A function f is linear if $f(a\mathbf{x} + b\mathbf{y}) = af(\mathbf{x}) + bf(\mathbf{y})$

Or equivalently f is linear if
1.) $f(a\mathbf{x}) = af(\mathbf{x})$ and 2.) $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$

Theorem: If f is linear, then $f(\mathbf{0}) = \mathbf{0}$

Proof: $f(\mathbf{0}) = f(0 \cdot \mathbf{0}) = 0 \cdot f(\mathbf{0}) = \mathbf{0}$

Example 0.) $f : R \to R$, $g(x) = 2x + 5$ is NOT linear

Proof 1: $g(0) = 5 \neq 0$

Proof 2:
$g(3 \cdot 4) = g(12) = 2(12) + 5 = 29$

$3g(4) = 3[2(4) + 5] = 3[13] = 39$

$29 \neq 39$

Hence $g(3 \cdot 4) \neq 3g(4)$

Proof 3:
$g(0 + 1) = g(1) = 2(1) + 5 = 7$

$g(0) + g(1) = [2(0) + 5] + [2(1) + 5] = 5 + 7 = 12$

$7 \neq 12$

Hence $g(0 + 1) \neq g(0) + g(1)$

Example 1.) $f : R \to R$, $f(x) = 2x$

Proof:
$f(ax + by) = 2(ax + by) = 2ax + 2by = af(x) + bf($

Example 3.) D : set of all differential functions $\to$ s
of all functions, $D(f) = f'$

Proof:
$D(af + bg) = (af + bg)' = af' + bg' = aD(f) + bD($

Example 4.) Given a, b real numbers,
I : set of all integrable functions on [a, b] $\to$ R
$I(f) = \int_a^b f$

Proof:
$I(sf + tg) = \int_a^b sf + tg = s\int_a^b f + t\int_a^b g = sI(f) + tI($

Example 5.) The inverse of a linear function is line
(when the inverse exists).

Proof: Suppose $f^{-1}(x) = c$, $f^{-1}(y) = d$.

Then $f(c) = x$ and $f(d) = y$ and
$f(ac + bd) = af(c) + bf(d) = ax + by$.

Hence $f^{-1}(ax + by) = ac + bd = af^{-1}(x) + bf^{-1}(y$

Example 6.) D : set of all twice differential functions $\rightarrow$ set of all functions, $L(f) = af'' + bf' + cf$

Proof:

$L(sf + tg) = a(sf + tg)'' + b(sf + tg)' + c(sf + tg)$

$$= saf'' + tag'' + sbf' + tbg' + scf + tcg$$

$$= s(af'' + bf' + cf) + t(ag'' + bg' + cg)$$

$$= sL(f) + tL(g)$$

Example 7.) The LaPlace transform $\mathcal{L}$: is linear

$\mathcal{L}$: set of all functions satisfying hypothesis of thm 6.1.2 $\rightarrow$ set of all functions,

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt = F(s)$$

Thm 6.1.2: The Laplace transform $\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt = F(s)$ exists for all $s > a$ if

1.) f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A.

2.) There exist constants K, a, M such that $|f(t)| \leq Ke^{at}$ for all $t \geq M$.

Theorem: Suppose f, f', ..., $f^{(n-1)}$ are continuo and $f^{(n)}$ is piecewise continuous on $0 \leq t \leq A$. Su pose there exists constance K, a, and M such th $|f(t)| \leq Ke^{at}$, $|f'(t)| \leq Ke^{at}$, ..., $|f^{(n-1)}(t)| \leq Ke$ for $t \geq M$. Then $\mathcal{L}(f^{(n)})$ exists for $s > a$ and is giv by

$\mathcal{L}(f^{(n)})$
$= s^n\mathcal{L}(f) - s^{n-1}f(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}($

LaPlace Transform

The LaPlace Transform is a method to change a d ferential equation to a linear equation.

Example:
Solve $2y'' + 3y' + 4y = 0$, $y(0) = 5$, $y'(0) = 6$.

1.) Take the LaPlace Transform of both sides of t equation:

2.) Use the fact that the LaPlace Transform is linear:

3.) Use thm to change this equation into an algebraic equation:

$\mathcal{L}(f^{(n)})$
$= s^n\mathcal{L}(f) - s^{n-1}f(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$

3.5) Substitute in the initial values.

4.) Solve the algebraic equation for $\mathcal{L}(y)$

Some algebra implies $\mathcal{L}(y) =$

5.) Solve for y by taking the inverse LaPlace tran
form of both sides (use a table):

To find the inverse LaPlace transform, you may ne
to use that the inverse LaPlace transform in line
You may also need to use partial fractions or oth
methods in order to write the right-hand side of (*)
a sum of functions whose inverse LaPlace transfor
is known.

Calculus pre-requisites you must know.

Derivative = slope of tangent line = rate.

Integral = area between curve and x-axis (where ar
can be negative).

A function is (Riemann) integrable if this area can
calculated using rectangles as in first year calculus