1a.) \( \mathcal{L}(0) = 0 \)

1b.) \( \mathcal{L}^{-1}(\frac{2}{(s-4)^2+5}) = \frac{2}{\sqrt{5}} e^{4t} \sin(t\sqrt{5}) \)

\[
\mathcal{L}^{-1}(\frac{2}{(s-4)^2+5}) = \frac{2}{\sqrt{5}} \mathcal{L}^{-1}(\frac{2\sqrt{5}}{(s-4)^2+5}) = \frac{2}{\sqrt{5}} e^{4t} \sin(t\sqrt{5})
\]

2.) Circle T for True or F for False:

Suppose \( y = f(t) \) is a solution to \( 3y'' + 10y = \cos(t) \), \( y(0) = 0 \), \( y'(0) = 0 \) and suppose \( y = g(t) \) is a solution to \( 3y'' + 10y = \cos(t) \), \( y(0) = 100 \), \( y'(0) = -200 \). For large values of \( t \), \( f(t) - g(t) \) is very small.

Note: no damping

F

3a.) Given \( 2y'' + 5y = \cos(wt) \), determine the value \( w \) for which undamped resonance occurs:

\( 2r^2 + 5 = 0 \). Thus \( r = i\sqrt{\frac{5}{2}} \). Hence homogeneous soln is \( y(t) = c_1 \cos(t\sqrt{\frac{5}{2}}) + c_2 \sin(t\sqrt{\frac{5}{2}}) \).

Hence a potential solution for the non-homogeneous equation \( 2y'' + 5y = \cos(t\sqrt{\frac{5}{2}}) \) would be of the form: \( t[A\cos(t\sqrt{\frac{5}{2}}) + B\sin(t\sqrt{\frac{5}{2}})] \).

Answer \( w = \sqrt{\frac{5}{2}} \)

3b.) Briefly describe in words the long-term behaviour of a solution to \( 2y'' + 5y = \cos(wt) \) for this value of \( w \).

The solution oscillates and the pseudo-amplitude gets increasingly larger, approaching infinity.

4.) A mass of 4 kg stretches a spring 5m. The mass is acted on by an external force of \( 6e^t \) N (newtons) and moves in a medium that imparts a viscous force of 8 N when the speed of the mass is 15 m/sec. The mass is pulled downward 1m below its equilibrium position, and then set in motion in the upward direction with a velocity of 10 m/sec. Formulate the initial value problem describing the motion of the mass.

\( m = 4 \). \( F_{\text{viscous}}(t) = -\gamma v(t) \), where \( v \) = velocity. Hence \( 8 = 15\gamma \) implies \( \gamma = \frac{8}{15} \). Also, \( mg = kL \). Hence \( k = \frac{ma}{L} = \frac{4(9.8)}{5} \).

Answer \( 4u''(t) + \frac{8}{15}u'(t) + \frac{4(9.8)}{5}u(t) = 6e^t \)
5.) Use ch 3 methods to solve the given initial value problem.
\[ y'' + 4y = \sin(t), \quad y(0) = 0, \quad y'(0) = 0 \]

Step 1.) Find the general solution to \( y'' + 4y = 0 \):

Guess \( y = e^{rt} \). Then \( r^2 e^{rt} + 4e^{rt} = 0 \) implies \( r^2 + 4 = 0 \) which implies \( r = \pm 2i \).

homogeneous solution: \( y(t) = c_1 \cos(2t) + c_2 \sin(2t) \)

Step 2.) Find ONE solution to \( y'' + 4y = \sin(t) \):

Educated guess: \( y = A \sin(t) \) (since no \( y' \) term).

\[
\begin{align*}
  y &= A \sin(t) \\
  y' &= A \cos(t) \\
  y'' &= -A \sin(t)
\end{align*}
\]

\(-A \sin(t) + 4A \sin(t) = \sin(t)\).

\(3A \sin(t) = \sin(t)\). Hence \( 3A = 1 \) and \( A = \frac{1}{3} \).

The general solution to NON-homogeneous equation is
\[
c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{3} \sin(t)
\]

Step 3.) Initial value problem:

Once general solution to problem is known, can solve initial value problem (i.e., use initial conditions to find \( c_1, c_2 \)).

\[
\begin{align*}
  y(t) &= c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{3} \sin(t) \\
  y'(t) &= -2c_1 \sin(2t) + 2c_2 \cos(2t) + \frac{1}{3} \cos(t) \\
  y(0) &= 0: \quad 0 = c_1 \\
  y'(0) &= 0: \quad 0 = 2c_2 + \frac{1}{3}. \text{ Hence } 2c_2 = -\frac{1}{3} \text{ and } c_2 = -\frac{1}{6}
\end{align*}
\]

Answer \( y(t) = -\frac{1}{6} \sin(2t) + \frac{1}{3} \sin(t) \)
6.) Use the LaPlace transform to solve the given initial value problem.

\[ y'' + 4y = \sin(t), \quad y(0) = 0, \quad y'(0) = 0 \]

\[
L(y'' + 4y) = L(\sin(t))
\]

\[
L(y'') + 4L(y) = \frac{1}{s^2 + 1}
\]

\[
s^2L(y) - sy(0) - y'(0) + 4L(y) = \frac{1}{s^2 + 1}
\]

\[
s^2L(y) + 4L(y) = \frac{1}{s^2 + 1}
\]

\[
L(y)[s^2 + 4] = \frac{1}{s^2 + 1}
\]

\[
L(y) = \frac{1}{(s^2+1)(s^2+4)}. \text{ Hence } y = L^{-1}\left(\frac{1}{(s^2+1)(s^2+4)}\right)
\]

Partial Fractions:

\[
\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}
\]

\[1 = (As + B)(s^2 + 4) + (Cs + D)(s^2 + 1)\]

\[1 = As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2 + Cs + D\]

\[1 = (A + C)s^3 + (B + D)s^2 + (4A + C)s + 4B + D\]

\[A + C = 0 \text{ and } 4A + C = 0.\]

Hence \(C = -A\) and \(4A - A = 0\). Hence \(3A = 0\) and \(A = 0\), \(C = 0\).

Alternatively note \(A = 0\), \(C = 0\) is “obviously a solution” and you only need one (plus it is “obvious” that there is only one solution). Note how “obvious” this is depends on your linear algebra background.

\[B + D = 0 \text{ and } 4B + D = 1\]

Hence \(D = -B\) and \(4B - B = 1\). Hence \(3B = 1\) and \(B = \frac{1}{3}\), \(D = -\frac{1}{3}\)

\[
\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3(s^2+1)} + \frac{-1}{3(s^2+4)}
\]

\[
y = L^{-1}\left(\frac{1}{3(s^2+1)}\right) - L^{-1}\left(\frac{1}{3(s^2+4)}\right) = \frac{1}{3}L^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}L^{-1}\left(\frac{1}{s^2+4}\right)
\]

\[= \frac{1}{3}\sin(t) - \frac{1}{6}L^{-1}\left(\frac{2}{s^2+4}\right) = \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t)
\]

\textbf{Answer} \quad \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t)\]
7.) Prove that if \( F(s) = \mathcal{L}(f(t)) \) exists for \( s > a \geq 0 \), and if \( c \) is a positive constant, then \( \mathcal{L}(u_c(t)f(t-c)) = e^{-cs}L(f(t)) \) with domain \( s > a \).

Hint: \( \int_{0}^{\infty} h(t)dt = \int_{0}^{c} h(t)dt + \int_{c}^{\infty} h(t)dt \) and use \( u \)-substitution (let \( u = t - c \)).

Proof: If the integral \( \int_{0}^{\infty} e^{-st}u_c(t)f(t-c)dt \) exists, then
\[
\mathcal{L}(u_c(t)f(t-c)) = \int_{0}^{\infty} e^{-st}u_c(t)f(t-c)dt \\
= \int_{0}^{c} e^{-st}u_c(t)f(t-c)dt + \int_{c}^{\infty} e^{-st}u_c(t)f(t-c)dt \\
= \int_{0}^{c} e^{-st} \cdot 0 \cdot f(t-c)dt + \int_{c}^{\infty} e^{-st} \cdot 1(t-c)dt \\
= 0 + \int_{c}^{\infty} e^{-st}f(t-c)dt
\]
Let \( u = t - c \), then \( du = dt \) and \( t = u + c \). When \( t = c \), \( u = c - c = 0 \)
\[
\int_{c}^{\infty} e^{-st}f(t-c)dt \\
= \int_{0}^{\infty} e^{-s(u+c)}f(u)du \\
= \int_{0}^{\infty} e^{-su}e^{-sc}f(u)du \\
= e^{-sc} \int_{0}^{\infty} e^{-su}f(u)du \quad \text{since} \quad e^{-sc} \text{ is a constant with respect to } u \\
= e^{-sc}L(f(u)) \\
= e^{-sc}L(f(t))
\]
Note \( F(s) = \mathcal{L}(f(t)) = \int_{0}^{\infty} e^{-su}f(u)du \) exists for \( s > a \).

Hence \( \mathcal{L}(u_c(t)f(t-c)) = \int_{0}^{\infty} e^{-st}u_c(t)f(t-c)dt \) exists for \( s > a \) and \( \mathcal{L}(u_c(t)f(t-c)) = e^{-sc}L(f(t)) \).