3.8: Mechanical and Electrical Vibrations

Trig background:

\[ \cos(y \mp x) = \cos(x \mp y) = \cos(x)\cos(y) \pm \sin(x)\sin(y) \]

Let \( A = R\cos(\delta) \), \( B = R\sin(\delta) \) in

\[ A\cos(\omega_0 t) + B\sin(\omega_0 t) \]
\[ = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t) \]
\[ = R\cos(\omega_0 t - \delta) \]

Amplitude = \( R \)
frequency = \( \omega_0 \) (measured in radians per unit time).
period = \( \frac{2\pi}{\omega_0} \)
phase (displacement) = \( \delta \)

Mechanical Vibrations:

\[ mu''(t) + \gamma u'(t) + ku(t) = F_{\text{external}}, \quad m, \gamma, k \geq 0 \]
\[ mg - kL = 0, \quad F_{\text{viscous}}(t) = \gamma u'(t) \]

\( m \) = mass,
\( k \) = spring force proportionality constant,
\( \gamma \) = damping force proportionality constant
g = 9.8 m/sec

Electrical Vibrations:

\[ L\frac{dI(t)}{dt} + RI(t) + \frac{1}{C}Q(t) = E(t), \quad L, R, C \geq 0 \text{ and } I = \frac{dQ}{dt} \]

\( L \) = inductance (henrys),
\( R \) = resistance (ohms)
\( C \) = capacitance (farads)
\( Q(t) \) = charge at time \( t \) (coulombs)
\( I(t) \) = current at time \( t \) (amperes)
\( E(t) \) = impressed voltage (volts).

1 volt = 1 ohm \cdot 1 ampere = 1 coulomb / 1 farad = 1 henry \cdot 1 amperes/ 1 second
\[ mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0 \]

\[ r_1, r_2 = -\dfrac{\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} \]

\[ \gamma^2 - 4km > 0: \quad u(t) = Ae^{r_1 t} + Be^{r_2 t} + \psi(t) \]

\[ \gamma^2 - 4km = 0: \quad u(t) = (A + Bt)e^{r_1 t} + \psi(t) \]

\[ \gamma^2 - 4km < 0: \quad u(t) = e^{-\dfrac{\gamma t}{2m}}(A\cos\mu t + B\sin\mu t) + \psi(t) \]

\[ = e^{-\dfrac{\gamma t}{2m}}R\cos(\mu t - \delta) + \psi(t) \]

where \( A = R\cos(\delta), B = R\sin(\delta) \)

\[ \mu = \text{quasi frequency}, \quad \frac{2\pi}{\mu} = \text{quasi period} \]

Note if \( \gamma = 0, \) then

Critical damping: \( \gamma = 2\sqrt{km} \)

Overdamped: \( \gamma > 2\sqrt{km} \)

Suppose a mass weighs 64 lbs stretches a spring 4 ft. If there is no damping and the spring is stretched an additional foot and set in motion with an upward velocity of \( \sqrt{8} \) ft/sec, find the equation of motion of the mass.

\[ Weight = mg: \quad m = \frac{\text{weight}}{g} = \frac{64}{32} = 2 \]

\[ mg - kL = 0 \implies k = \frac{mg}{L} = \frac{64}{4} = 16 \]

\[ mu''(t) + \gamma u'(t) + ku(t) = F_{external} \]

\[ [\gamma^2 - 4km < 0: \quad u(t) = e^{-\dfrac{\gamma t}{2m}}(A\cos\mu t + B\sin\mu t) \]

Hence \( u(t) = A\cos\mu t + B\sin\mu t \) since \( \gamma = 0 \].

\[ 2u''(t) + 16u(t) = 0 \]

\[ u''(t) + 8u(t) = 0 \]

\[ u(0) = 1, \quad u'(0) = -\sqrt{8} \]

\[ r^2 + 8 = 0 \implies r^2 = -8 \implies r = \sqrt{-8} = i\sqrt{8} = 0 + i\sqrt{8} \]