3.8: Mechanical and Electrical Vibrations

Trig background:

\[ \cos(y \mp x) = \cos(x \mp y) = \cos(x)\cos(y) \pm \sin(x)\sin(y) \]

Let \( A = R\cos(\delta), \) \( B = R\sin(\delta) \) in

\[ A \cos(\omega_0 t) + B \sin(\omega_0 t) \]
\[ = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t) \]
\[ = R\cos(\omega_0 t - \delta) \]

Amplitude = \( R \)

frequency = \( \omega_0 \) (measured in radians per unit time).

period = \( \frac{2\pi}{\omega_0} \)

phase (displacement) = \( \delta \)
Mechanical Vibrations:

\[ mu''(t) + \gamma u'(t) + ku(t) = F_{\text{external}}, \quad m, \gamma, k \geq 0 \]

\[ mg - kL = 0, \quad F_{\text{viscous}}(t) = \gamma u'(t) \]

\( m \) = mass,
\( k \) = spring force proportionality constant,
\( \gamma \) = damping force proportionality constant
\( g = 9.8 \text{ m/sec} \)

Electrical Vibrations:

\[ L \frac{dI(t)}{dt} + RI(t) + \frac{1}{C}Q(t) = E(t), \quad L, R, C \geq 0 \text{ and } I = \frac{dQ}{dt} \]

\( L \) = inductance (henrys),
\( R \) = resistance (ohms)
\( C \) = capacitance (farads)
\( Q(t) \) = charge at time \( t \) (coulombs)
\( I(t) \) = current at time \( t \) (amperes)
\( E(t) \) = impressed voltage (volts).

1 volt = 1 ohm \cdot 1 ampere = 1 coulomb / 1 farad = 1 henry \cdot 1 amperes/ 1 second
\[ mu''(t) + \gamma u'(t) + ku(t) = F_{\text{external}}, \quad m, \gamma, k \geq 0 \]

\[
r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}
\]

\[
\gamma^2 - 4km > 0: \quad u(t) = Ae^{r_1t} + Be^{r_2t} + \psi(t)
\]

\[
\gamma^2 - 4km = 0: \quad u(t) = (A + Bt)e^{r_1t} + \psi(t)
\]

\[
\gamma^2 - 4km < 0: \quad u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t) + \psi(t)
\]

\[
= e^{-\frac{\gamma t}{2m}}Rcos(\mu t - \delta) + \psi(t)
\]

where \( A = Rcos(\delta), \quad B = Rsin(\delta) \)

\[ \mu = \text{quasi frequency}, \quad \frac{2\pi}{\mu} = \text{quasi period} \]

Note if \( \gamma = 0 \), then

Critical damping: \( \gamma = 2\sqrt{km} \)

Overdamped: \( \gamma > 2\sqrt{km} \)
Suppose a mass weighs 64 lbs stretches a spring 4 ft. If there is no damping and the spring is stretched an additional foot and set in motion with an upward velocity of $\sqrt{8}$ ft/sec, find the equation of motion of the mass.

Weight = $mg$: $m = \frac{\text{weight}}{g} = \frac{64}{32} = 2$

$mg - kL = 0$ implies $k = \frac{mg}{L} = \frac{64}{4} = 16$

$mu''(t) + \gamma u'(t) + ku(t) = F_{\text{external}}$

$[\gamma^2 - 4km < 0$: $u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t)$

Hence $u(t) = Acos\mu t + Bsin\mu t$ since $\gamma = 0$.

$2u''(t) + 16u(t) = 0$

$u''(t) + 8u(t) = 0$

$u(0) = 1$, $u'(0) = -\sqrt{8}$

$r^2 + 8 = 0 \rightarrow r^2 = -8 \rightarrow r = \sqrt{-8} = i\sqrt{8} = 0 + i\sqrt{8}$