Thursday, March 24, 2010

Alexander Polynomial

The Alexander polynomial $\Delta(t)$ for a knot satisfies the skein relation

1. $\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\Delta L_0$

   $\begin{array}{c}
   \text{positive crossing} \\
   \text{negative crossing} \\
   \text{smoothed crossing}
   \end{array}$

2. $\Delta(O_1) = 1$

Conway Polynomial

Similar to Alexander polynomial, satisfying:

1. $\nabla(L_+) - \nabla(L_-) = z \nabla(L_0)$

2. $\nabla(O_1) = 1$

Note: One may get from the Conway polynomial to the Alexander polynomial by:

$z \rightarrow t^{1/2} - t^{-1/2}$

Computing Alexander/Conway polynomial for Split Link

$\nabla \left( \begin{array}{c}
   \text{positive crossing} \\
   \text{negative crossing}
   \end{array} \right) - \nabla \left( \begin{array}{c}
   \text{negative crossing} \\
   \text{smoothed crossing}
   \end{array} \right) = z \nabla \left( \begin{array}{c}
   \text{positive crossing} \\
   \text{negative crossing}
   \end{array} \right)$

$\therefore \nabla(\text{split link}) = 0$

$\Delta(\text{split link}) = 0$

Resolving Tree

By (1) of Conway polynomial, we have

$\nabla(L_+) = \nabla(L_-) + z \nabla(L_0)$

$\nabla(L_-) = \nabla(L_+) - z \nabla(L_0)$
Ex: Figure Eight Knot

\[ \Delta(4,1) = 1 - (t^{1/2} - t^{-1/2})^2 = 1 - t + \chi - t^{-1} = -t + 3 - t^{-1} \]

\[ \nabla(4,1) = 1 + Z(1 + 0 + (-Z)(1)) = 1 - Z^2 \]

Ex:

Note: We change the # of components each time we smooth a crossing.
Observe: \[ \Delta_k \in \begin{cases} \Delta(t, t^{-1}) & \text{if } k \text{ is a knot} \\ \Delta(t^m, t^{-m}) & \text{if } k \text{ is a link} \end{cases} \]

Recall: One may always change crossings of knot diagrams to obtain the unknot by making the diagram descending/ascending.

\[ \Rightarrow \text{All } \exists \text{ resolving tree} \]

\[ \text{lose crossing via smoothing} \]

\[ \text{get closer to descending} \]

\[ \text{Skein Relations & Tangles} \]

Suppose we have a tangle of the form \[ \text{i.e. assuming oriented parity} \]

Ex: \[ \text{positive } \pi \text{ since the crossing is positive} \]

In general: \[ A = a_0(\pi) + a_0(\pi) \]

Similarly, \[ B = b_0(\pi) + b_0(\pi) \]
Let's take a look at composite knots:

\[ D(A) \# D(B) \]

\[ = a_0(z) \Delta (K_1 \# K_2) + b_0(z) \nabla (K_1) \]

In general:

\[ \nabla (K_1 \# K_2) = \nabla (K_1) \cdot \nabla (K_2) \]

\[ \Delta (K_1 \# K_2) = \Delta (K_1) \cdot \Delta (K_2) \]

Note: This result only used existence of skein relation which resolves oriented 2-string tangles of type \( \text{unknot} \).

Mutations:

back of face...
hence the hair
Note: The $0, \pm 1$, and $\infty$-tangles are invariant under mutations.

Ex: $N(A+B)$

\[ N(A+B) = a_\infty b_0 + a_0 b_\infty \]

Similarly:

\[ N(A+B) = a_0 b_0 + a_\infty b_\infty = \nabla \]

(Similar results hold for other orientations/parities of $B$)

Result: Conway, Alexander, and other similarly defined polynomial invariants defined using skein relations do not detect mutations.

Homflypt Polynomial

\[ a^1 P(L_+) - a P(L_-) = \mathcal{Z} P(L_0) \]

$P(O_1) = 1$

$P \rightarrow \Delta$ by

\[
\begin{align*}
\{ a & \rightarrow 1 \\
\{ z & \rightarrow t^\frac{1}{\nu} - t^{-\frac{1}{\nu}} \}
\end{align*}
\]

$P \rightarrow$ Jones Polynomial by

\[
\begin{align*}
\{ a & \rightarrow t \\
\{ z & \rightarrow t^\frac{1}{\nu} - t^{-\frac{1}{\nu}} \}
\end{align*}
\]
Facts: ① ̂_\text{P}_k(\text{L} \# \text{L}_2) = \text{P}_k(\text{L}_1) \cdot \text{P}_k(\text{L}_2)
② Cannot distinguish mutants

Example: Let A = \text{IL} \Rightarrow A^{-1} = A^{-1} = -i l^{-1} - i l^{-1} p(l_+) - i l^{-1} p(l_-) = z p(l_0)
\quad l^{-1} p(l_+) + l p(l_) = i z p(l_0) = m p(l_0)

Note: Alexander polynomial does not detect chirality.
Homflypt polynomial often detects chirality:
\quad P_{k^*}(l, m) = P_k(l^{-1}, m)

Kauffman Bracket Polynomial (for unoriented links)
\quad \langle \begin{array}{c} A \\ \text{B} \end{array} \rangle = A \langle () \rangle + B \langle \equiv \rangle

Check R2, let B = A^{-1}
\quad \langle 0 \text{K} \rangle = d \langle \text{K} \rangle \quad \text{where} \quad d = -A^2 - A^{-2}

↑
Unkotted components

Note: Invariant under R2 & R3, not R1
"Invariant of framed links"