^2 + br + c = 0 \)

\[ r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

If \( r_1 \neq r_2 \), then \( b^2 - 4ac \neq 0 \). Hence a general solution is \( y = c_1e^{r_1t} + c_2e^{r_2t} \)

If \( b^2 - 4ac > 0 \), general solution is \( y = c_1e^{r_1t} + c_2e^{r_2t} \).

If \( b^2 - 4ac < 0 \), change format to linear combination of real-valued functions instead of complex valued functions by using Euler’s formula.

general solution is \( y = c_1e^{rt}\cos(nt) + c_2e^{rt}\sin(nt) \)

where \( r = d \pm in \)

If \( b^2 - 4ac = 0 \), \( r_1 = r_2 \), so need 2nd (independent) solution: \( te^{r_1t} \)

Hence general solution is \( y = c_1e^{r_1t} + c_2te^{r_1t} \).

Initial value problem: use \( y(t_0) = y_0, \quad y'(t_0) = y'_0 \) to solve for \( c_1, c_2 \) to find unique solution.
Derivation of general solutions:

If \( b^2 - 4ac > 0 \) we guessed \( e^{rt} \) is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If \( b^2 - 4ac < 0 \),

Changed format of \( y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \) to linear combination of real-valued functions instead of complex valued functions by using Euler’s formula:

\[
e^{it} = \cos(t) + i\sin(t)
\]

Hence \( e^{(d+in)t} = e^{dt} e^{int} = e^{dt} [\cos(nt) + i\sin(nt)] \)

Let \( r_1 = d + in \), \( r_2 = d - in \)

\[
y = c_1 e^{r_1 t} + c_2 e^{r_2 t}
\]

\[
= c_1 e^{dt} [\cos(nt) + i\sin(nt)] + c_2 e^{dt} [\cos(-nt) + i\sin(-nt)]
\]

\[
= c_1 e^{dt} \cos(nt) + ic_1 e^{dt} \sin(nt) + c_2 e^{dt} \cos(-nt) - ic_2 e^{dt} \sin(nt)
\]

\[
= (c_1 + c_2) e^{dt} \cos(nt) + i(c_1 - c_2) e^{dt} \sin(nt)
\]

\[
= k_1 e^{dt} \cos(nt) + k_2 e^{dt} \sin(nt)
\]

Section 3.4: If \( b^2 - 4ac = 0 \), then \( r_1 = r_2 \).

Hence one solution is \( y = e^{r_1 t} \). Need second solution.

If \( y = e^{rt} \) is a solution, \( y = c e^{rt} \) is a solution.

How about \( y = v(t) e^{rt} \)?

\[
y' = v'(t) e^{rt} + v(t) r e^{rt}
\]

\[
y'' = v''(t) e^{rt} + v'(t) r e^{rt} + v'(t) r e^{rt} + v(t) r^2 e^{rt}
\]

\[
= v''(t) e^{rt} + 2v'(t) re^{rt} + v(t) r^2 e^{rt}
\]

\[
ay'' + by' + cy = 0
\]

\[
a(v''e^{rt} + 2v' re^{rt} + vr^2 e^{rt}) + b(v'e^{rt} + vre^{rt}) + cve^{rt} = 0
\]

\[
a(v''(t) + 2v'(t)r + v(t)r^2) + b(v'(t) + v(t)r) + cv(t) = 0
\]

\[
av''(t) + 2av'(t)r + av(t)r^2 + bv'(t) + bv(t)r + cv(t) = 0
\]

\[
av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0
\]

\[
av''(t) + (2a(-\frac{b}{2a}) + b)v'(t) + 0 = 0
\]

since \( ar^2 + br + c = 0 \) and \( r = -\frac{b}{2a} \)

\[
av''(t) + (-b + b)v'(t) = 0.
\]

Thus \( av''(t) = 0 \).

Hence \( v''(t) = 0 \) and \( v'(t) = k_1 \) and \( v(t) = k_1 t + k_2 \)

Hence \( v(t)e^{r_1 t} = (k_1 t + k_2)e^{r_1 t} \) is a soln

Thus \( te^{r_1 t} \) is a nice second solution.

Hence general solution is \( y = c_1 e^{r_1 t} + c_2 te^{r_1 t} \)