**Combinatorial Knot Floer Homology**

Note these notes are based on notes by Candice Price (primary source), Peter Osvzath, and Rob Ghrist

# 1 Homology

http://www.ima.umn.edu/2008-2009/ND6.15-26.09/activities/Ghrist-Robert/lecture4.pdf

To calculate homology, we need something like

1.) Chains: These are often formal sums of generators.

If $S$ = set of generators, let $\mathbb{C}_n = \{\Sigma n_i g_i \mid n_i \in \mathbb{Z}, g_i \in S\}$

2.) Grading: A function f: Chains $\to \mathbb{Z}$

Let $\mathbb{C}_n = \{C \in \mathbb{C} \mid f(C) = n\}$.

3.) boundary maps: $\partial_n : \mathbb{C}_n \to \mathbb{C}_{n-1}$ such that $\partial^2 = 0$. Thus $im\partial_{n+1} \subset ker\partial_n$

Defn: $H_n = ker\partial_n / im\partial_{n+1}$.

Lemma: If $(C_*, \partial)$ is a finite-dimensional chain complex and $H_*$ its homology, then

$\chi(C_*) = \Sigma_{k=0}^{\infty} dim C_k = \Sigma_{k=0}^{\infty} dim H_k = \chi(H_*)$

# 2 Combinatorial Description

**Definition 1** (Planar grid diagram)**.** *A* **planar grid diagram** *$G$ lies in a square grid on the plane with $n \times n$ squares, where $n$ =* **grid number***. Each square is decorated with an $X$, an $O$ or nothing, such that*

- *every row contains exactly one $X$ and one $O$.*
- *every column contains exactly one $X$ and one $O$.*

We can label the $O$s and $X$s from 1 to $n$ and denote the collections $\mathbb{O} = \{O_i\}_{i=1}^n$ and $\mathbb{X} = \{X_i\}_{i=1}^n$.

A projection of an <u>oriented</u> link can be constructed in a grid diagram by

- draw horizontal segments from an element in $\mathbb{O}$ to an element in $\mathbb{X}$ in each row.
- draw vertical segments from an element in $\mathbb{X}$ to an element in $\mathbb{O}$ in each column.

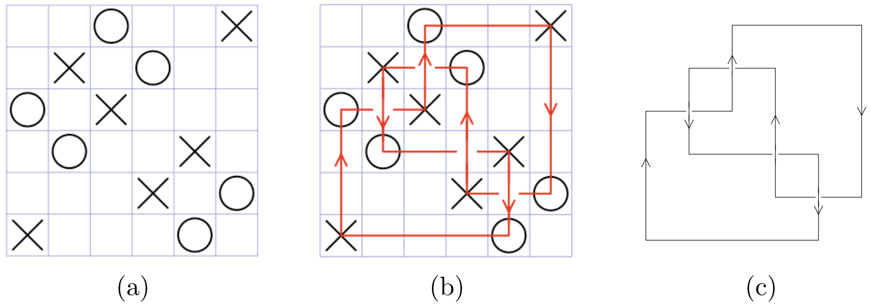


Figure 1: Projection of the $4_1$ knot drawn on a $6 \times 6$ grid diagram.

- At each intersection point we let the horizontal segment be the underpass and the vertical segment be the overpass.

We then take this diagram and turn it into a torus in the usual way by associating the top and bottom lines of the square and associating the left and right lines of the square with one another creating a new diagram known as the *torodial grid diagram.*

Grid diagrams were first studied by Brunn in 1898. They have resurfaced many times since, including in work of Birman and Menasco, and also in the guise of Legendrian knot projections.

Theorem. (Cromwell) An invariant associated to grid diagrams is a knot invariant precisely if it is invariant under the following three types of moves:

- Cyclic permutations.
- Commutation moves.
- Stabilizations.

See also Dynnikov arXiv:math/0208153

To this torodial grid diagram, we associate the chain complex $C(G)$.

**Chains:**

The set of generators for $C(G)$, denoted $\mathbf{S}$, consists of 1-1 correspondences between the horizontal and vertical circles created in the torodial grid diagram.

Thus for an $n \times n$ grid, $\mathbf{S}$ = the set of permutations of $\{1, ..., n\}$.

There are several different versions of knot Floer homology:

The simplest version of the chain complex associated to this diagram is a chain complex over field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ with generators in $\mathbf{S}$.

$$\widetilde{CF}(K) = \bigoplus_{\mathbf{p} \in \mathbf{S}} \mathbb{Z}/2\mathbb{Z} \cdot \mathbf{p}.$$

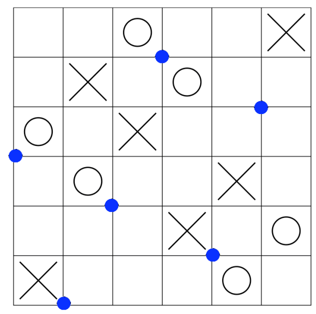


Figure 2: An example of the generator (4 1 3 6 2 5) [or (3 0 2 5 1 4) depending on your convention] for the $6 \times 6$ torodial grid diagram. Recall that the leftmost and rightmost lines, respectively top and bottom lines, are identified. Geometrically, we can think of these generators as $n$-tuples of intersection points between horizontal and vertical circles, with the property that no intersection point appears on more than one horizontal (or vertical) circle.

We will focus on this chain complex first, but its homology will need a "correction factor" to turn it into a knot invariant

**Grading**

Given two collections $A$ and $B$ of finitely many points in the plane,

Define $\mathcal{I}(A, B)$ as the number of pairs $(a_1, a_1) \in A$ and $(b_1, b_2) \in B$ with $a_1 < b_1$ and $a_2 < b_2$.

Define $\mathcal{J}(A, B) = \frac{\mathcal{I}(A,B)+\mathcal{I}(B,A)}{2}$.

We can extend this function bilinearly over formal sums of subsets.

Taking a fundamental domain for the torus, cut along a horizontal and vertical circle with the left and bottom edges included. Thus, given a generator $\mathbf{p}$, which is viewed as a collection of points with integer coordinates, and $\mathbb{O} = \{O_i\}_{i=1}^n$ as a collection of points in the plane with half-integer coordinates, the function $M : \mathbf{S} \to \mathbb{Z}$, known as the *Maslov grading*, is defined as follows:

$$M_{\mathbb{O}}(\mathbf{p}) = \mathcal{I}(\mathbf{p}, \mathbf{p}) - \mathcal{I}(\mathbf{p}, \mathbb{O}) - \mathcal{I}(\mathbb{O}, \mathbf{p}) + \mathcal{I}(\mathbb{O}, \mathbb{O}) + 1.$$

We also define the *Alexander filtration* for a knot $A : \mathbf{S} \to \mathbb{Z}$ where

$$A(\mathbf{p}) = \frac{M_{\mathbb{X}}(\mathbf{p}) - M_{\mathbb{O}}(\mathbf{p})}{2} - \frac{n-1}{2} \qquad \text{where } M_{\mathbb{X}}(\mathbf{p}) = \mathcal{I}(\mathbf{p}, \mathbf{p}) - \mathcal{I}(\mathbf{p}, \mathbb{X}) - \mathcal{I}(\mathbb{X}, \mathbf{p}) + \mathcal{I}(\mathbb{X}, \mathbb{X}) + 1$$

**Boundary maps**

To define the differential associated to this chain complex, the following definition is needed: Let $\mathbf{p}$ and $\mathbf{q} \in \mathbf{S}$. Then $\mathbf{p}$ and $\mathbf{q}$ can be *connected* by a rectangle if all but two coordinates of $\mathbf{p}$ are equal to two coordinates of $\mathbf{q}$. The rectangle $r$ *connects* $\mathbf{p}$ to $\mathbf{q}$ if:

1. all four corners of $r$ are intersection points in $\mathbf{p} \cup \mathbf{q}$, and

2. the bottom left corner of $r$ is a point from $\mathbf{p}$.

We denote this set of rectangles $Rect(\mathbf{p},\mathbf{q})$. Notice that if $\mathbf{q}$ and $\mathbf{p}$ can be connected by one rectangle then $|Rect(\mathbf{p},\mathbf{q})| = |Rect(\mathbf{q},\mathbf{p})| = 2$.

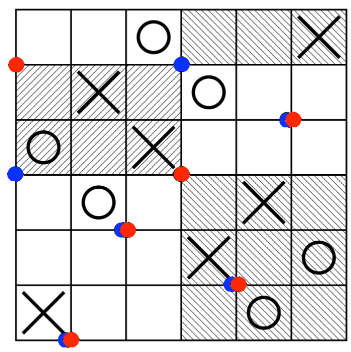


Figure 3: There are two shaded rectangles connecting $\mathbf{p}$ to $\mathbf{q}$.

The simplest version of the chain complex associated to this diagram is a chain complex over field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ with generators in $\mathbf{S}$.

$$\widetilde{CFK}(K) = \bigoplus_{\mathbf{p}\in\mathbf{S}} \mathbb{Z}/2\mathbb{Z}\cdot\mathbf{p}.$$

The associated differential is defined as:

$$\widetilde{\partial}(\mathbf{p}) = \sum_{\mathbf{q}\in\mathbf{S}} \#\left\{ r\in Rect(\mathbf{p},\mathbf{q}) \middle| \mathbb{X}\cap r=\emptyset, \mathbb{O}\cap r=\emptyset, \mathbf{p}\cap int\ r=\emptyset \right\}\cdot\mathbf{q}$$

keeps a count of the *empty* rectangles connecting generators. This differential has the following properties:

**Theorem 1.** *Let $\partial^-$ be defined as above. Then,*

- $\partial^-\circ\partial^- = 0$
- *$\partial^-$ drops the Maslov grading by 1.*
- *$\partial^-$ preserves the Alexander grading.*

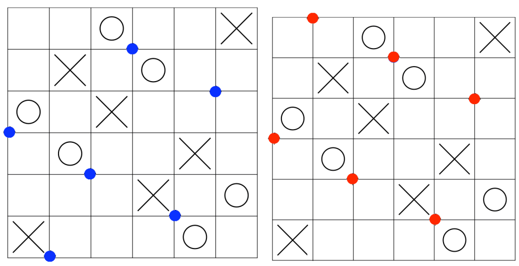
## 2.1 Homology

$\widetilde{HFK}(K) \cong \widehat{HFK}(K)\otimes H_*(T^{n-1})$

# 3 Invariance

**Lemma 3.1.** *The function $M$ is invariant under cyclic permutations.*

*Proof.* Fix $\mathbf{p} \in \mathbf{S}$, so that there is one component $a$ with coordinates $(m, 0)$. Let $\mathbf{p}'$ denote the same generator in the fundamental domain with the top and left edges included, so there is now a component $b$ with coordinates $(m, n)$. For each $i$ with $0 \le i < n, i \ne m$, there is one component $c_i$ in $\mathbf{p}$ and $\mathbf{p}'$ with first coordinate $i$.



(a) Generator $p$ (b) Generator $p'$

For each $i$, such that $m < i < n$, the pair $(a, c_i)$ contributes 1 to the count of $\mathcal{I}(\mathbf{p}, \mathbf{p})$. The corresponding pair $(c_i, b)$ does not contribute to $\mathcal{I}(\mathbf{p}', \mathbf{p}')$. Symmetrically, for each $i$ with $0 \le i < m$ the pair $(c_i, a)$ does not contribute to $\mathcal{I}(\mathbf{p}, \mathbf{p})$ but $(c_i, b)$ contributes to $\mathcal{I}(\mathbf{p}', \mathbf{p}')$. Thus,

$$\mathcal{I}(\mathbf{p}, \mathbf{p}) + \overbrace{m}^{\substack{\#\text{of } c_i \\ \text{that contribute to } \mathcal{I}(\mathbf{p}',\mathbf{p}') \\ \text{but not to } \mathcal{I}(\mathbf{p},\mathbf{p})}} = \mathcal{I}(\mathbf{p}', \mathbf{p}') + \underbrace{n - m - 1}_{\substack{\#\text{of } c_i \\ \text{that contribute to } \mathcal{I}(\mathbf{p},\mathbf{p}) \\ \text{but not to } \mathcal{I}(\mathbf{p},'\mathbf{p}')}} .$$

Now, notice that for $\frac{1}{2} \le i \le n - \frac{1}{2}$, there is an $O_i \in \mathbb{O}$ with first coordinate $i$. For $m + \frac{1}{2} \le i < n$ the pair $(a, O_i)$ contributes 1 to $\mathcal{I}(\mathbf{p}, \mathbb{O})$ whereas the corresponding pair $(b, O_i)$ does not contribute to $\mathcal{I}(\mathbf{p}', \mathbb{O})$. Symmetrically, for $\frac{1}{2} \le i \le m - \frac{1}{2}$, the pair $(a, O_i)$ does not contribute to $\mathcal{I}(\mathbb{O}, \mathbf{p})$ and each pair $(b, O_i)$ contributes 1 to $\mathcal{I}(\mathbb{O}, \mathbf{p}')$. Therefore

$$\mathcal{I}(\mathbf{p}', \mathbb{O}) + \mathcal{I}(\mathbb{O}, \mathbf{p}') + n - m = \mathcal{I}(\mathbf{p}, \mathbb{O}) + \mathcal{I}(\mathbb{O}, \mathbf{p}) + m.$$

To complete the rotation, we have to change $\mathbb{O}$ to $\mathbb{O}'$ by moving the bottom row which contains an $O$ with coordinates $(l - \frac{1}{2}, \frac{1}{2})$ to the top row, creating $O'$ with coordinates $(l - \frac{1}{2}, n + \frac{1}{2})$. Similar arguments will show that

$$\mathcal{I}(\mathbf{p}', \mathbb{O}') + \mathcal{I}(\mathbb{O}, \mathbf{p}') = \mathcal{I}(\mathbf{p}', \mathbb{O}) + \mathcal{I}(\mathbb{O}, \mathbf{p}') + 2l - n$$

$$2\mathcal{I}(\mathbb{O}', \mathbb{O}') = 2\mathcal{I}(\mathbb{O}, \mathbb{O}) + 2l - n - 1.$$

Plugging in these substitutions we see that $M(\mathbf{p}) = M(\mathbf{p}')$, as desired. $\square$

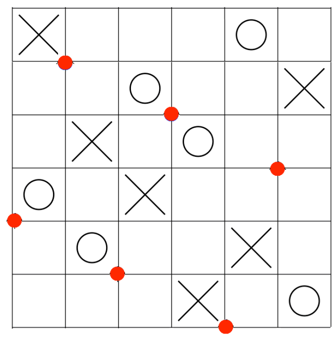


Figure 4: Complete rotation of generator $p$ to $p'$ now including $\mathbb{O}'$.

**Lemma 3.2.** *The function $A$ is under cyclic permutations.*

*Proof.* This is analogous to the proof of lemma 3.1 by definition of Alexander grading. $\square$

Etc...........................................................................

## 3.1 Other versions

There are several different versions of knot Floer homology:

The simplest version of the chain complex associated to this diagram is a chain complex over field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ with generators in $\mathbf{S}$.

$$\widetilde{CF}(K) = \bigoplus_{\mathbf{p} \in \mathbf{S}} \mathbb{Z}/2\mathbb{Z} \cdot \mathbf{p}.$$

Maslov grading $= M_{\mathbb{O}}(\mathbf{p}) = \mathcal{I}(\mathbf{p},\mathbf{p}) - \mathcal{I}(\mathbf{p},\mathbb{O}) - \mathcal{I}(\mathbb{O},\mathbf{p}) + \mathcal{I}(\mathbb{O},\mathbb{O}) + 1.$

*Alexander filtration* for a knot $= A(\mathbf{p}) = \dfrac{M_{\mathbb{X}}(\mathbf{p}) - M_{\mathbb{O}}(\mathbf{p})}{2} - \dfrac{n-1}{2}.$

The associated differential is defined as:

$$\widetilde{\partial}(\mathbf{p}) = \sum_{\mathbf{q} \in \mathbf{S}} \# \left\{ r \in Rect(\mathbf{p},\mathbf{q}) \middle| \mathbb{X} \cap r = \emptyset, \mathbb{O} \cap r = \emptyset, \mathbf{p} \cap int\ r = \emptyset \right\} \cdot \mathbf{q}$$

Note its homology needed a "correction factor" to turn it into a knot invariant

A more enhanced version of this chain complex works over a polynomial ring $\mathbb{F}[U_1, \dots, U_n]$ with generators also in $\mathbf{S}$ and is described as

$$CF^{-}(K) = \bigoplus_{\mathbf{p} \in \mathbf{S}} \mathbb{F}[U_1, \dots, U_n] \cdot \mathbf{p}.$$

where $n =$ the grid number.

The summand for this chain complex are generated by expressions $U_1^{m_1} \cdots U_n^{m_n} \cdot \mathbf{p}$, with $\mathbf{p} \in \mathbf{S}$, with maslov grading

$$d = M(\mathbf{p}) - 2\sum_{i=1}^{n} m_i$$

and alexander grading

$$a = A(\mathbf{p}) - \sum_{i=1}^{n} m_i.$$

The differential

$$\partial^-(p) = \sum_{q \in \mathbf{S}} \sum_{\{r \in Rect(\mathbf{p},\mathbf{q} | \mathbf{p} \notin int(r), X \notin r\}} U_1^{\#O_i \cap r} \cdot \ldots \cdot U_n^{\#O_n \cap r} \cdot \mathbf{q},$$

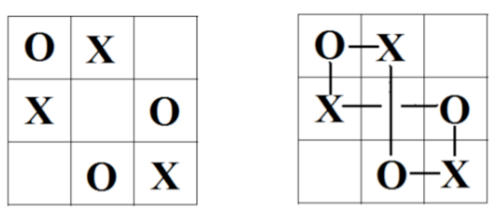
where $\#O_i \cap r$ denotes the number of times $O_i$ is in $r$. This number is either 1 or 0. Thus, $\partial^-$ keep account of the elements of $\mathbb{O}$ that are in rectangles connecting generators.

These differentials have the following properties:

**Theorem 2.** *Let $\partial^-$ and $\widehat{\partial}$ be defined as above. Then,*

- $\partial^- \circ \partial^- = 0, \quad \widehat{\partial} \circ \widehat{\partial} = 0$
- *Both differentials drop Maslov grading by 1.*
- *Both differentials preserve the Alexander grading.*

# 4 Examples



(a) (b)

Figure 5: (a) is an example of a $3 \times 3$ grid for the unknot with one crossing. We can see in (b) the knot drawn on the grid.

In this example is a calculation of $HFK^-$(1crossing knot). Let $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Looking at this $3 \times 3$ grid, figure 5, we see that there are six generators.

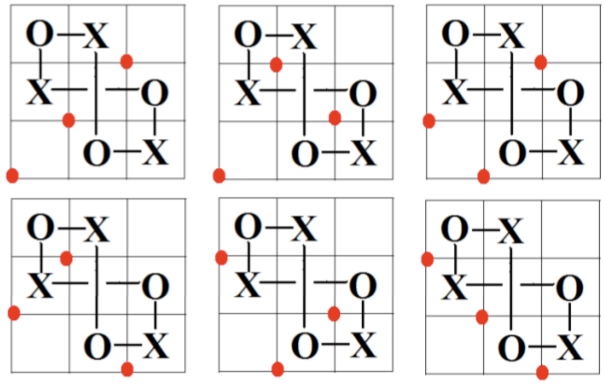



Figure 6: The generators for the grid diagram in figure 5 labelled from left to right and top to bottom: (1 2 3), (1 3 2), (2 1 3), (2 3 1), (3 1 2) and (3 2 1).

Using the following formulas for the Maslov and Alexander gradings,

$$\text{Maslov Grading: } M_{\mathbb{O}}(\mathbf{p}) = \mathcal{I}(\mathbf{p},\mathbf{p}) - \mathcal{I}(\mathbf{p},\mathbb{O}) - \mathcal{I}(\mathbb{O},\mathbf{p}) + \mathcal{I}(\mathbb{O},\mathbb{O}) + 1$$

$$\text{Alexander Grading: } A(\mathbf{p}) = \frac{M_{\mathbb{X}}(\mathbf{p}) - M_{\mathbb{O}}(\mathbf{p})}{2} - \frac{n-1}{2}$$

we can calculate the two gradings $(A(\cdot), M(\cdot))$, associated to each generator as seen in table1

| $\mathbf{p}$ | $\mathcal{I}(\mathbf{p},\mathbf{p})$ | $\mathcal{I}(\mathbf{p},\mathbb{O})$ | $\mathcal{I}(\mathbb{O},\mathbf{p})$ | $\mathcal{I}(\mathbf{p},\mathbb{X})$ | $\mathcal{I}(\mathbb{X},\mathbf{p})$ | $\mathcal{I}(\mathbb{O},\mathbb{O})$ | $\mathcal{I}(\mathbb{X},\mathbb{X})$ | $(A(\mathbf{p}), M(\mathbf{p}))$ |
|---|---|---|---|---|---|---|---|---|
| (1 2 3) | 3 | 4 | 1 | 4 | 1 | 1 | 1 | $(-1,0)$ |
| (1 3 2) | 2 | 4 | 1 | 4 | 1 | 1 | 1 | $(-1,-1)$ |
| (2 1 3) | 2 | 4 | 1 | 4 | 1 | 1 | 1 | $(-1,-1)$ |
| (2 3 1) | 1 | 3 | 0 | 4 | 1 | 1 | 1 | $(0,0)$ |
| (3 1 2) | 1 | 4 | 1 | 3 | 0 | 1 | 1 | $(-2,-2)$ |
| (3 2 1) | 0 | 3 | 0 | 3 | 0 | 1 | 1 | $(-1,-1)$ |

Table 1: This table shows the calculations needed to calculate the gradings of each of the generators in figure 6.

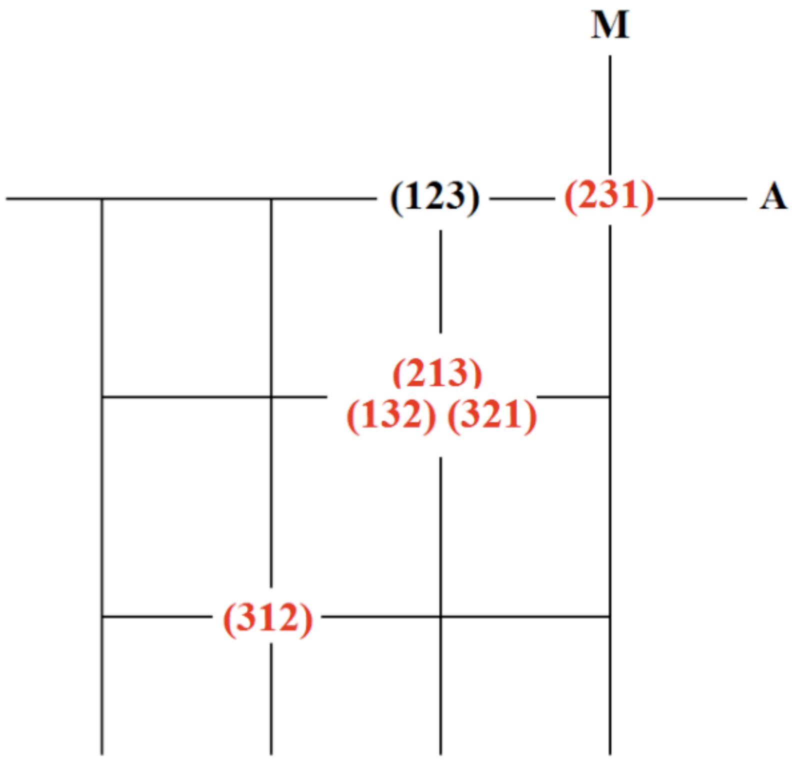



Figure 7: This figure is a graph with x-axis equal to the Alexander grading and the y-axis equal to the Maslov grading.

In order to calculate the differentials of each generator, by definition we count rectangles connecting one generator to another without any element of the generators or any element in $\mathbb{X}$. Recall that we are looking at rectangles on a torus and that we will have two rectangles connecting one generator to another.

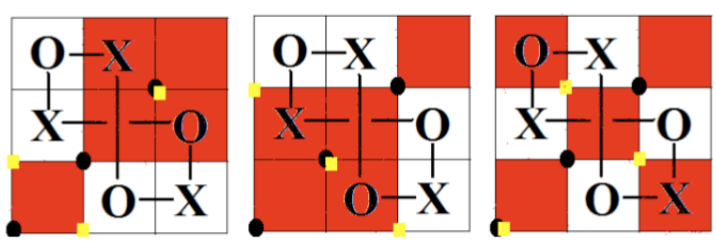


Figure 8: The differential for the generator (1 2 3) is calculated using the rectangles created from generators (2 1 3), (3 1 2) and (1 3 2), as seen in this figure. Thus $\partial^-(1\ 2\ 3) = (2\ 1\ 3) + (3\ 2\ 1) + (1\ 3\ 2\ )$.

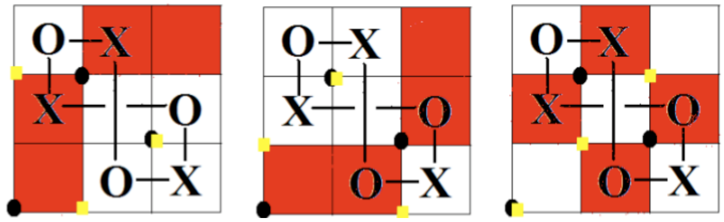



Figure 9: The differential for the generator (1 3 2) is calculated using the rectangles created from generators (3 1 2), (2 3 1) and (1 2 3), as seen in this figure. Thus $\partial^-(1\ 3\ 2\ ) = (U_3 + U_2)(2\ 3\ 1)$.

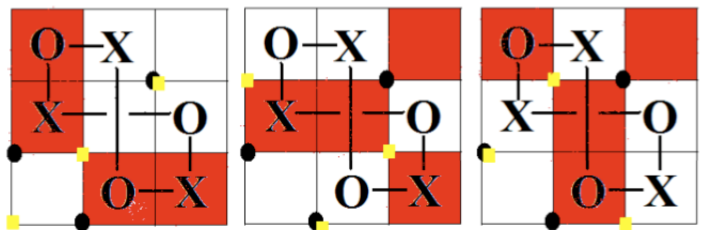



Figure 10: The differential for the generator (2 1 3) is calculated using the rectangles created from generators (1 2 3), (3 1 2) and (2 3 1), as seen in this figure. Thus $\partial^-(2\ 1\ 3) = (U_1 + U_2)(2\ 3\ 1)$.

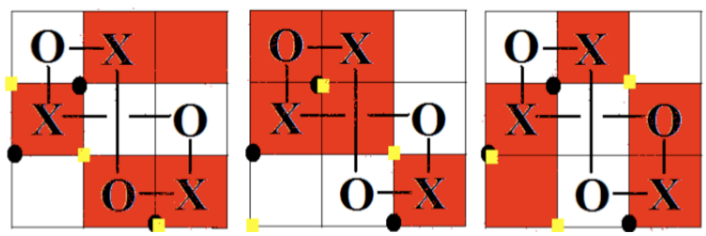



Figure 11: The differential for the generator (2 3 1) is calculated using the rectangles created from generators (3 2 1), (1 3 2) and (2 1 3), as seen in this figure. Thus $\partial^-(2\ 3\ 1) = 0$.

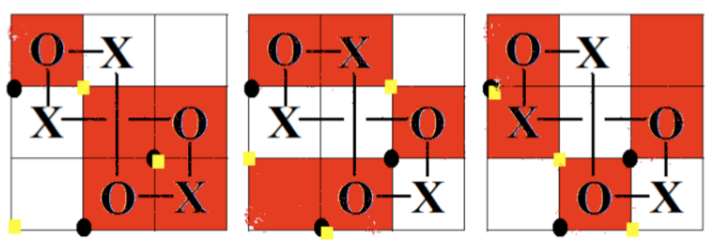



Figure 12: The differential for the generator (3 1 2) is calculated using the rectangles created from generators (1 3 2), (2 1 3) and (3 2 1), as seen in this figure. Thus$\partial^-(3\ 1\ 2) = U_1(1\ 3\ 2) + U_2(2\ 1\ 3) + U_3(3\ 2\ 1)$.

Thus, table 2 shows us the differential we get for each generator. Using this information, we can calculate the homology of our chain complex. We know from homology theory the following:

$$HFK = \frac{ker\partial_n}{img\partial_{n+1}}.$$

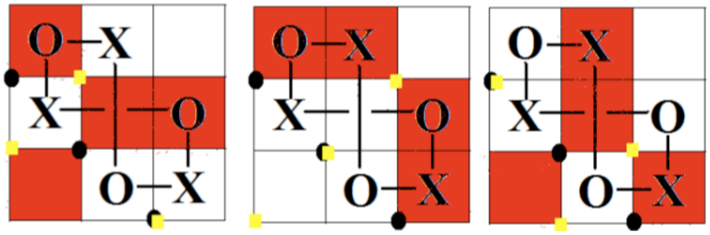



Figure 13: The differential for the generator (3 2 1) is calculated using the rectangles created from generators (2 3 1), (1 2 3) and (3 1 2), as seen in this figure. Thus $\partial^-(3\ 2\ 1) = (U_1 + U_3)(2\ 3\ 1)$.

$$\partial^-(123) = (213) + (321) + (132)$$

$$\partial^-(132) = (U_3 + U_2)(231)$$

$$\partial^-(213) = (U_1 + U_2)(231)$$

$$\partial^-(231) = 0$$

$$\partial^-(312) = U_1(132) + U_2(213) + U_3(321)$$

$$\partial^-(321) = (U_1 + U_3)(231)$$

Table 2: Differential for each of the generators.

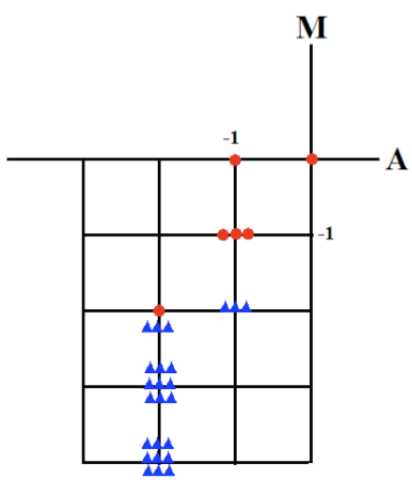



Figure 14: The figure is a graph that shows the generators as red circles and the generators times $U_1, U_2, U_3$ as blue triangles.

Looking at the $(0,0)$ homology level, we see that $\partial^-(2\ 3\ 1) = 0$. The $ker(\partial_{(0,0)}) =< (2\ 3\ 1) >$. And since there does not exist a map above this map, we see that $HFK^-_{(0,0)}(unknot) =< (2\ 3\ 1) >= \mathbb{F}$. Continuing in this matter we look at the $HFK^-_{(-1,0)}, HFK^-_{(-1,-1)}$ and $HFK^-_{(-1,-2)}$. At the

$(-1,0)$ level there is one generator: $(1\ 2\ 3)$. At the $(-1,-1)$ level there are three generators: $(1\ 3\ 2), (2\ 1\ 3), (3\ 2\ 1)$. At the $(-1,-2)$ level, there are also three elements, the generator $(2\ 3\ 1)$ times $U_1, U_2,$ and $U_3$ respectively. This gives us the following sequence:

$$0 \xrightarrow{\partial_3^-} \mathbb{F} \xrightarrow{\partial_2^-} \mathbb{F}^3 \xrightarrow{\partial_1^-} \mathbb{F}^3 \xrightarrow{\partial_0^-} 0$$

where the mapping notation is simplified.

We know that this sequence begins and ends where it does with 0 because we have no other elements with Alexander grading $= -1$. Thus looking at $\partial_3^-$ we see that the $img(\partial_3^-) = 0$. Notice, $ker(\partial_2^-) = 0$. Thus,

$$HFK^-_{(-1,0)}(unknot) = ker(\partial_2^-)/img(\partial_3^-) = 0.$$

Calculating the image of $\partial_2^-$ we see that $img(\partial_2^-) = \mathbb{F}$ since $ker(\partial_2^-) = 0$. Now, calculating $ker(\partial_1^-)$ we see that $\partial_1^-((1\ 3\ 2) + (2\ 1\ 3) + (3\ 2\ 1)) = 0$. Thus, $ker(\partial_1^-) = n((1\ 3\ 2) + (2\ 1\ 3) + (3\ 2\ 1))$, i.e. $ker(\partial_1^-) = \mathbb{F}$. Thus

$$HFK^-_{(-1,-1)}(unknot) = \frac{\mathbb{F}}{\mathbb{F}} = 0.$$

Notice then we can we see that $img(\partial_1^-) = \mathbb{F}^2$. To calculate the kernal of $\partial_0^-$ notice that $\partial_0^-(U_i(2\ 3\ 1)) = 0$ for $i = 1, 2, 3$. Thus we have that $ker(\partial_0^-) = n_i U_i(2\ 3\ 1)$ for $i = 1, 2, 3$ and $n_i \in \mathbb{F}$. Thus we have

$$HFK^-_{(-1,-2)}(unknot) = \frac{\mathbb{F}^3}{\mathbb{F}^2} = \mathbb{F}.$$

We can continue in this manner to find

$$HFK^-_{(-2,-2)}(unknot) = 0,\ HFK^-_{(-2,-3)}(unknot) = 0,\ HFK^-_{(-2,-4)}(unknot) = \mathbb{F}$$

giving us fig**??**.

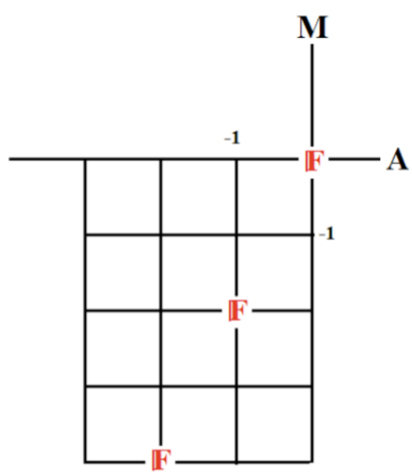



Figure 15: Graph of the homology groups of $3 \times 3$ grid diagram of the unknot.

So at each level $\{-n, -2n\}$ we have $U_i^n$ times the generator (2 3 1) giving a non-trivial homology. Thus

$$HFK^-_{(-n,-2n)}(unknot) = \mathbb{F}[U].$$

Yet even with the combinatorial tool, the homology quickly becomes difficult to calculate. Notice that the number of generators for an $n \times n$ grid is $n!$. Thus several computations tools have been developed to help with calculations.

# 5 Computational Tools

GridLink is a program by Marc Culler that calculates $\widehat{HFK}$ for knots only. Also, Jean-Marie Droz of the University of Zurich (working along with Anna Beliakova) wrote a Python program to compute the (hat-version) Heegaard-Floer Knot Homology, which has been integrated into Dror Bar-Nataan's Knot Atlas page:

http://katlas.math.toronto.edu/wiki/Heegaard_Floer_Knot_Homology.

Curently, there are no computational tools to find the Floer homology of a 2 or more component link.

# References

[1] C. Manolescu, P. Ozsváth, Z. Szabó , and D. Thurston, *On combinatorial link Floer homology*, **Geometry & Topology** 11 (2007) 2339-2350.

[2] C. Manolescu, P. Ozsváth, and S. Sarkar. *A Combinatorial Description of Knot Floer Homology*, Annals of Mathematics, Vol. 169 (2009), 633-639.

[MOST] , C. Manolescu, P. Ozsváth, Z. Szabó , and D. Thurston, *On combinatorial link Floer homology*, **Geometry & Topology** 11 (2007) 2339-2350.

[MOS] C. Manolescu, P. Ozsváth, and S. Sarkar. *A Combinatorial Description of Knot Floer Homology*, **Annals of Mathematics**, Vol. 169 (2009), 633-639.

[OS] P.Oszváth and Z. Szabó. *An Introduction to Heegaard Floer Homology*, Floer Homology, Gauge Theory, and Low Dimensional Topology, Proceedings of the Clay Mathematics Institute 2004 Summer School. Alfréd Rényi Institute of Mathematics, Budapest, Hungary.pp. 3–28.

[OS2] Oszváth, Peter and Szabó, Zoltan. *Knots with unknotting number one and Heegaard Floer homology*, preprint http://arxiv.org/pdf/math.GT/0401426.pdf

[OS3] Oszváth, Peter and Szabó, Zoltan. *Skein relation paper*, preprint http://arxiv.org/pdf/0707.1165.pdf

[OS4] Oszváth, Peter and Szabó, Zoltan. *Holomorphic disk and three manifold invariants: properties and applications* preprint http://arxiv.org/abs/math.SG/0105202

[OS5] Oszváth, Peter and Szabó, Zoltan. Heegaard Diagrams and Floer Homology. preprint http://arxiv.org/abs/math.GT/0602232

[O1] Ozsváth, Peter. "Knot Floer Homology", Advanced Summer School in Knot Theory. ICTP, Trieste, Italy. 18-21 May 2009

[O2] Ozsváth, Peter. "Heegaard Floer homology", Low Dimensional Topology Workshop. MSRI, Berkley, CA. 14 Aug. 2008

[Sarkar] Topics in Heegaard Floer Homology, PhD thesis http://arxiv.org/abs/0906.2579

[SD] Donaldson, Simon. *Self-dual connections and the topology of smooth 4-manifolds*. **Bull. Amer. Math. Soc. (N.S.)** Volume 8, Number 1 (1983), 81-83.

[Scharlemann] Scharlemann, M. "Unknotting Number One Knots Are Prime." **Invent. Math.** 82, 37-55, 1985.

[Wikipeadia] Wikipedia page on Gauge Theory, http://en.wikipedia.org/wiki/Gauge_theory