Linear algebra pre-requisites you must know.

$\mathbf{b_1}, ..., \mathbf{b_n}$ are linearly independent if

$$c_1\mathbf{b_1} + c_2\mathbf{b_2} + ... + c_n\mathbf{b_n} = d_1\mathbf{b_1} + d_2\mathbf{b_2} + ... + d_n\mathbf{b_n}$$

implies $c_1 = d_1,\ c_2 = d_2 ...,\ c_n = d_n$.

or equivalently,

$\mathbf{b_1}, ..., \mathbf{b_n}$ are linearly independent if

$c_1\mathbf{b_1} + c_2\mathbf{b_2} + ... + c_n\mathbf{b_n} = 0$ implies $c_1 = c_2 = ...c_n$.

Example 1: $\mathbf{b_1} = (1,0,0)$, $\mathbf{b_2} = (0,1,0)$, $\mathbf{b_3} = (0,0,1)$. $\blacksquare$

$$(1,2,3) \neq (1,2,4).$$

If $(a,b,c) = (1,2,3)$ then $a = 1,\ b = 2,\ c = 3$.

Example 2: $\mathbf{b_1} = 1,\ \mathbf{b_2} = t,\ \mathbf{b_3} = t^2$.

$$1 + 2t + 3t^2 \neq 1 + 2t + 4t^2.$$

If $a + bt + ct^2 = 1 + 2t + 3t^2$ then $a = 1,\ b = 2,\ c = 3$.

Application: Partial Fractions

$$\frac{4}{(x^2+1)(x-3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-3}$$

$$= \frac{(Ax+B)(x-3)+C(x^2+1)}{(x^2+1)(x-3)}$$

Hence $\frac{4}{(x^2+1)(x-3)} = \frac{(Ax+B)(x-3)+C(x^2+1)}{(x^2+1)(x-3)}$

Thus $4 = (Ax + B)(x - 3) + C(x^2 + 1)$

$$4 = Ax^2 + Bx - 3Ax - 3B + Cx^2 + C$$

$$4 = (A + C)x^2 + (B - 3A)x - 3B + C$$

I.e., $0x^2+0x+4 = (A + C)x^2 + (B - 3A)x - 3B + C$

Thus $0 = A + C,\ 0 = B - 3A,\ 4 = -3B + C$.

$C = -A,\ B = 3A,$
$4 = -3(3A) + -A$ implies $4 = -10A$.
Hence $A = -\frac{2}{5},\ B = 3(-\frac{2}{5}) = -\frac{6}{5},\ C = \frac{2}{5}$.

Thus, $\frac{4}{(x^2+1)(x-3)} = \frac{-\frac{2}{5}x-\frac{6}{5}}{x^2+1} + \frac{\frac{2}{5}}{x-3}$

$$= \frac{-2x-6}{5(x^2+1)} + \frac{2}{5(x-3)}$$