Math 34 Differential Equations Exam #1
March 4, 2005

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[27] 1.) A mass weighing 1 kg stretches a spring 9.8m. If the mass is pulled down an additional 2m and then set in motion with an upward velocity of 2m/sec, and if there is no damping, determine the position u of the mass at any time t. Find the frequency, period, and amplitude of the motion.

$m = 1$ kg $\qquad mg - kL = 0 \Rightarrow k = \frac{mg}{L} = \frac{(1)(9.8)}{9.8} = 1$

$\gamma = 0$

$mu'' + \gamma u' + ku = 0$

$u'' + u = 0$
$u(0) = 2$
$u'(0) = -2$

$r^2 + 1 = 0$

$r^2 = -1$

$r = \pm\sqrt{-1} = \pm i$

$u(t) = A\cos(t) + B\sin(t)$

$u'(t) = -A\sin(t) + B\cos(t)$

$u(0) = 2: \quad 2 = A\cdot 1 + B\cdot 0 \Rightarrow A = 2$

$u'(0) = -2: \quad -2 = -A\cdot 0 + B\cdot 1 \Rightarrow B = -2$

$u(t) = 2\cos(t) - 2\sin(t)$

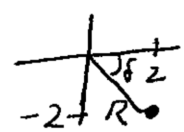
$u(t) = A\cos(t) + B\sin(t)$
$= R\cos\delta\cos t + R\sin\delta\sin t$
$= R\cos(t - \delta)$

where $A = R\cos\delta$
$B = R\sin\delta$

$\Rightarrow R = \sqrt{A^2 + B^2} = \sqrt{2^2 + (-2)^2}$
$= \sqrt{4+4} = \sqrt{8}$

$\tan\delta = \frac{B}{A} = \frac{-2}{2} = -1$

$\delta = -\frac{\pi}{4}$



$u(t) = \sqrt{8}\cos\left(t + \frac{\pi}{4}\right)$

Answer

position: $u(t) = \sqrt{8}\cos\left(t + \frac{\pi}{4}\right)$

frequency = 1 $\qquad$ period = 2π $\qquad$ amplitude = $\sqrt{8}$

[18] 2.) Find the general solution to the following differential equation:

$$4y' = t(y^2 - 4)$$

$$4\frac{dy}{dt} = t(y^2 - 4)$$

$$\int \frac{4\,dy}{y^2-4} = \int t\,dt$$

$$\int \frac{4\,dy}{(y-2)(y+2)} = \frac{1}{2}t^2 + C$$

$$\frac{A}{y-2} + \frac{B}{y+2} = \frac{4}{(y-2)(y+2)}$$

$$A(y+2) + B(y-2) = 4$$

$$Ay + 2A + By - 2B = 4$$

$$y(A+B) + 2A - 2B = 0y + 4$$

$$A + B = 0 \Rightarrow B = -A$$

$$2A - 2B = 4$$

$$2A - 2(-A) = 4A = 4 \Rightarrow A = 1$$

$$B = -A = -1$$

$$\int \frac{dy}{y-2} + \int \frac{-dy}{y+2} = \frac{1}{2}t^2 + C$$

$$\ln|y-2| - \ln|y+2| = \frac{1}{2}t^2 + C$$

$$\ln\left|\frac{y-2}{y+2}\right| = \frac{1}{2}t^2 + C$$

$$\left(\frac{y-2}{y+2}\right) = Ke^{(\frac{1}{2}t^2)}$$

$$(y-2) = (y+2)Ke^{(t^2/2)}$$

$$y - yKe^{(t^2/2)} = 2Ke^{t^2/2} + 2$$

$$y(1 - Ke^{(t^2/2)}) = 2Ke^{t^2/2} + 2$$

$$y = \frac{2Ke^{(t^2/2)} + 2}{1 - Ke^{(t^2/2)}}$$

Answer 2.) or ↓ or

[18] 3.) Find the general solution to the following differential equation:

$$ty' + 3y = t^5$$

$$y' + \left(\frac{3}{t}\right)y = t^4 \quad \times t^3$$

$$u = e^{\int \frac{3}{t}} = e^{3\ln|t|} = e^{\ln|t|^3} = |t|^3$$

use $u = t^3$

$$t^3 y' + 3t^2 y = t^7$$

$$\int (t^3 y)' = \int t^7$$

$$t^3 y = \frac{t^8}{8} + c$$

$$y = \frac{t^5}{8} + ct^{-3}$$

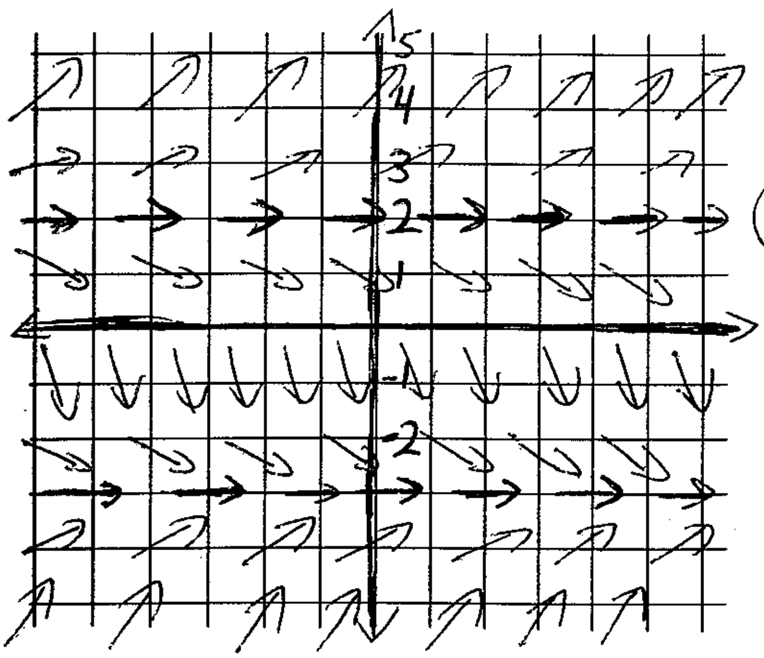
or

Answer 3.) ______________________

[15] 4.) Draw a direction field for the following differential equation:

$$y' = (y+3)(y-2)$$

Find the equilibrium solution(s) and determine if asymtptotically stable, semistable, or unstable.



$y = 2$ asymptototically unstable sol'n

$y = -3$ asymptotically stable sol'n

[9] 5.) Suppose that the general solution to $y'' - y = 0$ is $c_1 e^t + c_2 e^{-t}$. Find the general solution to $y'' - y = cos(t)$

$\psi = A \cos t$

$\psi' = -A \sin t$

$\psi'' = -A \cos t$

$y'' - y = \cos t$

$-A\cos t - A\cos t = \cos t$

$-2A\cos t = \cos t$

$-2A = 1 \Rightarrow A = -1/2$

Answer 5.) $y = -\frac{1}{2}\cos t + c_1 e^{t} + c_2 e^{-t}$

[6] 6.) Calculate the Wronskian of $f(x) = e^x$ and $g(x) = e^{x-1}$. Are f and g linearly dependent or linearly independent?

$$\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} e^x & e^{x-1} \\ e^x & e^{x-1} \end{vmatrix} =$$

$$= e^x e^{x-1} - e^x e^{x-1} = 0 \quad \text{for all } x$$

Hence f & g are linearly dependent

7.) Match the following differential equation to its graph:

C [3] 7i.) $y'' + 2y' + y = 0$, $y(0) = 0.1$, $y'(0) = 0.2$ $\quad r^2+2r+1 = (r+1)^2$ one real sol. critically damp

B [3] 7ii) $y'' + 2y' + 10y = 0$, $y(0) = 0.1$, $y'(0) = 0.2$ $\quad r^2+2r+10 = 0$, $r = \frac{-2 \pm \sqrt{4-4(1)(10)}}{2}$ 2 comp

A [3] 7iii) $y'' + 10y = 0$, $y(0) = 0.1$, $y'(0) = 0.2$

$r^2+10=0$ no damping