[5] 1a.) Define: A function $f$ is linear if

$$f(ax + by) = af(x) + bf(y)$$

where $a$, $b$ are scalars (or real numbers or complex numbers for this class).

Circle T for True or F for False:

[3] 1b.) If $y = \phi_1(t)$ and $y = \phi_2(t)$ are solutions to a second order homogeneous differential equation, then $c_1\phi_1 + c_2\phi_2$ is also a solution.  

F

[3] 1c.) If $y = \phi_1(t)$ and $y = \phi_2(t)$ are solutions to a second order linear homogeneous differential equation, then $c_1\phi_1 + c_2\phi_2$ is also a solution.  

T

[3] 1d.) $\ln(t)y'' - \frac{y'}{t} + y\sqrt{t} = e^t\cos(t)$ is a second order linear differential equation.  

T

[3] 1e.) If $p$, and $g$ are continuous, then there exists a unique solution to

$$y' + p(t)y = g(t), \ y(0) = 2.$$  

T

[3] 1f.) A first order linear differential equation has a unique solution such that $y(0) = 2$.  

F

Choose 4 problems from problems 2 - 6. You may do all the problems for up to 4 pts extra credit. If you do not choose your best 4 problems, I will substitute your extra problem for your lowest scoring problem, but with a 3 point penalty (if it improves your grade).

Circle the numbers corresponding to your 4 chosen problems: 2 3 4 5 6

Extra credit problem (choose 1 from problems 2 - 6): _____________
2a.) Match the following differential equation to its direction field. Indicate all equilibrium solutions (if any) and state whether **stable, unstable or semi-stable**. If a differential equation has no equilibrium solutions, state so.

- **E = I.** \( y' = 1 - y \)
- **D = II.** \( y' = -1 + y \)
- **A = III.** \( y' = y(y + 2) \)
- **C = IV.** \( y' = y^2(2 - y) \)
- **B = V.** \( y' = t + y \)

III = A.) stable: \( y = -2 \); unstable: \( y = 0 \)

no equilibrium soln.

IV = C.) semi-stable: \( y = 0 \); stable: \( y = 2 \)

II = D.) unstable: \( y = 1 \)

I = E.) stable \( y = 1 \)

2b.) Match the following differential equation initial value problem to its graph:

- **C = I.** \( y'' + y' + 49y = 0, \quad y(0) = 0, \quad y'(0) = 5 \)
  \( r^2 + r + 49 = 0 \) implies \( r = a \pm bi \)
  General solution:
  \[ y = e^{at}[c_1\cos(bt) + c_2\sin(bt)] \]
  Note \( a \) is negative

- **D = II.** \( y'' + y' + 49y = 0, \quad y(0) = 1, \quad y'(0) = 5 \)
  \( r^2 + 49 = 0 \) implies \( r = \pm 7i \)
  (NOTE: no damping)
  General solution:
  \[ y = c_1\cos(bt) + c_2\sin(bt) \]

- **B = III.** \( y'' + 49y = 0, \quad y(0) = 0, \quad y'(0) = 5 \)

- **A = IV.** \( y'' + 49y = 0, \quad y(0) = 1, \quad y'(0) = 5 \)
3.) Solve the differential equation \( t^3y' + 3t^2y = \frac{\ln(e)}{t^2-4} \). Simplify your answer.

Note this is a first order LINEAR differential equation. Hence you can use an integrating factor in order to write the LHS as the derivative of a product.

Shortcut for this problem: Note the LHS is already the derivative of a product

\[
t^3y' + 3t^2y = \frac{\ln(e)}{t^2-4}.
\]

\[
(t^3y)' = \frac{1}{t^2-4}
\]

\[
\int (t^3y)' dt = \int \frac{1}{t^2-4} dt
\]

\[
t^3y = \int \frac{1}{t^2-4} dt
\]

\[
\int \frac{1}{t^2-4} dt = \int \left[ \frac{1}{4(t-2)} - \frac{1}{4(t+2)} \right] dt = \frac{1}{4} \ln|t-2| - \frac{1}{4} \ln|t+2| + C
\]

\[
= \frac{1}{4} (\ln|t-2| - \ln|t+2|) + C
\]

\[
= \frac{1}{4} \ln\left| \frac{t-2}{t+2} \right| + C
\]

Hence \( t^3y = \frac{1}{4} \ln\left| \frac{t-2}{t+2} \right| + C \)

Hence \( y = \frac{1}{4t^3} \ln\left| \frac{t-2}{t+2} \right| + Ct^{-3} \)

Note to integrate the RHS, we needed to use partial fractions:

\[
\frac{A}{t-2} + \frac{B}{t+2} = \frac{1}{t^2-4}
\]

\[
A(t + 2) + B(t - 2) = 1
\]

\[
At + 2A + Bt - 2B = 1
\]

\[
t(A + B) + 2A - 2B = 1
\]

Thus \( A + B = 0, 2A - 2B = 1 \). Thus \( A = \frac{1}{4}, B = -\frac{1}{4} \)

Answer: \( y = \frac{\ln\left| \frac{t-2}{t+2} \right| + C}{4t^3} \)
4.) Solve \( \frac{y''}{y} - \frac{1}{y^2} = 0 \), \( y(2) = 1 \), \( y'(2) = -1 \)

Non-linear 2nd order differential equation. Hence you have only one option. Since you have no idea how to solve a non-linear 2nd order differential equation, you must transform it into something you can solve: a first order differential equation.

Let \( v = y' \), \( v' = y'' \)

Hence \( \frac{y''}{y} - \frac{1}{y^2} = 0 \) becomes \( \frac{v'}{v} - \frac{1}{y^2} = 0 \)

Now we have a first order differential equation, but it involves 3 variables. Since \( v' = \frac{dv}{dt} \), our equation involves the variables \( v, t, \) and \( y \). Fortunately we know how to eliminate one of these variables:

Recall \( v = y' = \frac{dy}{dt} \). Hence \( v' = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = \frac{dv}{dy} v \).

Thus the equation \( \frac{v'}{v} - \frac{1}{y^2} = 0 \) becomes \( \frac{dv}{dy} v - \frac{1}{y^2} = 0 \)

We can simplify and separate variables.

\[
\frac{dv}{dy} = \frac{1}{y^2}
\]

\[
\int dv = \int y^{-2}dy
\]

\[
y' = v = -y^{-1} + c_1
\]

\[
y(2) = 1, \quad y'(2) = -1: \text{ when } t = 2, \quad -1 = -1 + c_1. \quad \text{Thus } c_1 = 0
\]

\[
\frac{dy}{dt} = -y^{-1}
\]

\[
\int ydy = \int -dt
\]

\[
\frac{1}{2} y^2 = -t + c_2
\]

\[
y^2 = -2t + c_2
\]

\[
y = \pm \sqrt{-2t + c_2}
\]

\[
y(2) = 1: \quad 1 = \pm \sqrt{-4 + c_2}. \quad \text{Thus } c_2 = 5 \quad \text{and} \quad y = \sqrt{-2t+5}
\]

Answer: \( y = \sqrt{-2t+5} \)
5.) A mass of 10 kg stretches a spring 9.8 m. The mass is pushed upward, contracting the spring a distance of one meter and set in motion with an upward velocity of 4 m/sec. If the mass moves in a medium that imparts a viscous force of 100 N when the speed of the mass is 5 m/sec, find the equation of motion of the mass.

\[ m u''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0 \]
\[ mg - kL = 0, \quad F_{damping}(t) = -\gamma u'(t) \]

\( m \) = mass,
\( k \) = spring force proportionality constant,
\( \gamma \) = damping force proportionality constant

\( m = 10, \)
\( mg = kL: \quad 98 = k(9.8) \) implies \( k = 10 \)

\( F_{damping}(t) = -\gamma u'(t): \quad 100 = 5\gamma. \) Thus \( \gamma = 20 \)

\[ 10u''(t) + 20u'(t) + 10u(t) = 0 \]

\[ u''(t) + 2u'(t) + u(t) = 0, \quad u(0) = -1, \quad u'(0) = -4 \]

If \( u = e^{rt} \), then \( u' = re^{rt} \) and \( u'' = r^2e^{rt}. \)

Hence \( r^2e^{rt} + 2re^{rt} + e^{rt} = 0 \)

\( r^2 + 2r + 1 = 0 \)

\( (r + 1) = 0. \) Hence \( r = -1 \)

Hence general solution is \( u(t) = c_1e^{-t} + c_2te^{-t} \)

\( u(0) = -1, \quad u'(0) = -4 \)

\( u(t) = c_1e^{-t} + c_2te^{-t} \)

\( u'(t) = -c_1e^{-t} + c_2e^{-t} - c_2te^{-t} \)

\( -1 = c_1 \)

\( -4 = -c_1 + c_2, \quad -4 = 1 + c_2. \) Thus \( c_2 = -5 \)

\( u(t) = -e^{-t} - 5te^{-t} \)

Answer: \( u(t) = -e^{-t} - 5te^{-t} \)
6.) Show that $L : \text{set of all twice differentiable functions} \rightarrow \text{set of all functions},
L(f) = af'' + bf' + cf$ is a linear function.

Hint: Calculate $L(rf +tg)$ where $r, t$ are real numbers and $f, g$ are twice differentiable functions.

$L(rf +tg) = a[rf +tg]'' + b[rf +tg]' + c[rf +tg]$

$= arf'' + atg'' + brf' + btg' + crf + ctg$

$= arf'' + brf' + crf + atg'' + btg' + ctg$

$= r[af'' + bf' + cf] + t[ag'' + bg' + cg]$

$= rL(f) + tL(g)$

Hence $L$ is a linear function.

If $y = \phi(t)$ is a solution to $af'' + bf' + cf = 0$, then $L(\phi) = \underline{0}$.

If $y = \psi(t)$ is a solution to $af'' + bf' + cf = 0$, then $L(\psi) = \underline{0}$.

$L(c_1\phi + c_2\psi) = \underline{0}$.

Is $c_1\phi + c_2\psi$ a solution to $af'' + bf' + cf = 0$? \underline{yes}