To solve differential equations:

**First order differential equation:**

Method 1: Separate variables.

Method 2: If linear \[ y' + p(t)y = g(t) \], multiply equation by an integrating factor \( u = e^{\int p(t)dt} \).

\[
(y'u + p(t)uy) = ug(t) \\
(uy)' = ug(t)
\]

**Second order differential equation:**

Method 1: If there is no independent variable (t) OR if there is no dependent variable (y), transform into first order differential equation:

let \( v = \frac{dy}{dt} = y' \). Then \( v' = \) ________________

If there are 3 variables, note: \( \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = v \frac{dv}{dy} \)

Method 2 (linear equation with constant coefficients):
If the second order differential equation is

\[ ay'' + by' + cy = 0, \]

then \( y = e^{rt} \) is a solution

Need to have two independent solutions.
\[
ay'' + by' + cy = 0, \quad y = e^{rt}, \text{ then}
\]
\[
ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0 \implies ar^2 + br + c = 0,
\]

Suppose \( r = r_1, r_2 \) are solutions to \( ar^2 + br + c = 0 \)

If \( r_1 \neq r_2 \), then \( b^2 - 4ac \neq 0 \). Hence a general solution is \( y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \)

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If \( b^2 - 4ac > 0 \), general solution is \( y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \).

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If \( b^2 - 4ac < 0 \), change format to linear combination of real-valued functions instead of complex valued functions by using Euler’s formula.

general solution is \( y = c_1 e^{dt \cos(nt)} + c_2 e^{dt \sin(nt)} \) where \( r = d \pm in \)

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If \( b^2 - 4ac = 0 \), \( r_1 = r_2 \), so need 2nd (independent) solution: \( te^{r_1 t} \)

Hence general solution is \( y = c_1 e^{r_1 t} + c_2 te^{r_1 t} \).

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Initial value problem: use \( y(0) = y_0, y'(0) = y'_0 \) to solve for \( c_1, c_2 \) to find unique solution.
Derivation of general solutions:

If $b^2 - 4ac > 0$ we guessed $e^{rt}$ is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

If $b^2 - 4ac < 0,$

Changed format of $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler’s formula:

$$e^{it} = \cos(t) + i\sin(t)$$

Hence $e^{(d+in)t} = e^{dt} e^{int} = e^{dt} [\cos(nt) + i\sin(nt)]$

Let $r_1 = d + in,$ $r_2 = d - in$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$= c_1 e^{dt} [\cos(nt) + i\sin(nt)] + c_2 e^{dt} [\cos(-nt) + i\sin(-nt)]$

$= c_1 e^{dt} \cos(nt) + ic_1 e^{dt} \sin(nt) + c_2 e^{dt} \cos(nt) - ic_2 e^{dt} \sin(nt)$

$=(c_1 + c_2) e^{dt} \cos(nt) + i(c_1 - c_2) e^{dt} \sin(nt)$

$= k_1 e^{dt} \cos(nt) + k_2 e^{dt} \sin(nt)$
If \( b^2 - 4ac = 0 \), then \( r_1 = r_2 \).

Hence one solution is \( y = e^{r_1 t} \) Need second solution.

If \( y = e^{rt} \) is a solution, \( y = ce^{rt} \) is a solution.

How about \( y = v(t)e^{rt} \)?

\[
y' = v'(t)e^{rt} + v(t)re^{rt}
\]

\[
y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}
\]

\[
= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}
\]

\[
ay'' + by' + cy = 0
\]

\[
a(v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}) + b(v'(t)e^{rt} + v(t)re^{rt}) + c(v(t)e^{rt}) = 0
\]

\[
a(v'(t) + 2v(t)r + v(t)r^2) + b(v'(t) + v(t)r) + cv(t) = 0
\]

\[
av''(t) + 2av'(t)r + av(t)r^2 + bv'(t) + bv(t)r + cv(t) = 0
\]

\[
av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0
\]

\[
av''(t) + (2a(\frac{-b}{2a}) + b)v'(t) + 0 = 0
\]

Since \( ar^2 + br + c = 0 \) and \( r = \frac{-b}{2a} \)
\[ av''(t) + (-b + b)v'(t) = 0 \]

\[ av''(t) = 0 \]

Hence \( v''(t) = 0 \)

\[ v'(t) = k_1 \]

\[ v(t) = k_1 t + k_2 \]

Hence \( v(t)e^{r_1 t} = (k_1 t + k_2)e^{r_1 t} \) is a soln

Hence \( te^{r_1 t} \) is a nice second solution.

Hence general solution is \( y = c_1 e^{r_1 t} + c_2 te^{r_1 t} \)

3.6 Nonhomogeneous Equations: \( ay'' + by' + cy = g(t) \)

Method of Undetermined Coefficients.