Solving first order differential equation:

Method 1 (sect. 2.2): Separate variables.

Method 2 (sect. 2.1): If linear $[y'(t)+p(t)y(t) = g(t)]$, multiply equation by an integrating factor
$u(t) = e^{\int p(t)dt}$.

$$\begin{gathered} y' + py = g \\ y'u + upy = ug \\ (uy)' = ug \\ \int (uy)' = \int ug \\ uy = \int ug \\ \text{etc...} \end{gathered}$$

Method 3 (sect. 2.4): Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n > 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

If $v = \frac{dx}{dt}$, can use the following to simplify (especially if there are 3 variables).

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

integration techniques: u-substitution, integration by parts, partial fractions.

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direction field = slope field = graph of $\frac{dv}{dt}$ in t, v-plane.

*** can use slope field to determine behavior of v including as $t \to \infty$.
Equilibrium Solution = constant solution

stable, unstable, semi-stable.

Solving second order differential equation:

p. 135: $y'' = f(t, y')$, $y'' = f(y, y')$,

Transform to first order: Let $v = y'$.

If needed, note $v' = \frac{dv}{dt} = \frac{dv}{dt}\frac{dy}{dy} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$.

Note this trick sometimes helpful for first order equations.

Ch 3: linear $ay'' + by' + cy = 0$,

Need to have two independent solutions.

If ϕ_1, ϕ_2 are solutions to a LINEAR HOMOGENEOUS differential equation, $c_1\phi_1 + c_2\phi_2$ is also a solution

Existence and Uniqueness

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$\begin{gathered} y' + p(t)y = g(t), \\ y(t_0) = y_0 \end{gathered}$$

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$\begin{gathered} y'' + p(t)y' + q(t)y = g(t), \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{gathered}$$

Definition: The Wronskian of two differential functions, f and g is

$$W(f, g) = fg' - f'g = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Thm 3.2.4: Given (1) the hypothesis of thm 3.2.1
(2) ϕ_1 and ϕ_2 are 2 sol'ns to $y'' + p(t)y' + q(t)y = 0$ (*)
(3) $W(\phi_1, \phi_2)(t_0) \neq 0$, for some $t_0 \in (a, b)$,
then if f is a solution to (*), then $f = c_1\phi_1 + c_2\phi_2$ for some c_1 and c_2.

Thm 2.4.2: Suppose $z = f(t, y)$ and $z = \frac{\partial f}{\partial y}(t, y)$ are continuous on $(a, b) \times (c, d)$ and the point $(t_0, y_0) \in (a, b) \times (c, d)$, then there exists an interval $(t_0 - h, t_0 + h) \subset (a, b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Note the initial value problem

$$y' = y^{\frac{1}{3}}, \ y(0) = 0$$

has an infinite number of different solutions.

$$\begin{gathered} y^{-\frac{1}{3}} dy = dt \\ \tfrac{3}{2} y^{\frac{2}{3}} = t + C \\ y = \pm(\tfrac{2}{3}t + C)^{\frac{3}{2}} \\ y(0) = 0 \text{ implies } C = 0 \end{gathered}$$

Thus $y = \pm(\frac{2}{3}t)^{\frac{3}{2}}$ are solutions.

$y = 0$ is also a solution, etc.

Compare to Thm 2.4.2:
$f(t, y) = y^{\frac{1}{3}}$ is continuous near (0, 0)
But $\frac{\partial f}{\partial y}(t, y) = \frac{1}{3} y^{\frac{-2}{3}}$ is not continuous near (0, 0) since it isn't defined at (0, 0).