

3.7 Mechanical Vibrations:

$$mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0$$

$$mg - kL = 0, \quad F_{damping}(t) = -\gamma u'(t)$$

m = mass,

k = spring force proportionality constant,

γ = damping force proportionality constant

$g = 9.8 \text{ m/sec}^2$ or 32 ft/sec^2 . Weight = mg .

Electrical Vibrations:

$$L \frac{dI(t)}{dt} + RI(t) + \frac{1}{C}Q(t) = E(t), \quad L, R, C \geq 0 \text{ and } I = \frac{dQ}{dt}$$

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t)$$

L = inductance (henrys),

R = resistance (ohms)

C = capacitance (farads)

$Q(t)$ = charge at time t (coulombs)

$I(t)$ = current at time t (amperes)

$E(t)$ = impressed voltage (volts).

1 volt = 1 ohm · 1 ampere = 1 coulomb / 1 farad =
1 henry · 1 amperes / 1 second

Suppose a mass weighing 64 lbs stretches a spring 4 ft. If there is no damping and the spring is stretched an additional foot and set in motion with an upward velocity of $\sqrt{8}$ ft/sec, find the equation of motion of the mass.

$$\text{Weight} = mg: m = \frac{\text{weight}}{g} = \frac{64}{32} = 2$$

$$mg - kL = 0 \text{ implies } k = \frac{mg}{L} = \frac{64}{4} = 16$$

$$mu''(t) + \gamma u'(t) + ku(t) = F_{\text{external}}$$

$$[\gamma^2 - 4km < 0: u(t) = e^{-\frac{\gamma t}{2m}} (A \cos \mu t + B \sin \mu t)$$

Hence $u(t) = A \cos \mu t + B \sin \mu t$ since $\gamma = 0$].

$$2u''(t) + 16u(t) = 0$$

$$u''(t) + 8u(t) = 0, \quad u(0) = 1, u'(0) = -\sqrt{8}$$

$$r^2 + 8 = 0 \rightarrow r = \pm\sqrt{-8} = \pm i\sqrt{8} = 0 \pm i\sqrt{8}$$

$$u(t) = c_1 e^{it\sqrt{8}} + c_2 e^{-it\sqrt{8}}$$

$$u(t) = A \cos \sqrt{8}t + B \sin \sqrt{8}t$$

$$u(0) = 1: 1 = A \cos(0) + B \sin(0) = A$$

$$u'(t) = -\sqrt{8}A \sin \sqrt{8}t + \sqrt{8}B \cos \sqrt{8}t$$

$$u'(0) = -\sqrt{8}: -\sqrt{8} = -\sqrt{8}A \sin(0) + \sqrt{8}B \cos(0)$$

$$B = -1 \quad \text{Thus } u(t) = \cos \sqrt{8}t - \sin \sqrt{8}t$$

$$\begin{aligned}
 u(t) &= \cos(t\sqrt{8}) - \sin(t\sqrt{8}) = R\cos(t\sqrt{8} - \delta) \\
 &= \sqrt{2}\cos\left(t\sqrt{8} - \frac{7\pi}{4}\right) = \sqrt{2}\cos\left(t\sqrt{8} + \frac{\pi}{4}\right)
 \end{aligned}$$

$$A = 1 = R\cos(\delta) \text{ and } B = -1 = R\sin(\delta)$$

$$\text{Thus } R = \sqrt{A^2 + B^2} = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$$

$$\tan(\delta) = \frac{R\sin(\delta)}{R\cos(\delta)} = \frac{B}{A} = \frac{-1}{1}. \text{ Thus } \delta = -\frac{\pi}{4} = \frac{7\pi}{4}$$

Trig background:

$$\cos(y \mp x) = \cos(x \mp y) = \cos(x)\cos(y) \pm \sin(x)\sin(y)$$

Let $A = R\cos(\delta)$, $B = R\sin(\delta)$ in

$$\begin{aligned}
 &A\cos(\omega_0 t) + B\sin(\omega_0 t) \\
 &= R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t) \\
 &= R\cos(\omega_0 t - \delta)
 \end{aligned}$$

Amplitude = R

frequency = ω_0 (measured in radians per unit time).

period = $\frac{2\pi}{\omega_0}$ phase (displacement) = δ

$A = R\cos(\delta)$, $B = R\sin(\delta)$ implies

$$\begin{aligned}
 A^2 + B^2 &= R^2\cos^2(\delta) + R^2\sin^2(\delta) \\
 &= R^2(\cos^2(\delta) + \sin^2(\delta)) = R^2
 \end{aligned}$$

Homogeneous equation (no external force):

$$mu''(t) + \gamma u'(t) + ku(t) = 0, \quad m, \gamma, k \geq 0$$

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

$$\gamma^2 - 4km > 0: u(t) = Ae^{r_1 t} + Be^{r_2 t}$$

$$\gamma^2 - 4km = 0: u(t) = (A + Bt)e^{r_1 t}$$

$$\begin{aligned} \gamma^2 - 4km < 0: u(t) &= e^{-\frac{\gamma t}{2m}} (A \cos \mu t + B \sin \mu t) \\ &= e^{-\frac{\gamma t}{2m}} R \cos(\mu t - \delta) \\ &\text{where } A = R \cos(\delta), B = R \sin(\delta) \end{aligned}$$

$\mu =$ quasi frequency, $\frac{2\pi}{\mu} =$ quasi period

Note if $\gamma = 0$, then

Critical damping: $\gamma = 2\sqrt{km}$

Overdamped: $\gamma > 2\sqrt{km}$