Thm: Suppose \( c_1 \phi_1(t) + c_2 \phi_2(t) \) is a general solution to
\[
ay'' + by' + cy = 0,
\]

If \( \psi \) is a solution to
\[
ay'' + by' + cy = g(t) \quad [*],
\]

Then \( \psi + c_1 \phi_1(t) + c_2 \phi_2(t) \) is also a solution to \([*]\).

Moreover if \( \gamma \) is also a solution to \([*]\), then there exist constants \( c_1, c_2 \) such that
\[
\gamma = \psi + c_1 \phi_1(t) + c_2 \phi_2(t)
\]

Or in other words, \( \psi + c_1 \phi_1(t) + c_2 \phi_2(t) \) is a general solution to \([*]\).

Proof: Let \( h = c_1 \phi_1(t) + c_2 \phi_2(t) \). Since \( h \) is a solution to the differential equation, \( ay'' + by' + cy = 0 \),

Since \( \psi \) is a solution to \( ay'' + by' + cy = g(t) \),
We will now show that $\psi + c_1 \phi_1(t) + c_2 \phi_2(t) = \psi + h$
is also a solution to [*].
Since $\gamma$ a solution to $ay'' + by' + cy = g(t)$,

We will first show that $\gamma - \psi$ is a solution to the differential equation $ay'' + by' + cy = 0$.

Since $\gamma - \psi$ is a solution to $ay'' + by' + cy = 0$ and $c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to $ay'' + by' + cy = 0$,

there exist constants $c_1, c_2$ such that

$$\gamma - \psi = c_1\phi_1(t) + c_2\phi_2(t).$$

Thus $\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$. 
**Thm:** Suppose that $f_1$ is a solution to $ay'' + by' + cy = g_1(t)$ and $f_2$ is a solution to $ay'' + by' + cy = g_2(t)$, then $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$.

**Proof:**

Since $f_1$ is a solution to $ay'' + by' + cy = g_1(t)$,

Since $f_2$ is a solution to $ay'' + by' + cy = g_2(t)$,

We will now show that $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$.

**Sidenote:** The proofs above work even if $a, b, c$ are functions of $t$ instead of constants.
Examples:

Find a suitable form for ψ for the following differential equations:

1.) \( y'' - 4y' - 5y = 4e^{2t} \)

2.) \( y'' - 4y' - 5y = 4\sin(3t) \)

3.) \( y'' - 4y' - 5y = t^2 - 2t + 1 \)

4.) \( y'' - 5y = 4\sin(3t) \)

5.) \( y'' - 4y' = t^2 - 2t + 1 \)

6.) \( y'' - 4y' - 5y = 4(t^2 - 2t - 1)e^{2t} \)
7.) \( y'' - 4y' - 5y = 4\sin(3t)e^{2t} \)

8.) \( y'' - 4y' - 5y = 4(t^2 - 2t - 1)\sin(3t)e^{2t} \)

9.) \( y'' - 4y' - 5y = 4\sin(3t) + 4\sin(3t)e^{2t} \)

10.) \[
    y'' - 4y' - 5y \\
    = 4\sin(3t)e^{2t} + 4(t^2 - 2t - 1)e^{2t} + t^2 - 2t - 1
    
11.) \( y'' - 4y' - 5y = 4\sin(3t) + 5\cos(3t) \)

12.) \( y'' - 4y' - 5y = 4e^{-t} \)
To solve $ay'' + by' + cy = g_1(t) + g_2(t) + ...g_n(t)$ [**] 

1.) Find the general solution to $ay'' + by' + cy = 0$: 
$$c_1\phi_1 + c_2\phi_2$$

2.) For each $g_i$, find a solution to $ay'' + by' + cy = g_i$: 
$$\psi_i$$

This includes plugging guessed solution into $ay'' + by' + cy = g_i$ to find constant(s).

The general solution to [**] is 
$$c_1\phi_1 + c_2\phi_2 + \psi_1 + \psi_2 + ...\psi_n$$

3.) If initial value problem:

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find $c_1, c_2$).