3.7 Mechanical Vibrations:
\[ mu''(t) + \gamma u'(t) + ku(t) = F_{\text{external}}, \quad m, \gamma, k \geq 0 \]
\[ mg - kL = 0, \quad F_{\text{damping}}(t) = -\gamma u'(t) \]
m = mass,
k = spring force proportionality constant,
\( \gamma = \) damping force proportionality constant
\( g = 9.8 \text{ m/sec}^2 \) or 32 ft/sec^2. Weight = \( mg \).

Electrical Vibrations:
\[ L \frac{dI(t)}{dt} + RI(t) + \frac{1}{C}Q(t) = E(t), \quad L, R, C \geq 0 \text{ and } I = \frac{dQ}{dt} \]
\[ LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t) \]

L = inductance (henrys),
R = resistance (ohms)
C = capacitance (farads)
\( Q(t) = \) charge at time t (coulombs)
\( I(t) = \) current at time t (amperes)
\( E(t) = \) impressed voltage (volts).

1 volt = 1 ohm \cdot 1 ampere = 1 coulomb / 1 farad = 1 henry \cdot 1 amperes / 1 second

Suppose a mass weighing 64 lbs stretches a spring 4 ft. If there is no damping and the spring is stretched an additional foot and set in motion with an upward velocity of \( \sqrt{8} \) ft/sec, find the equation of motion of the mass.

Weight = \( mg \): \( m = \frac{\text{weight}}{g} = \frac{64}{32} = 2 \)
\[ mg - kL = 0 \text{ implies } k = \frac{mg}{L} = \frac{64}{4} = 16 \]
\[ mu''(t) + \gamma u'(t) + ku(t) = F_{\text{external}} \]
\[ [\gamma^2 - 4km < 0: u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t) \]
Hence \( u(t) = Acos\mu t + Bsin\mu t \) since \( \gamma = 0 \).

\[ 2u''(t) + 16u(t) = 0 \]
\[ u''(t) + 8u(t) = 0, \quad u(0) = 1, \quad u'(0) = -\sqrt{8} \]
\[ r^2 + 8 = 0 \rightarrow r = \pm i\sqrt{8} = \pm i\sqrt{8} = 0 \pm i\sqrt{8} \]
\[ u(t) = c_1e^{it\sqrt{8}} + c_2e^{-it\sqrt{8}} \]
\[ u(t) = A\cos\sqrt{8}t + B\sin\sqrt{8}t \]
\[ u(0) = 1: 1 = A\cos(0) + B\sin(0) = A \]
\[ u'(t) = -\sqrt{8}Asin\sqrt{8}t + \sqrt{8}Bcos\sqrt{8}t \]
\[ u'(0) = -\sqrt{8}: -\sqrt{8} = -\sqrt{8}Asin(0) + \sqrt{8}Bcos(0) \]
\[ B = -1 \quad \text{Thus } u(t) = \cos\sqrt{8}t - \sin\sqrt{8}t \]
\( u(t) = \cos(t\sqrt{8}) - \sin(t\sqrt{8}) = R\cos(t\sqrt{8} - \delta) \)
\[ = \sqrt{2}\cos(t\sqrt{8} - \frac{7\pi}{4}) = \sqrt{2}\cos(t\sqrt{8} + \frac{\pi}{4}) \]

A = 1 = R\cos(\delta) and B = -1 = R\sin(\delta)

Thus \( R = \sqrt{A^2 + B^2} = \sqrt{(1)^2 + (-1)^2} = \sqrt{2} \)

\[ \tan(\delta) = \frac{R\sin(\delta)}{R\cos(\delta)} = \frac{B}{A} = \frac{-1}{1}. \text{ Thus } \delta = -\frac{\pi}{4} = \frac{7\pi}{4} \]

Trig background:

\( \cos(y \mp x) = \cos(x \mp y) = \cos(x)\cos(y) \pm \sin(x)\sin(y) \)

Let \( A = R\cos(\delta), B = R\sin(\delta) \) in

\[ A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t) = R\cos(\omega_0 t - \delta) \]

Amplitude = \( R \)

frequency = \( \omega_0 \) (measured in radians per unit time).

period = \( \frac{2\pi}{\omega_0} \) phase (displacement) = \( \delta \)

\( A = R\cos(\delta), B = R\sin(\delta) \) implies

\[ A^2 + B^2 = R^2\cos^2(\delta) + R^2\sin^2(\delta) = R^2(\cos^2(\delta) + \sin^2(\delta)) = R^2 \]

Homogeneous equation (no external force):

\[ mu''(t) + \gamma u'(t) + ku(t) = 0, \ m, \gamma, k \geq 0 \]

\( r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} \)

\( \gamma^2 - 4km > 0: u(t) = Ae^{r_1t} + Be^{r_2t} \)

\( \gamma^2 - 4km = 0: u(t) = (A + Bt)e^{r_1t} \)

\( \gamma^2 - 4km < 0: u(t) = e^{-\frac{\gamma t}{2m}}(A\cos \mu t + B\sin \mu t) = e^{-\frac{\gamma t}{2m}}R\cos(\mu t - \delta) \)

where \( A = R\cos(\delta), B = R\sin(\delta) \)

\( \mu \) = quasi frequency, \( \frac{2\pi}{\mu} \) = quasi period

Note if \( \gamma = 0 \), then

Critical damping: \( \gamma = 2\sqrt{km} \)

Overdamped: \( \gamma > 2\sqrt{km} \)