Linear algebra pre-requisites you must know.

$\mathbf{b_1}, ..., \mathbf{b_n}$ are linearly independent if

$$c_1\mathbf{b_1} + c_2\mathbf{b_2} + ... + c_n\mathbf{b_n} = d_1\mathbf{b_1} + d_2\mathbf{b_2} + ... + d_n\mathbf{b_n}$$
$$\text{implies } c_1 = d_1,\ c_2 = d_2 ...,\ c_n = d_n.$$

or equivalently,

$\mathbf{b_1}, ..., \mathbf{b_n}$ are linearly independent if

$c_1\mathbf{b_1} + c_2\mathbf{b_2} + ... + c_n\mathbf{b_n} = 0$ implies $c_1 = c_2 = ...c_n$.

Example 1: $\mathbf{b_1} = (1, 0, 0)$, $\mathbf{b_2} = (0, 1, 0)$, $\mathbf{b_3} = (0, 0, 1)$.

$$(1, 2, 3) \neq (1, 2, 4).$$

$$\text{If } (a, b, c) = (1, 2, 3) \text{ then } a = 1,\ b = 2,\ c = 3.$$

Example 2: $\mathbf{b_1} = 1$, $\mathbf{b_2} = t$, $\mathbf{b_3} = t^2$.

$$1 + 2t + 3t^2 \neq 1 + 2t + 4t^2.$$

$$\text{If } a + bt + ct^2 = 1 + 2t + 3t^2 \text{ then } a = 1,\ b = 2,\ c = 3.$$

Application: Partial Fractions

$$\frac{4}{(x^2+1)(x-3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-3}$$

$$= \frac{(Ax+B)(x-3)+C(x^2+1)}{(x^2+1)(x-3)}$$

Hence $\frac{4}{(x^2+1)(x-3)} = \frac{(Ax+B)(x-3)+C(x^2+1)}{(x^2+1)(x-3)}$

Thus $4 = (Ax+B)(x-3) + C(x^2+1)$

$$4 = Ax^2 + Bx - 3Ax - 3B + Cx^2 + C$$

$$4 = (A+C)x^2 + (B-3A)x - 3B + C$$

I.e., $0x^2+0x+4 = (A+C)x^2 + (B-3A)x - 3B + C$

Thus $0 = A + C$, $0 = B - 3A$, $4 = -3B + C$.

$C = -A$, $B = 3A$,
$4 = -3(3A) + -A$ implies $4 = -10A$.
Hence $A = -\frac{2}{5}$, $B = 3(-\frac{2}{5}) = -\frac{6}{5}$, $C = \frac{2}{5}$.

Thus, $\frac{4}{(x^2+1)(x-3)} = \frac{-\frac{2}{5}x-\frac{6}{5}}{x^2+1} + \frac{\frac{2}{5}}{x-3}$

$$= \frac{-2x-6}{5(x^2+1)} + \frac{2}{5(x-3)}$$

Linear Functions

A function f is linear if $f(a\mathbf{x} + b\mathbf{y}) = af(\mathbf{x}) + bf(\mathbf{y})$

Or equivalently f is linear if
1.) $f(a\mathbf{x}) = af(\mathbf{x})$ and 2.) $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$

Theorem: If f is linear, then $f(\mathbf{0}) = \mathbf{0}$

Proof: $f(\mathbf{0}) = f(0 \cdot \mathbf{0}) = 0 \cdot f(\mathbf{0}) = \mathbf{0}$

Example 1.) $f : R \to R$, $f(x) = 2x$

Proof:
$f(ax + by) = 2(ax + by) = 2ax + 2by = af(x) + bf(y)$

Example 2.) $f : R^2 \to R^2$,
$$f((x_1, x_2)) = (2x_1, x_1 + x_2)$$

Proof: Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$

$a\mathbf{x}+b\mathbf{y} = a(x_1, x_2)+b(y_1, y_2) = (ax_1, ax_2)+(by_1, by_2) =$
$(ax_1 + by_1, ax_2 + by_2)$

$$f(ax_1 + by_1, ax_2 + by_2)$$

$$= (2(ax_1 + by_1), ax_1 + by_1 + ax_2 + by_2)$$

$$= (2ax_1 + 2by_1, ax_1 + ax_2 + by_1 + by_2)$$

$$= (2ax_1, ax_1 + ax_2) + (2by_1, by_1 + by_2)$$

$$= a(2x_1, x_1 + x_2) + b(2y_1, y_1 + y_2)$$

$$= af((x_1, x_2)) + bf((y_1, y_2))$$

Example 3.) D : set of all differential functions $\to$ set of all functions, $D(f) = f'$

Proof:
$D(af + bg) = (af + bg)' = af' + bg' = aD(f) + bD(g)$

Example 4.) Given a, b real numbers,
I : set of all integrable functions on [a, b] $\to$ R , $I(f) = \int_a^b f$

Proof: $I(sf + tg) = \int_a^b sf + tg = s\int_a^b f + t\int_a^b g = sI(f) + tI(g)$

Example 5.) The inverse of a linear function is linear (when the inverse exists).

Suppose $f^{-1}(x) = c$, $f^{-1}(y) = d$.

Then $f(c) = x$ and $f(d) = y$ and
$f(ac + bd) = af(c) + bf(d) = ax + by$.

Hence $f^{-1}(ax + by) = ac + bd = af^{-1}(x) + bf^{-1}(y)$.

Example 6.) D : set of all twice differential functions $\rightarrow$ set of all functions, $L(f) = af'' + bf' + cf$

Proof:

$$
\begin{aligned}
L(sf + tg) &= a(sf + tg)'' + b(sf + tg)' + c(sf + tg) \\
&= saf'' + tag'' + sbf' + tbg' + scf + tcg \\
&= s(af'' + bf' + cf) + t(ag'' + bg' + cg) \\
&= sL(f) + tL(g)
\end{aligned}
$$

Consequence 1: If ψ_1, ψ_2 are solutions to $af'' + bf' + cf = 0$, then $3\psi_1 + 5\psi_2$ is also a solution to $af'' + bf' + cf = 0$,

Proof: Since ψ_1, ψ_2 are solutions to $af''+bf'+cf = 0$, $L(\psi_1) = 0$ and $L(\psi_2) = 0$.

Hence $L(3\psi_1 + 5\psi_2) = 3L(\psi_1) + 5L(\psi_2)$

$$= 3(0) + 5(0) = 0.$$

Thus $3\psi_1+5\psi_2$ is also a solution to $af''+bf'+cf = 0$

Consequence 2:
If ψ_1 is a solution to $af'' + bf' + cf = h$
and ψ_2 is a solution to $af'' + bf' + cf = k$,
then $3\psi_1+5\psi_2$ is a solution to $af''+bf'+cf = 3h+5k$,

Since ψ_1 is a solution to $af''+bf'+cf = h$, $L(\psi_1) = h$.

Since ψ_2 is a solution to $af''+bf'+cf = k$, $L(\psi_2) = k$.

Hence $L(3\psi_1 + 5\psi_2) = 3L(\psi_1) + 5L(\psi_2)$

$$= 3h + 5k.$$

Thus $3\psi_1 + 5\psi_2$ is also a solution to

$$af'' + bf' + cf = 3h + 5k$$

Example 7.) The LaPlace transform $\mathcal{L}$: is linear

Theorem: Suppose f, f', ..., $f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on $0 \le t \le A$. Suppose there exists constance K, a, and M such that $|f(t)| \le Ke^{at}$, $|f'(t)| \le Ke^{at}$, ..., $|f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}(f^{(n)})$ exists for $s > a$ and is given by

$$\mathcal{L}(f^{(n)})$$
$$= s^n \mathcal{L}(f) - s^{n-1} f(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

LaPlace Transform

The LaPlace Transform is a method to change a differential equation to a linear equation.

Example:
Solve $2y'' + 3y' + 4y = 0$, $y(0) = 5$, $y'(0) = 6$.

1.) Take the LaPlace Transform of both sides of the equation:

2.) Use the fact that the LaPlace Transform is linear:

3.) Use thm to change this equation into an algebraic equation:

$$\mathcal{L}(f^{(n)})$$
$$= s^n \mathcal{L}(f) - s^{n-1} f(0) - ... - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

3.5) Substitute in the initial values.

4.) Solve the algebraic equation for $\mathcal{L}(y)$

Some algebra implies $\mathcal{L}(y) =$

5.) Solve for y by taking the inverse LaPlace transform of both sides (use a table):

To find the inverse LaPlace transform, you may need to use that the inverse LaPlace transform in linear. You may also need to use partial fractions or other methods in order to right the righthand side of (*) as a sum of functions whose inverse LaPlace transform is known.

Calculus pre-requisites you must know.

Derivative = slope of tangent line = rate.

Integral = area between curve and x-axis (where area can be negative).

Integration by parts:

Derivative of a product: $(uv)' = uv' + vu'$

$$uv' = (uv)' - vu'$$
$$\int uv' = \int (uv)' - \int vu'$$
$$\int uv' = (uv) - \int vu'$$

Example: $\int e^{2x} sin(3x)$

Let $u = sin(3x)$, $dv = e^{2x}$

then $du = 3cos(3x)$, $v = \frac{1}{2}e^{2x}$

then $d^2u = -9sin(3x)$, $\int v = \frac{1}{4}e^{2x}$

$\int e^{2x} sin(3x) = \frac{1}{2} sin(3x)e^{2x} - \int \frac{3}{2}e^{2x} cos(3x)$

$$= \frac{1}{2} sin(3x)e^{2x} - [\frac{3}{4} cos(3x)e^{2x} - \int \frac{-9}{4} sin(3x)e^{2x}$$

$\int e^{2x} sin(3x) = \frac{1}{2} sin(3x)e^{2x} - \frac{3}{4} cos(3x)e^{2x} - \frac{9}{4} \int sin(3x)e^{2x}$

$\frac{13}{4} \int e^{2x} sin(3x) = \frac{1}{2} sin(3x)e^{2x} - \frac{3}{4} cos(3x)e^{2x}$

$\int e^{2x} sin(3x) = \frac{4}{13}[\frac{1}{2} sin(3x)e^{2x} - \frac{3}{4} cos(3x)e^{2x}]$

Exercise: Calculate $\int e^{x} cos(2x)$