

Thm: Suppose $c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$

If ψ is a solution to

$$ay'' + by' + cy = g(t) \text{ [*]},$$

Then $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to [*].

Moreover if γ is also a solution to [*], then there exist constants c_1, c_2 such that

$$\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$$

Or in other words, $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to [*].

Proof: Let $h = c_1\phi_1(t) + c_2\phi_2(t)$. Since h is a solution to the differential equation, $ay'' + by' + cy = 0$,

Since ψ is a solution to $ay'' + by' + cy = g(t)$,

We will now show that $\psi + c_1\phi_1(t) + c_2\phi_2(t) = \psi + h$ is also a solution to [*].

Since γ a solution to $ay'' + by' + cy = g(t)$,

We will first show that $\gamma - \psi$ is a solution to the differential equation $ay'' + by' + cy = 0$.

Since $\gamma - \psi$ is a solution to $ay'' + by' + cy = 0$ and

$c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$

there exist constants c_1, c_2 such that

$$\gamma - \psi = \underline{\hspace{15em}}$$

Thus $\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$.

Thm: Suppose that f_1 is a a solution to $ay'' + by' + cy = g_1(t)$ and f_2 is a a solution to $ay'' + by' + cy = g_2(t)$, then $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$

Proof:

Since f_1 is a solution to $ay'' + by' + cy = g_1(t)$,

Since f_2 is a solution to $ay'' + by' + cy = g_2(t)$,

We will now show that $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$.

Sidenote: The proofs above work even if a, b, c are functions of t instead of constants.

Examples:

Find a suitable form for ψ for the following differential equations:

1.) $y'' - 4y' - 5y = 4e^{2t}$

2.) $y'' - 4y' - 5y = 4\sin(3t)$

3.) $y'' - 4y' - 5y = t^2 - 2t + 1$

4.) $y'' - 5y = 4\sin(3t)$

5.) $y'' - 4y' = t^2 - 2t + 1$

6.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)e^{2t}$

$$7.) y'' - 4y' - 5y = 4\sin(3t)e^{2t}$$

$$8.) y'' - 4y' - 5y = 4(t^2 - 2t - 1)\sin(3t)e^{2t}$$

$$9.) y'' - 4y' - 5y = 4\sin(3t) + 4\sin(3t)e^{2t}$$

$$10.) y'' - 4y' - 5y \\ = 4\sin(3t)e^{2t} + 4(t^2 - 2t - 1)e^{2t} + t^2 - 2t - 1$$

$$11.) y'' - 4y' - 5y = 4\sin(3t) + 5\cos(3t)$$

$$12.) y'' - 4y' - 5y = 4e^{-t}$$

To solve $ay'' + by' + cy = g_1(t) + g_2(t) + \dots g_n(t)$ [**]

1.) Find the general solution to $ay'' + by' + cy = 0$:

$$c_1\phi_1 + c_2\phi_2$$

2.) For each g_i , find a solution to $ay'' + by' + cy = g_i$:

$$\psi_i$$

The general solution to [**] is

$$c_1\phi_1 + c_2\phi_2 + \psi_1 + \psi_2 + \dots\psi_n$$