3.7 Mechanical Vibrations:

$$mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0$$

$$mg - kL = 0, \qquad F_{damping}(t) = -\gamma u'(t)$$

m= mass,

k = spring force proportionality constant,

γ = damping force proportionality constant

$g = 9.8$ m/sec^2 or 32 ft/sec^2. Weight = mg.

Electrical Vibrations:

$$L\frac{dI(t)}{dt} + RI(t) + \frac{1}{C}Q(t) = E(t), \quad L, R, C \geq 0 \text{ and } I = \frac{dQ}{dt}$$

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t)$$

L = inductance (henrys),

R = resistance (ohms)

C = capacitance (farads)

$Q(t)$ = charge at time t (coulombs)

$I(t)$ = current at time t (amperes)

$E(t)$ = impressed voltage (volts).

1 volt = 1 ohm · 1 ampere = 1 coulomb / 1 farad = 1 henry · 1 amperes/ 1 second

Suppose a mass weighing 64 lbs stretches a spring 4 ft. If there is no damping and the spring is stretched an additional foot and set in motion with an upward velocity of $\sqrt{8}$ ft/sec, find the equation of motion of the mass.

$Weight = mg$: $m = \frac{weight}{g} = \frac{64}{32} = 2$

$mg - kL = 0$ implies $k = \frac{mg}{L} = \frac{64}{4} = 16$

$mu''(t) + \gamma u'(t) + ku(t) = F_{external}$

[$\gamma^2 - 4km < 0$: $u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t)$ Hence $u(t) = Acos\mu t + Bsin\mu t$ since $\gamma = 0$].

$2u''(t) + 16u(t) = 0$

$u''(t) + 8u(t) = 0, \qquad u(0) = 1,\ u'(0) = -\sqrt{8}$

$r^2 + 8 = 0 \to r = \pm\sqrt{-8} = \pm i\sqrt{8} = 0 \pm i\sqrt{8}$

$u(t) = c_1 e^{it\sqrt{8}} + c_2 e^{-it\sqrt{8}}$

$u(t) = Acos\sqrt{8}t + Bsin\sqrt{8}t$

$u(0) = 1$: $1 = Acos(0) + Bsin(0) = A$

$u'(t) = -\sqrt{8}Asin\sqrt{8}t + \sqrt{8}Bcos\sqrt{8}t$

$u'(0) = -\sqrt{8} : -\sqrt{8} = -\sqrt{8}Asin(0) + \sqrt{8}Bcos(0)$

$B = -1$ Thus $u(t) = cos\sqrt{8}t - sin\sqrt{8}t$

$$u(t) = cos(t\sqrt{8}) - sin(t\sqrt{8}) = Rcos(t\sqrt{8} - \delta)$$
$$= \sqrt{2}cos(t\sqrt{8} - \tfrac{7\pi}{4}) = \sqrt{2}cos(t\sqrt{8} + \tfrac{\pi}{4})$$

$A = 1 = Rcos(\delta)$ and $B = -1 = Rsin(\delta)$

Thus $R = \sqrt{A^2 + B^2} = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$

$tan(\delta) = \frac{Rsin(\delta)}{Rcos(\delta)} = \frac{B}{A} = \frac{-1}{1}$. Thus $\delta = -\frac{\pi}{4} = \frac{7\pi}{4}$

Trig background:

$cos(y \mp x) = cos(x \mp y) = cos(x)cos(y) \pm sin(x)sin(y)$

Let $A = Rcos(\delta)$, $B = Rsin(\delta)$ in

$$Acos(\omega_0 t) + Bsin(\omega_0 t)$$
$$= Rcos(\delta)cos(\omega_0 t) + Rsin(\delta)sin(\omega_0 t)$$
$$= Rcos(\omega_0 t - \delta)$$

Amplitude $= R$

frequency $= \omega_0$ (measured in radians per unit time).

period $= \frac{2\pi}{\omega_0}$ phase (displacement) $= \delta$

$A = Rcos(\delta)$, $B = Rsin(\delta)$ implies

$$A^2 + B^2 = R^2cos^2(\delta) + R^2sin^2(\delta)$$
$$= R^2(cos^2(\delta) + sin^2(\delta)) = R^2$$

Homogeneous equation (no external force):

$$mu''(t) + \gamma u'(t) + ku(t) = 0, \quad m, \gamma, k \geq 0$$

$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$

$\gamma^2 - 4km > 0$: $u(t) = Ae^{r_1 t} + Be^{r_2 t}$

$\gamma^2 - 4km = 0$: $u(t) = (A + Bt)e^{r_1 t}$

$\gamma^2 - 4km < 0$: $u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t)$
$$= e^{-\frac{\gamma t}{2m}} Rcos(\mu t - \delta)$$
where $A = Rcos(\delta)$, $B = Rsin(\delta)$

$\mu =$ quasi frequency, $\frac{2\pi}{\mu} =$ quasi period

Note if $\gamma = 0$, then

Critical damping: $\gamma = 2\sqrt{km}$

Overdamped: $\gamma > 2\sqrt{km}$