$$g(t) = \begin{cases} 0 & t < 4 \\ 2 & 4 \le t < 10 \\ t & t \ge 10 \end{cases}$$

Hence $g(t) = 2u_4(t) + (t-2)u_{10}(t)$

Solve $3y'' + y' + y = 2u_4(t) + (t-2)u_{10}(t)$,
$y(0) = 0$, $y'(0) = 0$.

$$3\mathcal{L}(y'') + \mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(2u_4(t)) + \mathcal{L}((t-2)u_{10}(t))$$

Thm: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$.

Thus $\mathcal{L}(u_c(t)f(t)) =$ ____________________.

$$3[s^2\mathcal{L}(y) - sy(0) - y'(0)] + s\mathcal{L}(y) - y(0) + \mathcal{L}(y) \\ = e^{-4s}\mathcal{L}(2) + e^{-10s}\mathcal{L}((t+8))$$

$$3[s^2\mathcal{L}(y)] + s\mathcal{L}(y) + \mathcal{L}(y) = 2e^{-4s}\mathcal{L}(1) + e^{-10s}\mathcal{L}(t) + 8e^{-10s}\mathcal{L}(1)$$

$$\mathcal{L}(y)[3s^2 + s + 1] = e^{-4s}\tfrac{2}{s} + e^{-10s}\tfrac{1}{s^2} + e^{-10s}\tfrac{8}{s}$$

$$\mathcal{L}(y) = e^{-4s}\frac{2}{s[3s^2+s+1]} + e^{-10s}\frac{1}{s^2[3s^2+s+1]} + 8e^{-10s}\frac{1}{s[3s^2+s+1]}$$

$$y = 2\mathcal{L}^{-1}(e^{-4s}\tfrac{1}{s[3s^2+s+1]}) + \mathcal{L}^{-1}(e^{-10s}\tfrac{1}{s^2[3s^2+s+1]}) \\ + 8\mathcal{L}^{-1}(e^{-10s}\tfrac{1}{s[3s^2+s+1]})$$

$$y = u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$$

where $f(t) = \mathcal{L}^{-1}(\frac{1}{s[3s^2+s+1]})$ and $h(t) = \mathcal{L}^{-1}(\frac{1}{s^2[3s^2+s+1]})$

$$\frac{1}{s[3s^2+s+1]} = \frac{A}{s} + \frac{Bs+C}{3s^2+s+2}$$

$$1 = A(3s^2+s+1) + (Bs+C)s$$

$$0s^2 + 0s + 1 = (3A+B)s^2 + (A+C)s + A$$

$$0 = 3A + B,\ 0 = A + C,\ 1 = A$$

Hence $A = 1$, $B = -3A = -3$, $C = -A = -1$

$$f(t) = \mathcal{L}^{-1}(\tfrac{1}{s[3s^2+s+1]})$$
$$= \mathcal{L}^{-1}(\tfrac{1}{s} + \tfrac{-3s-1}{3s^2+s+1})$$
$$= \mathcal{L}^{-1}(\tfrac{1}{s} + \tfrac{-3s-1}{3s^2+s+1})$$
$$= 1 + \mathcal{L}^{-1}(\tfrac{-3s-1}{3[s^2+\frac{1}{3}s+\frac{1}{3}]}))$$
$$= 1 + \mathcal{L}^{-1}(\tfrac{-3s-1}{3[(s^2+\frac{1}{3}s+\underline{\qquad})-\underline{\qquad}+\frac{1}{3}]})$$
$$= 1 + \mathcal{L}^{-1}(\tfrac{-3s-1}{3[(s+\frac{1}{6})^2-\frac{1}{36}+\frac{1}{3}]})$$
$$= 1 + \mathcal{L}^{-1}(\tfrac{-3(s+\frac{1}{3})}{3[(s+\frac{1}{6})^2+\frac{11}{36}]})$$
$$= 1 + \mathcal{L}^{-1}(\tfrac{-(s+\frac{1}{6}-\frac{1}{6}+\frac{1}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]})$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6}+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{6}\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{\sqrt{11}}\frac{\sqrt{11}}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

Thm: $\mathcal{L}^{-1}(F(s-c)) = e^{ct}\mathcal{L}^{-1}(F(s))$

$$= 1 + e^{-\frac{1}{6}t}\mathcal{L}^{-1}\left(\frac{-s}{[s^2+\frac{11}{36}]}\right) - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}t}\mathcal{L}^{-1}\left(\frac{\frac{\sqrt{11}}{6}}{[s^2+\frac{11}{36}]}\right)$$

$$= 1 - e^{-\frac{1}{6}t}cos\frac{\sqrt{11}}{6}t - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}t}sin\frac{\sqrt{11}}{6}t$$

$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$

$\frac{1}{s^2[3s^2+s+1]} = \frac{As+D}{s^2} + \frac{Bs+C}{3s^2+s+2}$

$1 = (As+D)(3s^2+s+1) + (Bs+C)s^2$

$0s^3+0s^2+0s+1 = (3A+B)s^3 + (A+3D+C)s^2 + (A+D)s + D$

$0 = 3A+B$, $0 = A+3D+C$, $0 = A+D$, $1 = D$.

Hence $D = 1$, $A = -D = -1$, $C = -A-3D = 1-3 = -2$, $B = -3A = 3$.

$$\frac{1}{s^2[3s^2+s+1]} = \frac{-s+1}{s^2} + \frac{3s-2}{3s^2+s+1} = \frac{-s}{s^2} + \frac{1}{s^2} + \frac{3(s-\frac{2}{3})}{3[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{1}{6}-\frac{2}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} = \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{5}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$

$$= \mathcal{L}^{-1}\left(\frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= -1 + t + e^{-\frac{1}{6}t}cos\frac{\sqrt{11}}{6}t - \frac{5}{\sqrt{11}}e^{-\frac{1}{6}t}sin\frac{\sqrt{11}}{6}t$$

Hence the final answer is

$y = u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$

$$= u_4(t)[1-e^{-\frac{1}{6}(t-4)}cos\frac{\sqrt{11}}{6}(t-4) - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}(t-4)}sin\frac{\sqrt{11}}{6}(t-4)]+$$
$$u_{10}[-1+t-10+e^{-\frac{1}{6}(t-10)}cos\frac{\sqrt{11}}{6}(t-10) - \frac{5}{\sqrt{11}}e^{-\frac{1}{6}(t-10)}sin\frac{\sqrt{11}}{6}(t-10)]$$
$$+8u_{10}[1-e^{-\frac{1}{6}(t-10)}cos\frac{\sqrt{11}}{6}(t-10) - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}(t-10)}sin\frac{\sqrt{11}}{6}(t-10)]$$
