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Introduction to Vassiliev Knot Invariants

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Preface

This text provides an introduction to the theory of finite type (Vassiliev) knot invariants, with a stress on its combinatorial aspects. It is intended for readers with no or little background in this area, and we care more about a clear explanation of the basic notions and constructions than about widening the exposition to more recent and more advanced material. Our aim is to lead the reader to understanding through pictures and calculations rather than through explaining abstract theories. For this reason we allow ourselves to give the detailed proofs only if they are transparent and instructive, referring more advanced readers to the original papers for more technical considerations.

Historical remarks. The notion of finite type knot invariants was independently invented by Victor Vassiliev (Moscow) and Mikhail Goussarov (St. Petersburg) in the end of 1980's and first published in their respective papers [**Va1**] (1990) and [**G1**] (1991).[1]

Neither Vassiliev nor Goussarov thought very much of their discovery. M. Goussarov (1958–1999), doing first-class work, somehow did not care about putting his thoughts on paper. V. Vassiliev, at the time, was not much interested in knot invariants, either: his main concern was the general theory of discriminants in the spaces of smooth maps. It was V. I. Arnold [**Ar2**] who understood the revolutionary importance of finite type invariants in knot theory. He coined the name "Vassiliev invariants" and explained them to Joan Birman, one of the world's leading experts in knots. Since that time, the term "Vassiliev invariants" has become standard, although nowadays some people use the expression "Vassiliev-Goussarov invariants", which is closer to the historical truth.

Vassiliev's approach is based on the study of discriminants in the (infinite-dimensional) spaces of smooth maps from one manifold into another. By definition, the discriminant consists of all maps with singularities.

For example, consider the space of all smooth maps from the circle into 3-space $\mathcal{M} = \{f : S^1 \to \mathbb{R}^3\}$. If f is an embedding (has no singular points), then it represents a knot. The complement to the set of all knots is the discriminant $\Sigma \subset \mathcal{M}$. It consists of all smooth maps from S^1 into $\mathbb{R}^3$ that have singularities, either local, where $f' = 0$, or non-local, where f is not injective. Two knots are equivalent if and only if they can be joined by a path in the space $\mathcal{M}$ that does not intersect the discriminant. Therefore, knot types are in one-to-one correspondence with the connected components of the complement $\mathcal{M} \setminus \Sigma$, and knot invariants with values in an Abelian group

[1] According to O. Viro, Goussarov first mentioned finite type invariants in his talk at Rokhlin's seminar as early as in 1987.

$\mathbb{G}$ are nothing but cohomology classes in $H^0(\mathcal{M} \setminus \Sigma, \mathbb{G})$. To compute this cohomology, V. Vassiliev constructs a simplicial resolution of the singular variety Σ, then introduces a filtration of this resolution by finite-dimensional subvarieties that gives rise to a spectral sequence containing, in particular, the spaces of finite type invariants.

J. Birman and X.-S. Lin [**BL**] have contributed a lot to the simplification of Vassiliev's original techniques. They explained the relation between Jones polynomial and finite type invariants and emphasized the role of the algebra of chord diagrams. M. Kontsevich [**Kon1**] introduced an analytical tool (*Kontsevich's integral*) to prove that the study of Vassiliev invariants can be reduced to the combinatorics of chord diagrams.

D. Bar-Natan was the first mathematicians who undertook a thorough study of Vassiliev invariants as such. In his preprint [**BN0**] and PhD thesis [**BNt**] he found the relationship between finite type invariants and topological quantum field theory elaborated by his scientific advisor E. Witten [**Wit1, Wit2**]. Bar-Natan's paper [**BN1**] (whose preprint edition [**BN1a**] appeared in 1992) is still the most authoritative source on the fundamentals of the theory of Vassiliev invariants. About the same time, T. Le and J. Murakami [**LM2**], relying on V. Drinfeld's work [**Dr1, Dr2**], proved the rationality of the Kontsevich integral.

Among more recent important developments in the area of finite type knot invariants let us mention:

- The existence of non-Lie-algebraic weight systems (P. Vogel [**Vo1**], J. Lieberum [**Lieb**]).
- J. Kneissler's fine analysis [**Kn1, Kn2, Kn3**] of the structure of the algebra Λ introduced by P. Vogel [**Vo1**].
- Gauss diagram formulas invented by M. Polyak and O. Viro [**PV1**] and the proof by M. Goussarov [**G3**] that all finite type invariants can be obtained in this way.
- D. Bar-Natan's proof that Vassiliev invariants for *braids* separate braids [**BN4**][2]
- Habiro's theory of claspers [**Ha2**] (see also [**G4**]).
- V. Vassiliev's papers [**Va4, Va5**] where a general technique for deriving combinatorial formulas for cohomology classes in the complements to discriminants, and in particular, for finite type invariants, is proposed.
- Explicit formulas for the Kontsevich integral of some knots and links ([**BGRT, BLT, BNL, Roz2, Kri2, Mar, GK**].

[2] This fact is also an immediate consequence of T. Kohno's theorem [**Koh2**], proved before the notion of finite type invariants was introduced.

- V. Turchin's paper [**Tu**] where he introduces a Hogde decomposition in the homology of knots and states an important conjecture about the detectability of knot orientation by finite type invariants.

An important source of regularly updated information on finite type invariants is the online *Bibliography of Vassiliev invariants* started by D. Bar-Natan and currently living at

`http://www.pdmi.ras.ru/~duzhin/VasBib/`

On July 12, 2005 it contained 600 entries, out of which 36 were added during the seven months of 2005. The study of finite type invariants is going on at a steady pace. However, notwithstanding all efforts the most important problem risen in 1990:

Is it true that Vassiliev invariants distinguish knots?

— is still open. The partial question, *s it true that Vassiliev invariants can detect knot orientation?* — is open, too.

Prerequisites. We assume that the reader has a basic knowledge of calculus on manifolds (vector fields, differential forms, Stokes' theorem), general algebra (groups, rings, modules, Lie algebras, fundamentals of homological algebra), linear algebra (vector spaces, linear operators, tensor algebra, elementary facts about representations) and topology (topological spaces, homotopy, homology, Euler characteristic). More advanced algebraic material (bialgebras, free algebras, universal enveloping algebras etc.) which is of primary importance in this book, can be found in the Appendix at the end of the book. No knowledge of knot theory is presupposed, although it may be useful.

Contents. The book consists of four parts, divided into fourteen chapters.

The first part opens with a short introduction into the theory of knots and their classical polynomial invariants and closes with the definition of Vassiliev invariants.

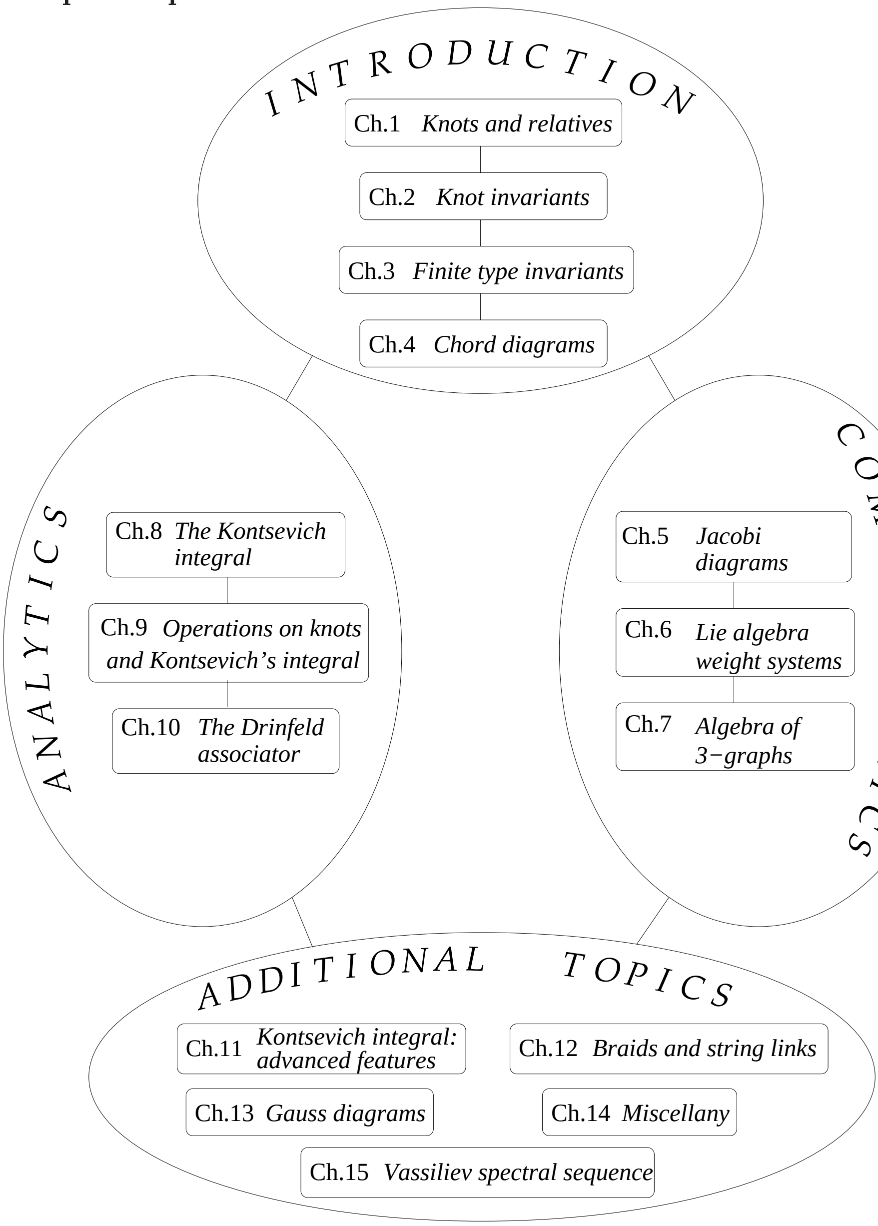
In part 2, we systematically study the graded Hopf algebra naturally associated with the filtered space of Vassiliev invariants, which appears in three different disguises: as the algebra of chord diagrams $\mathcal{A}$, as the algebra of closed diagrams $\mathcal{C}$, and as the algebra of open 1-3-diagrams $\mathcal{B}$. After that, we study the auxiliary algebra Γ generated by regular trivalent graphs and closely related to the algebras $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ as well as to Vogel's algebra Λ. In the last chapter we discuss the weight systems defined by Lie algebras, both universal and depending on a chosen representation.

Part 3 is dedicated to a detailed exposition of the Kontsevich integral; it contains the proof of the main theorem of the theory of Vassiliev knot invariants that reduces their study to combinatorics of chord diagrams and related algebras. Chapter 11 is about more advanced material related to the Kontsevich integral: the wheels formula, the Rozansky rationality conjecture etc.

The last part of the book is devoted to various topics left out in the previous exposition, in particular, Gauss diagram formulas for Vassiliev invariants, the Melvin–Morton conjecture, the Drinfeld associator, the Goussarov–Habiro theory, the size of the space of Vassiliev invariants etc.

The book is intended to be a textbook, so we have included many exercises, both embedded in text and constituting a separate section at the end of each chapter. Open problems are marked with an asterisk.

Chapter dependence.



If this text is used as an actual textbook, then a one semester course may consist of the introduction (chapters 1–4) followed by either combinatorial (5–7) or analytical (8–10) chapters according to the preferences of the instructor and the students. If time permits, the course may be enhanced by some of the additional topics. For a one year course we would recommend to follow the numerical order of chapters 1–9 and conclude with some additional topics to teacher's taste.

Acknowledgements. Our work on this book actually began in August 1992, when our colleague Inna Scherbak returned to Pereslavl-Zalessky from the First European Mathematical Congress in Paris and brought a photocopy of Arnold's lecture notes about the new-born theory of Vassiliev knot invariants. We spent several months filling our waste-paper baskets with pictures of chord diagrams, before our first joint article [**CD1**] was ready.

In the preparation of the present text, we have extensively used our papers (joint, single-authored and with other coauthors, see bibliography) and in particular, lecture notes of the course "Vassiliev invariants and combinatorial structures" that one of us (S. D.) delivered at the Graduate School of Mathematics, University of Tokyo, in Spring 1999. It is our pleasure to thank V. I. Arnold, D. Bar-Natan, J. Birman, C. De Concini, O. Dasbach, A. Durfee, V. Goryunov, O. Karpenkov, T. Kerler, T. Kohno, S. Lando, M. Polyak, I. Scherbak, A. Vaintrob, A. Varchenko, V. Vassiliev, and S. Willerton for many useful comments concerning the subjects touched upon in the book. We are likewise indebted to the anonymous referees whose criticism and suggestions helped us to improve the text.

Our work was supported by several grants: INTAS 00-0259, NWO-RFBR 047.008.005, RFBR 00-15-96084 and 01-01-00660 (S. D.), Professional Development Grants of OSU, Mansfield (2002, 2004, S. Ch.). Part of the work was accomplished when the first author was vising MSRI (summer 2004), the second author was visiting the Ohio State University (autumn quarter 2003) and all the three authors, at various times, visited the Max-Planck-Institut für Mathematik. We are grateful to all these institutions for excellent working conditions and a stimulating atmosphere.

Part 1

Fundamentals

Chapter 1

Knots and their relatives

This book is about knots. It is, however, hardly possible to speak about knots without mentioning other one-dimensional topological objects embedded into the three-dimensional space. Therefore, in this introductory chapter we give basic definitions and constructions pertaining to knots and their relatives: links, braids and tangles.

The table of knots at the end of this chapter (page 26) will be used throughout the book as a source of examples and exercises.

1.1. Definitions and examples

A *knot* is a closed non-self-intersecting curve in 3-space. In this book, we shall mainly study smooth oriented knots. A precise definition can be given as follows.

1.1.1. Definition. A *parameterized knot* is an embedding of the circle S^1 into the Euclidean space $\mathbb{R}^3$.

Recall that an *embedding* is a smooth map which is injective and whose differential is nowhere zero. In our case, the non-vanishing of the differential means that the tangent vector to the curve is non-zero. In the above definition and everywhere in the sequel, the word *smooth* means *infinitely differentiable*.

A choice of an orientation for the parameterizing circle

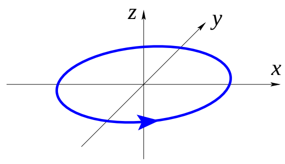
$$S^1 = \{(\cos t, \sin t) \,|\, t \in \mathbb{R}\} \subset \mathbb{R}^2$$

gives an orientation to all the knots simultaneously. We shall always assume that S^1 is oriented counterclockwise. We shall also fix an orientation of the 3-space; each time we pick a basis for $\mathbb{R}^3$ we shall assume that it is consistent with the orientation.

If coordinates x, y, z are chosen in $\mathbb{R}^3$, a knot can be given by three smooth periodic functions of one variable $x(t)$, $y(t)$, $z(t)$.

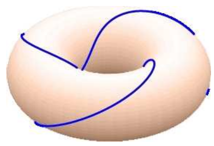
1.1.2. Example. The simplest knot is represented by a plane circle:

$$\begin{aligned} x &= \cos t, \\ y &= \sin t, \\ z &= 0. \end{aligned}$$



1.1.3. Example. The curve that goes 3 times around and 2 times across a standard torus in $\mathbb{R}^3$ is called the *left trefoil knot*, or the $(2,3)$-*torus knot*:

$$\begin{aligned} x &= (2+\cos 3t)\cos 2t, \\ y &= (2+\cos 3t)\sin 2t, \\ z &= \sin 3t. \end{aligned}$$



1.1.4. Exercise. Give the definition of a (p,q)-torus knot. What are the appropriate values of p and q for this definition?

It will be convenient to identify knots that only differ by a change of a parametrization. An *oriented knot* is an equivalence class of parameterized knots under orientation-preserving diffeomorphisms of the parameterizing circle. Allowing *all* diffeomorphisms of S^1 in this definition, we obtain *unoriented knots*. Alternatively, an unoriented knot can be defined as the *image* of an embedding of S^1 into $\mathbb{R}^3$; an oriented knot is then an image of such an embedding together with the choice of one of the two possible directions on it.

We shall distinguish oriented/unoriented knots from parameterized knots in the notation: oriented and unoriented knots will be usually denoted by capital letters, while for the individual embeddings lowercase letters will be used. As a rule, the word "knot" will mean "oriented knot", unless it is clear from the context that we deal with unoriented knots, or consider a specific choice of parametrization.

1.2. Isotopy

The study of parametrized knots falls within the scope of differential geometry. The *topological* study of knots requires an equivalence relation which would not only discard the specific choice of parametrization, but also model the physical transformations of a closed piece of rope in space.

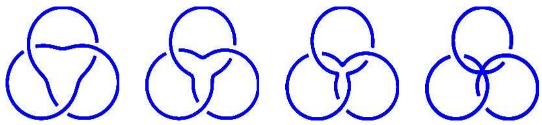
By a *smooth family of maps*, or a *map smoothly depending on a parameter*, we understand a smooth map $F : S^1 \times I \to \mathbb{R}^3$, where $I \subset \mathbb{R}$ is an interval. Assigning a fixed value a to the second argument of F, we get a map $f_a : S^1 \to \mathbb{R}^3$.

1.2.1. Definition. A smooth isotopy of a knot $f : S^1 \to \mathbb{R}^3$, is a smooth family of knots f_u, with u a real parameter, such that for some value $u = a$ we have $f_a = f$.

For example, the formulae

$$\begin{aligned} x &= (u + \cos 3t)\cos 2t, \\ y &= (u + \cos 3t)\sin 2t, \\ z &= \sin 3t, \end{aligned}$$

where $u \in (1, +\infty)$, represent a smooth isotopy of the trefoil knot 1.1.3, which corresponds to $u = 2$. In the pictures below the space curves are shown by their projection to the (x, y) plane:

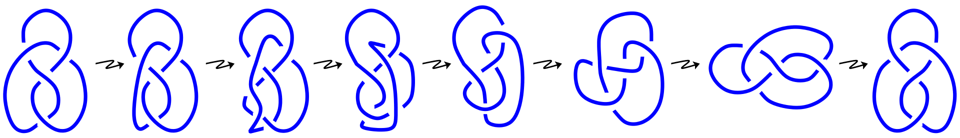


$u = 2$ $u = 1.5$ $u = 1.2$ $u = 1$

For any $u > 1$ the resulting curve is smooth and has no self-intersections, but as soon as the value $u = 1$ is reached we get a singular curve with three coinciding cusps[1] corresponding to the values $t = \pi/3$, $t = \pi$ and $t = 5\pi/3$. This curve is not a knot.

1.2.2. Definition. Two parameterized knots are said to be *isotopic* if one can be transformed into another by means of a smooth isotopy. Two oriented knots are isotopic if they represent the classes of isotopic parameterized knots; the same definition is valid for unoriented knots.

Example. This picture shows an isotopy of the figure eight knot into its mirror image:



1.2.3. There are other notions of knot equivalence, namely, *ambient equivalence* and *ambient isotopy*, which, for smooth knots, are the same thing as isotopy. Here are the definitions. A proof that they are equivalent to our definition of isotopy can be found in [**BZ**].

[1]A *cusp* of a spatial curve is a point where the curve can be represented as $x = s^2$, $y = s^3$, $z = 0$ in some local coordinates.

Definition. Two parameterized knots, f and g, are *ambient equivalent* if there is a commutative diagram

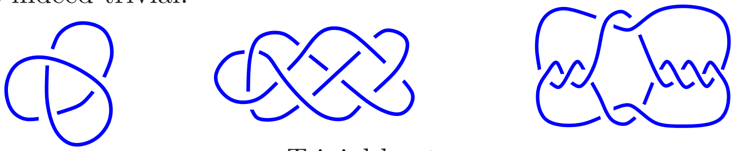
$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "\varphi"'] & \mathbb{R}^3 \arrow[d, "\psi"] \\ S^1 \arrow[r, "g"] & \mathbb{R}^3 \end{tikzcd}$$

where φ and ψ are orientation preserving diffeomorphisms of the circle and the 3-space, respectively.

Definition. Two parameterized knots, f and g, are *ambient isotopic* if there is a smooth family of diffeomorphisms of the 3-space $\psi_t : \mathbb{R}^3 \to \mathbb{R}^3$ with $\psi_0 = \mathrm{id}$ and $\psi_1 \circ f = g$.

1.2.4. A knot, equivalent to the plane circle of Example 1.1.2 is referred to as a *trivial knot*, or an *unknot*.

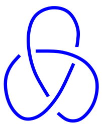
Sometimes, it is not immediately clear from a diagram of a trivial knot that it is indeed trivial:



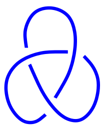
Trivial knots

There are algorithmic procedures to detect whether a given knot diagram represents an unknot; one of them is, based on W. Thurston's ideas, is implemented in J. Weeks' computer program `SnapPea`, see [**Wee**].

Here are several other examples of knots.



Left trefoil



Right trefoil

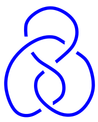
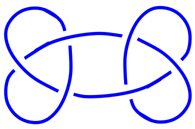
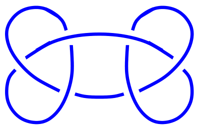


Figure 8 knot



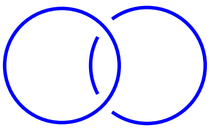
Granny knot



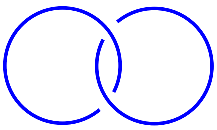
Square knot

Knots are a special case of links.

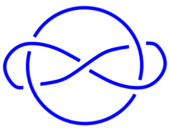
1.2.5. Definition. A *link* is a smooth embedding $S^1 \sqcup \cdots \sqcup S^1 \to \mathbb{R}^3$, where $S^1 \sqcup \cdots \sqcup S^1$ is the disjoint union of several circles.



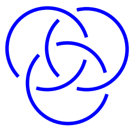
Trivial 2-component link



Hopf link



Whitehead link



Borromean rings

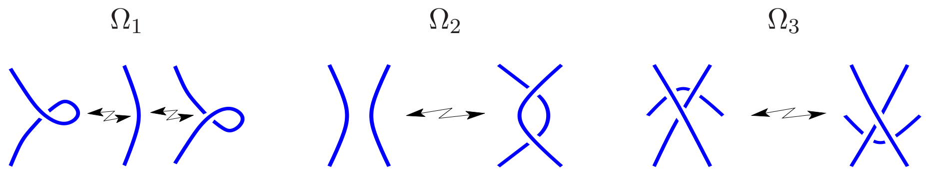
Equivalence of links is defined in the same way as for knots — with the exception that now one may choose whether to distinguish or not between the components of a link and thus speak about the equivalence of links with *numbered* or *unnumbered* components.

In the future, we shall often say "knot (link)" instead of "equivalence class", or "topological type of knots (links)".

1.3. Plane knot diagrams

Knots are best represented graphically by means of *knot diagrams*. A knot diagram is a plane curve whose only singularities are transversal double points (crossings), together with the choice of one branch of the curve at each crossing. The chosen branch is called an *overcrossing*; the other branch is referred to as an *undercrossing*. A knot diagram is thought of as a projection of a knot along some "vertical" direction; overcrossings and undercrossings indicate which branch is "higher" and which is "lower". To indicate the orientation, an arrow is added to the knot diagram.

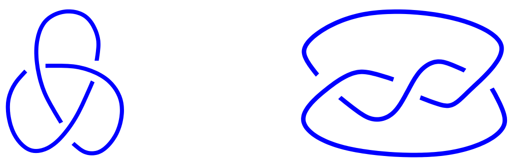
1.3.1. Theorem (Reidemeister [**Rei**], proofs can be found in [**BZ, Mur2**]). *Two unoriented knots K_1 and K_2, are equivalent if and only if a diagram of K_1 can be transformed into a diagram of K_2 by a sequence of ambient isotopies of the plane and local moves of the following three types:*



Reidemeister moves

To adjust the assertion of this theorem to the oriented case, each of the three Reidemeister moves has to be equipped with orientations in all possible ways. Smaller sufficient sets of oriented moves exist; one such set will be given later in terms of Gauss diagrams (see p. 35).

Exercise. Determine the sequence of Reidemeister moves that relates the two diagrams of the trefoil knot below:



1.3.2. Local writhe. Crossing points on a diagram come in two species, positive and negative:

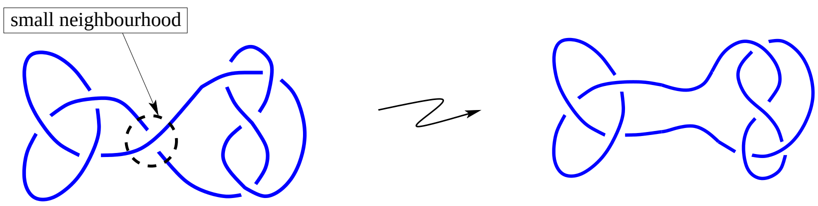
Positive crossing Negative crossing

Although this sign is defined in terms of the knot orientation, it is easy to check that it does not change if the orientation is reversed. For *links* with more than one component, the choice of orientation is essential.

The *local writhe* of a crossing is defined as $+1$ or -1 for positive or negative points, respectively. The *writhe* (or *total writhe*) of a diagram is the sum of the writhes of all crossing points, or, equivalently, the difference between the number of positive and negative crossings. Of course, the same knot may be represented by diagrams with different total writhes. In Chapter 2 we shall see how the writhe can be used to produce knot invariants.

1.3.3. Alternating knots. A knot diagram is called *alternating* if its overcrossings and undercrossing alternate while we travel along the knot. A knot is called *alternating* if it has an alternating diagram. A knot diagram is called *reducible* if it becomes disconnected after the removal of a small neighbourhood of some crossing.

The number of crossings in a reducible diagram can be decreased by a move shown in the picture:



reducible diagram reduction

A diagram which is not reducible is called *reduced.* As there is no immediate way to simplify a reduced diagram, the following conjecture naturally arises (P. G. Tait, 1898).

The Tait conjecture. *A reduced alternating diagram has the minimal number of crossings among all diagrams of the given knot.*

This conjecture stood open for almost 100 years. It was proved only in 1986 (using the newly invented Jones polynomial) simultaneously and independently by L. Kauffman [**Ka6**], K. Murasugi [**Mur1**], and M. Thistlethwaite [**Th**] (see Exercise (28) in Chapter 2).

1.4. Inverses and mirror images

Change of orientation (taking the inverse) and taking the mirror image are two basic operations on knots which are induced by orientation reversing smooth involutions on S^1 and $\mathbb{R}^3$ respectively. Every such involution on S^1 is conjugate to the reversal of the parametrization; on $\mathbb{R}^3$ it is conjugate to a reflection in a plane mirror.

Let K be a knot. Composing the parametrization reversal of S^1 with the map $f : S^1 \to \mathbb{R}^3$ representing K, we obtain the *inverse* K^* of K. The *mirror image* of K, denoted by $\overline{K}$, is a composition of the map $f : S^1 \to \mathbb{R}^3$ with a reflection in $\mathbb{R}^3$. Both change of orientation and taking the mirror image are involutions on the set of (equivalence classes of) knots. They generate a group isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$; the symmetry properties of a knot K depend on the subgroup that leaves it invariant. The group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ has 5 (not necessarily proper) subgroups, which give rise to 5 symmetry classes of knots.

1.4.1. Definition. A knot is called:

- *invertible*, if $K^* = K$,
- *plus-amphicheiral*, if $\overline{K} = K$,
- *minus-amphicheiral*, if $\overline{K} = K^*$,
- *fully symmetric*, if $K = K^* = \overline{K} = \overline{K}^*$,
- *totally asymmetric*, if all knots K, K^*, $\overline{K}$, $\overline{K}^*$ are different.

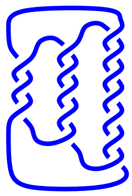
The word *amphicheiral* means either plus- or minus-amphicheiral. For invertible knots, this is the same. Amphicheiral and non-amphicheiral knots are also referred to as *achiral* and *chiral* knots, respectively.

The 5 symmetry classes of knots are summarized in the following table. The word "minimal" means "with the minimal number of crossings"; σ and τ denote the involutions of taking the mirror image and the inverse respectively.

Subgroup	**Orbit**	**Symmetry type**	**Min example**
$\{1\}$	$\{K, \overline{K}, K^*, \overline{K}^*\}$	totally asymmetric	9_{32}, 9_{33}
$\{1, \sigma\}$	$\{K, K^*\}$	+amphicheiral, non-inv	12^a_{427}
$\{1, \tau\}$	$\{K, \overline{K}\}$	invertible, chiral	3_1
$\{1, \sigma\tau\}$	$\{K, K^*\}$	−amphicheiral, non-inv	8_{17}
$\{1, \sigma, \tau, \sigma\tau\}$	$\{K\}$	fully symmetric	4_1

Example. The trefoil knots are invertible, because the rotation through 180° around an axis in $\mathbb{R}^3$ changes the direction of the arrow on the knot.

The existence of non-invertible knots was first proved by H. Trotter [**Tro**] in 1964. The simplest instance of Trotter's theorem is a *pretzel knot with parameters* $(3, 5, 7)$:



Among the knots with up to 8 crossings (see Table 1.5.2.1 on page 26) there is only one non-invertible knot: 8_{17}, which is, moreover, minus-amphicheiral. These facts were proved in 1979 by A. Kawauchi [**Ka1**].

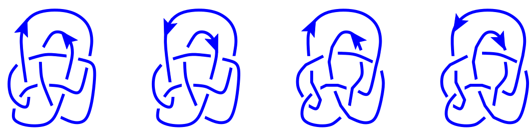
Example. The trefoil knots are not amphicheiral, hence the distinction between the left and the right trefoil. A proof of this fact, based on the calculation of the Jones polynomial, will be given in Sec. 2.4.

Remark. Knot tables only list knots up to taking inverses and mirror images. In particular, there is only one entry for the trefoil knots. Either of them is often referred to as *the trefoil*.

Example. The figure eight knot is amphicheiral. The isotopy between this knot and its mirror image is shown on page 19.

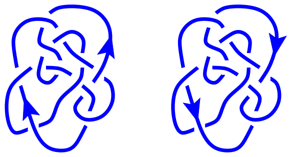
Among the 35 knots with up to 8 crossings shown in Table 1.5.2.1, there are exactly 7 amphicheiral knots: 4_1, 6_3, 8_3, 8_9, 8_{12}, 8_{17}, 8_{18}, out of which 8_{17} is minus-amphicheiral, the rest, as they are invertible, are both plus- and minus-amphicheiral.

The simplest totally asymmetric knots appear in 9 crossings, they are 9_{32} and 9_{33}. The following are all non-equivalent:



9_{33} $\quad$ 9_{33}^* $\quad$ $\overline{9}_{33}$ $\quad$ $\overline{9}_{33}^*$

Here is the simplest plus-amphicheiral non-invertible knot, together with its inverse:

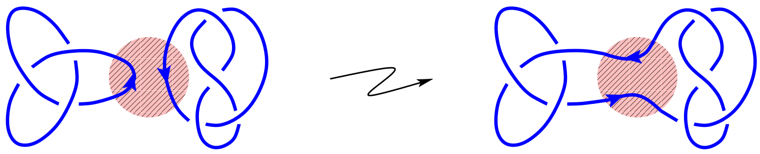


12^a_{427} $\quad$ ${12^a_{427}}^*$

In practice, the easiest way to find the symmetry type of a given knot or link is by using the computer program Knotscape [**HT**], which can handle link diagrams with up to 49 crossings.

1.5. Knot tables

1.5.1. Connected sum. There is a natural way to fuse two knots into one: cut each of the two knots at some point, then connect the two pairs of loose ends. This must be done with some caution: first, by a smooth isotopy, both knots should be deformed so that for a certain plane projection they look as shown in the picture below on the left, then they should be changed inside the dashed disk as shown on the right:



The connected sum makes sense only for oriented knots. It is well-defined and commutative on the equivalence classes of knots.

1.5.2. Definition. A knot is called *prime* if it cannot be represented as the connected sum of two nontrivial knots.

Each knot is a connected sum of prime knots, and this decomposition is unique (see [**CrF**] for a proof). In particular, this means that a trivial knot cannot be decomposed into a sum of two nontrivial knots. Therefore, in order to classify all knots, it is enough to have a table of all prime knots.

Prime knots are tabulated according to the minimal number of crossings that their diagrams can have. Within each group of knots with the same crossing number, knots are numbered in some, usually rather arbitrary, way. In Table 1.5.2.1, we use the widely adopted numbering that goes back to the table compiled by Alexander and Briggs in 1927 [**AB**], then repeated (in an extended and modified way) by D. Rolfsen in [**Rol**]. We also follow

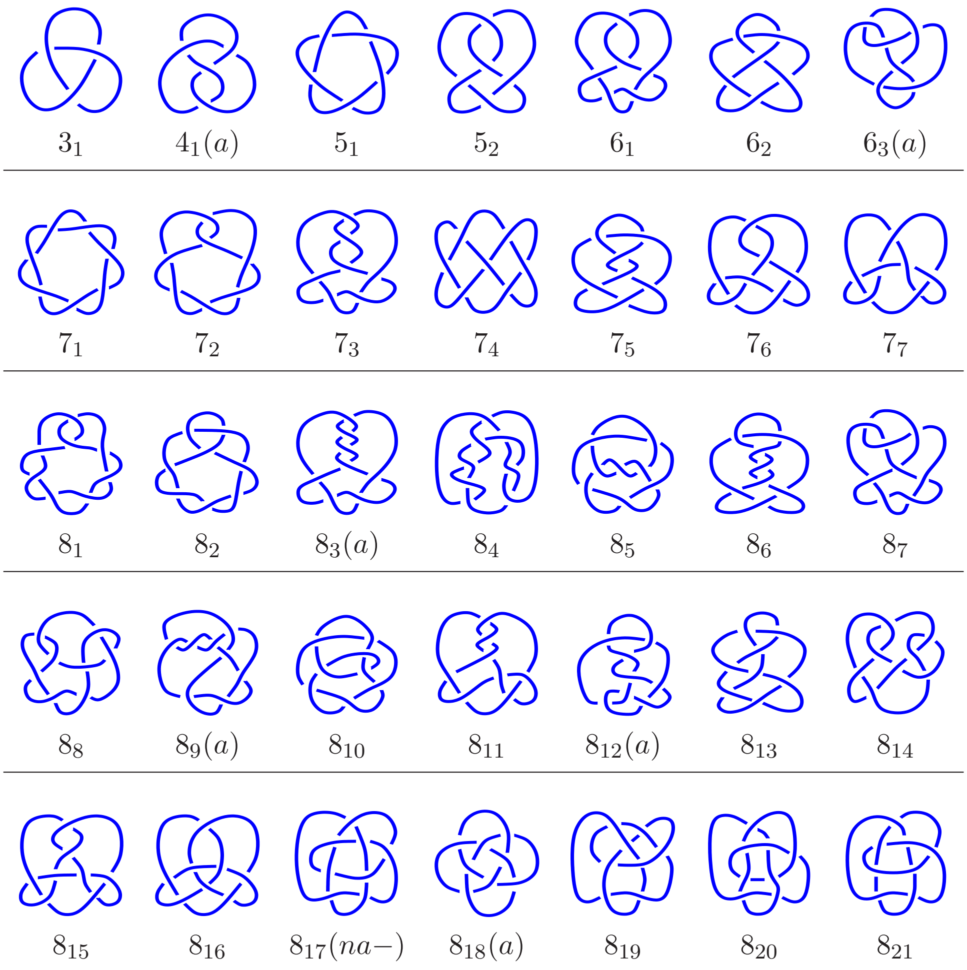
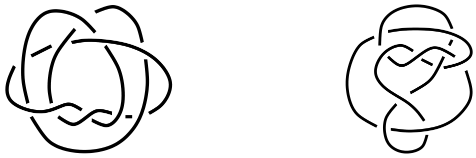


Table 1.5.2.1. Prime knots, up to orientation and mirror images, with up to 8 crossings. Amphicheiral knots are marked by 'a', the (only) non-invertible minus-amphicheiral knot by 'na-'.

Rolfsen's conventions in the choice of the version of non-amphicheiral knots: for example, our 3_1 is the left, not the right, trefoil.

Rolfsen's table of knots, authoritative as it is, contained an error. It is the famous *Perko pair* (knots 10_{161} and 10_{162} in Rolfsen) — two equivalent knots that were thought to be different for 75 years since 1899:



The equivalence of these two knots was established in 1973 by K. A. Perko [**Per1**], a lawyer from New York who studied mathematics at Princeton in 1960–1964 [**Per2**] but later chose jurisprudence to be his profession.[2]

Complete tables of knots are currently known up to crossing number 16 [**HTW**]. For knots with 11 through 16 crossings it is nowadays customary to use the numbering of Knotscape [**HT**] where the tables are built into the software. For each crossing number, Knotscape has a separate list of alternating and non-alternating knots. For example, the notation 12^a_{427} used in Section 1.4, refers to the item number 427 in the list of alternating knots with 12 crossings.

1.6. Algebra of knots

Denote by $\mathcal{K}$ the set of the equivalence classes of knots. It forms a commutative monoid (semigroup with a unit) under the connected sum of knots, and, therefore we can construct the monoid algebra $\mathbb{Z}\mathcal{K}$ of $\mathcal{K}$. By definition, elements of $\mathbb{Z}\mathcal{K}$ are formal finite linear combinations $\sum \lambda_i K_i$, $\lambda_i \in \mathbb{Z}$, $K_i \in \mathcal{K}$, the product is defined by $(K_1, K_2) \mapsto K_1 \# K_2$ on knots and then extended by linearity to the entire space $\mathbb{Z}\mathcal{K}$. This algebra $\mathbb{Z}\mathcal{K}$ will be referred to as the *algebra of knots*.

The algebra of knots provides a convenient language for the study of *knot invariants* (see the next chapter): in these terms, a knot invariant is nothing but a linear functional on $\mathbb{Z}\mathcal{K}$. Ring homomorphisms from $\mathbb{Z}\mathcal{K}$ to some ring are referred to as *multiplicative invariants*; later, in Section 4.3, we shall see the importance of this notion.

In the sequel, we shall introduce more operations in this algebra, as well as in the dual algebra of knot invariants. We shall also study a filtration on $\mathbb{Z}\mathcal{K}$ that will give us the notion of a finite type knot invariant.

1.7. Tangles, string links and braids

A *tangle* is a generalization of a knot which at the same time is simpler and more complicated than a knot: on one hand, knots are a particular case of tangles, on the other hand, knots can be represented as combinations of (simple) tangles.

[2] The combination of a professional lawyer and an amateur mathematician in one person is not new in the history of mathematics (think of Pierre Fermat!).

1.7.1. Definition. A *(parameterized) tangle* is a smooth embedding of a one-dimensional compact oriented manifold, $\boldsymbol{X}$, possibly with boundary, into a *box*

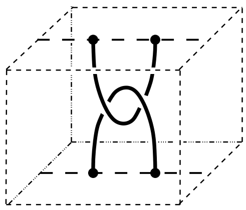
$$\{(x, y, z) \mid w_0 \leqslant x \leqslant w_1\,,\ -1 \leqslant y \leqslant 1\,,\ h_0 \leqslant z \leqslant h_1\} \subset \mathbb{R}^3,$$

where $w_0, w_1, h_0, h_1 \in \mathbb{R}$, such that the boundary of $\boldsymbol{X}$ is sent into the intersection of the (open) upper and lower faces of the box with the plane $y = 0$. An *oriented tangle* is a tangle considered up to an orientation-preserving change of parametrization; an *unoriented tangle* is an image of a parameterized tangle.

The boundary points of $\boldsymbol{X}$ are divided into the top and the bottom part; within each of these groups the points are ordered, say, from the left to the right. The manifold $\boldsymbol{X}$, with the set of its boundary points divided into two ordered subsets, is called the *skeleton* of the tangle.

The number $w_1 - w_0$ is called the *width*, and the number $h_1 - h_0$ is the *height* of the tangle.

Speaking of embeddings of manifolds with boundary, we mean that such embedding send boundaries to boundaries and interiors — to interiors. Here is an example of a tangle, shown together with its box:



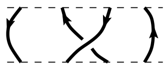
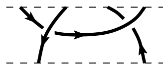
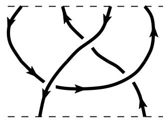
Usually the boxes will be omitted in the pictures.

We shall always identify tangles obtained by translations of boxes. Further, it will be convenient to have two notions of equivalences for tangles. Two tangles will be called *fixed-end isotopic* if one can be transformed into the other by a boundary-fixing isotopy of its box. We shall say that two tangles are simply *isotopic*, or *equivalent* if they become fixed-end isotopic after a suitable re-scaling of their boxes of the form

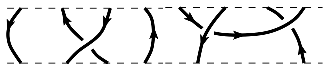
$$(x, y, z) \to (f(x), y, g(z)),$$

where f and g are strictly increasing functions.

1.7.2. Operations. In the case when the bottom of a tangle T_1 coincides with the top of another tangle T_2 of the same width (for oriented tangles we require the consistency of orientations, too), one can define the *product* $T_1 \cdot T_2$ by putting T_1 on top of T_2 (and, if necessary, smoothing out the corners at the joining points):

$T_1 =$  ; $T_2 =$  ; $T_1 \cdot T_2 =$  .

Another operation, *tensor product*, is defined by placing one tangle next to the other tangle of the same height:

$T_1 \otimes T_2 =$  .

Both operations give rise to products on equivalence classes of tangles. The product of two equivalence classes is defined whenever the bottom of one tangle and the top of the other consist of the same number of points (with matching orientations in the case of oriented tangles), the tensor product is defined for any pair of equivalence classes.

1.7.3. Special types of tangles. Knots, links and braids are particular cases of tangles. For example, an n-component link is just a tangle whose skeleton is a union of n circles (and whose box is disregarded).

Let us fix n distinct points p_i on the top boundary of a box of unit width and let q_i be the projections of the p_i to the bottom boundary of the box. We choose the points p_i (and, hence, the q_i) to lie in the plane $y = 0$.

Definition. A *string link on n strings* is an (unoriented) tangle whose skeleton consists of n intervals, the ith interval connecting p_i with q_i. A string link on one string is called a *long knot* .

Definition. A string link on n strings whose tangent vector is never horizontal is called a *pure braid on n strands*.

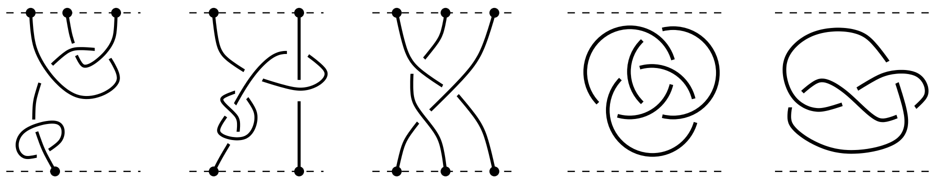
One difference between pure braids and string links is that the components of a string link can be knotted. However, there are string links with unknotted strands that are not equivalent to braids.

Let σ be a permutation of the set of n elements.

Definition. A *braid on n strands* is an (unoriented) tangle whose skeleton consists of n intervals, the ith interval connecting p_i with $q_{\sigma(i)}$, with the property that the tangent vector to it is never horizontal.

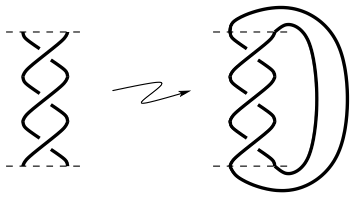
Pure braids are a specific case of braids with σ the identity permutation. Note that with our definition of equivalence, an isotopy between two braids can pass through tangles with points where the tangent vector is horizontal. Often, in the definition of the equivalence for braids it is required that that an isotopy consist entirely of braids; the two approaches are equivalent.

The above definitions are illustrated by the following comparison chart:

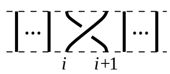


A tangle A string link A braid A link A knot

1.7.4. Braids. Braids are useful in the study of links, because any link can be represented as a *closure* of a braid (Alexander's theorem [**Al1**]):

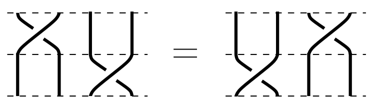


Braids are in many respects easier to work with, as they form groups under tangle multiplication: the set of equivalence classes of braids on n strands is the *braid group* denoted by B_n. A convenient set of generators for the group B_n consists of the elements σ_i, $i = 1, \ldots, n-1$:

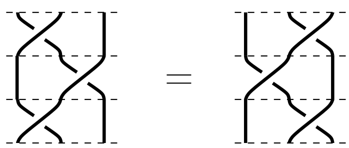


which satisfy the following complete set of relations.

Far commutativity, $\sigma_i \sigma_j = \sigma_j \sigma_i$, for $|i - j| > 1$.



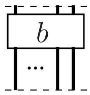
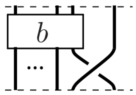
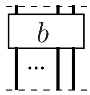
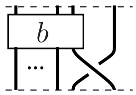
Braiding relation, $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, for $i = 1, 2, \ldots, n-2$.



Assigning to each braid in B_n the corresponding permutation σ, we get an epimorphism $\pi : B_n \to S_n$ of the braid group on n strands onto the symmetric group on n letters. The kernel of π consists of pure braids and is denoted by P_n.

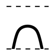
Theorem (Markov [**Mark, Bir1**]). *Two closed braids are equivalent (as links) if and only if the braids are related by a finite sequence of the following Markov moves:*

(M1) $b \longleftrightarrow aba^{-1}$ *for any* $a, b \in B_n$*;*

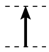
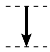
(M2) $B_n \ni$  $\longleftrightarrow$  $\in B_{n+1}$,  $\longleftrightarrow$  .

1.7.5. Elementary tangles. A link can be cut into several *simple* tangles by a finite set of horizontal planes, and the link is equal to the product of all such tangles. Every simple tangle is a tensor product of the following *elementary* tangles.

Unoriented case:

id := , X_+ := , X_- := , max := , min :=  .

Oriented case:

id := , id* := , X_+ := , X_- := ,

$\overrightarrow{\max}$:= , $\overleftarrow{\max}$:= , $\underrightarrow{\min}$:= , $\underleftarrow{\min}$:=  .

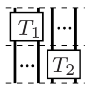
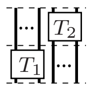
For example, the generator $\sigma_i \in B_n$ of the braid group is a simple tangle represented as the tensor product, $\sigma_i = \mathrm{id}^{\otimes(i-1)} \otimes X_+ \otimes \mathrm{id}^{\otimes(n-i-1)}$.

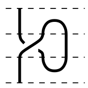
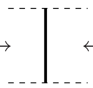
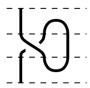
1.7.6. Exercise. Decompose the tangle  into elementary tangles.

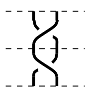
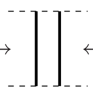
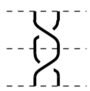
1.7.7. The Turaev moves. Having presented a tangle as a product of simple tangles it is natural to ask for an analogue of Reidemeister's (1.3.1) and Markov's (1.7.4) theorems, that is, a criterion for two such presentations to give isotopic tangles. Here is the answer.

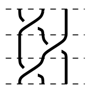
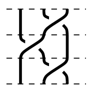
Theorem ([**Tur3**]). *Two products of simple tangles are isotopic if and only if they are related by a finite sequence of the following Turaev moves.*

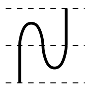
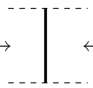
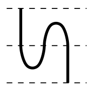
Unoriented case:

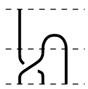
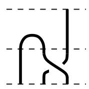
(T0)  $\longleftrightarrow$  *Note that the number of strands at top or bottom of either tangle T_1 or T_2, or both might be zero.*

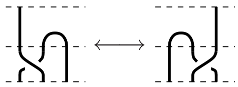
(T1)  $\longleftrightarrow$  $\longleftrightarrow$  (id$\otimes$max)$\cdot$($X_+\otimes$id)$\cdot$(id$\otimes$min)=id= =(id$\otimes$max)$\cdot$($X_-\otimes$id)$\cdot$(id$\otimes$min)

(T2)  $\longleftrightarrow$  $\longleftrightarrow$  $X_+\cdot X_-$=id$\otimes$id=$X_-\cdot X_+$

(T3)  $\longleftrightarrow$  ($X_+\otimes$id)$\cdot$(id$\otimes X_+$)$\cdot$($X_+\otimes$id)=(id$\otimes X_+$)$\cdot$($X_+\otimes$id)$\cdot$(id$\otimes X_+$)

(T4)  $\longleftrightarrow$  $\longleftrightarrow$  (max$\otimes$id)$\cdot$(id$\otimes$min)=id=(id$\otimes$max)$\cdot$(min$\otimes$id)

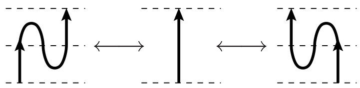
(T5)  $\longleftrightarrow$  (id$\otimes$max)$\cdot$($X_+\otimes$id)=(max$\otimes$id)$\cdot$(id$\otimes X_-$)

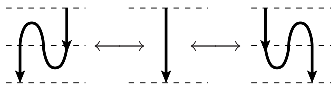
(T5′)  $(\mathrm{id}\otimes\max)\cdot(X_-\otimes\mathrm{id})=(\max\otimes\mathrm{id})\cdot(\mathrm{id}\otimes X_+)$

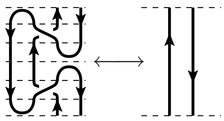
Oriented case:

(T0) *Same as in the unoriented case with arbitrary orientations of participating strings.*

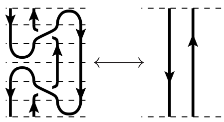
(T1 — T3) *Same as in the unoriented case with orientations of all strings from bottom to top.*

(T4)  $(\overrightarrow{\max}\otimes\mathrm{id})\cdot(\mathrm{id}\otimes\underrightarrow{\min})=\mathrm{id}=(\mathrm{id}\otimes\overleftarrow{\max})\cdot(\underleftarrow{\min}\otimes\mathrm{id})$

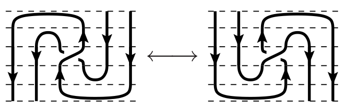
(T4′)  $(\overleftarrow{\max}\otimes\mathrm{id}^*)\cdot(\mathrm{id}^*\otimes\underleftarrow{\min})=\mathrm{id}^*=(\mathrm{id}^*\otimes\overrightarrow{\max})\cdot(\underrightarrow{\min}\otimes\mathrm{id}^*)$

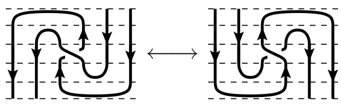
(T5) 

$$(\overleftarrow{\max}\otimes\mathrm{id}\otimes\mathrm{id}^*)\cdot(\mathrm{id}^*\otimes X_-\otimes\mathrm{id}^*)\cdot(\mathrm{id}^*\otimes\mathrm{id}\otimes\underleftarrow{\min})\cdot$$
$$\cdot(\mathrm{id}^*\otimes\mathrm{id}\otimes\overrightarrow{\max})\cdot(\mathrm{id}^*\otimes X_+\otimes\mathrm{id}^*)\cdot(\underrightarrow{\min}\otimes\mathrm{id}\otimes\mathrm{id}^*)=\mathrm{id}\otimes\mathrm{id}^*$$

(T5′) 

$$(\mathrm{id}^*\otimes\mathrm{id}\otimes\overrightarrow{\max})\cdot(\mathrm{id}^*\otimes X_+\otimes\mathrm{id}^*)\cdot(\underrightarrow{\min}\otimes\mathrm{id}\otimes\mathrm{id}^*)\cdot$$
$$\cdot(\overleftarrow{\max}\otimes\mathrm{id}\otimes\mathrm{id}^*)\cdot(\mathrm{id}^*\otimes X_-\otimes\mathrm{id}^*)\cdot(\mathrm{id}^*\otimes\mathrm{id}\otimes\underleftarrow{\min})=\mathrm{id}^*\otimes\mathrm{id}$$

(T6) 

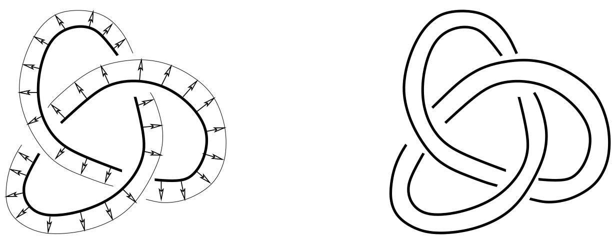
(T6′) 

$$(\overleftarrow{\max}\otimes\mathrm{id}^*\otimes\mathrm{id}^*)\cdot(\mathrm{id}^*\otimes\overleftarrow{\max}\otimes\mathrm{id}\otimes\mathrm{id}^*\otimes\mathrm{id}^*)\cdot$$
$$\cdot(\mathrm{id}^*\otimes\mathrm{id}^*\otimes X_\pm\otimes\mathrm{id}^*\otimes\mathrm{id}^*)\cdot$$
$$\cdot(\mathrm{id}^*\otimes\mathrm{id}^*\otimes\mathrm{id}\otimes\underleftarrow{\min}\otimes\mathrm{id}^*)\cdot(\mathrm{id}^*\otimes\mathrm{id}^*\otimes\underleftarrow{\min})\ =$$
$$=\ (\mathrm{id}^*\otimes\mathrm{id}^*\otimes\overrightarrow{\max})\cdot(\mathrm{id}^*\otimes\mathrm{id}^*\otimes\mathrm{id}\otimes\overrightarrow{\max}\otimes\mathrm{id}^*)\cdot$$
$$\cdot(\mathrm{id}^*\otimes\mathrm{id}^*\otimes X_\pm\otimes\mathrm{id}^*\otimes\mathrm{id}^*)\cdot$$
$$\cdot(\mathrm{id}^*\otimes\underrightarrow{\min}\otimes\mathrm{id}\otimes\mathrm{id}^*\otimes\mathrm{id}^*)\cdot(\underrightarrow{\min}\otimes\mathrm{id}^*\otimes\mathrm{id}^*)$$

1.8. Variations

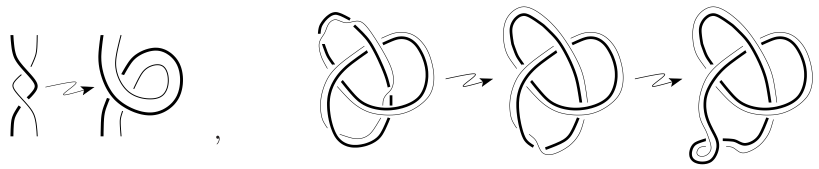
1.8.1. Framed knots. A *framed knot* is a knot equipped with a *framing*, that is, a smooth family of non-zero vectors perpendicular to the knot. Two framings are considered as equivalent, if one can be transformed to another by a smooth deformation. Up to this equivalence relation, a framing is uniquely determined by one integer: the linking number between the knot itself and the curve formed by a small shift of the knot in the direction of the framing. This integer, called the *self-linking number*, can be arbitrary.

One way to choose a framing is to use the *blackboard framing*, defined by a plane knot projection, with the vector field everywhere parallel to the projection plane, for example



A framed knot can also be visualized as a *ribbon knot*, that is, a narrow knotted strip (see the right picture above).

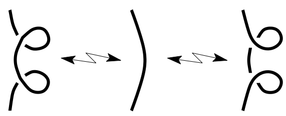
An arbitrary framed knot can be represented by a plane diagram with the blackboard framing. This is achieved by choosing an arbitrary projection and then performing local moves to straighten out the twisted band:



For framed knots (with blackboard framing) the Reidemeister theorem 1.3.1 does not hold since the first Reidemeister move Ω_1 changes the blackboard framing. Here is an appropriate substitute.

1.8.2. Theorem (framed Reidemeister theorem). *Two knot diagrams with blackboard framing D_1 and D_2 are equivalent if and only if D_1 can be transformed into D_2 by a sequence of plane isotopies and local moves of three types $F\Omega_1$, Ω_2, and Ω_3, where*

$F\Omega_1$:



while Ω_2 and Ω_3 are usual Reidemeister moves defined in 1.3.1.

One may also consider framed tangles. These are defined in the same manner as framed knots, with the additional requirement that at each boundary point of the tangle the normal vector is equal to $(\varepsilon, 0, 0)$ for some $\varepsilon > 0$. Framed tangles can be represented by tangle diagrams with blackboard framing. For such tangles there is an analogue of Theorem 1.7.7 — the Turaev move (T1) should be replaced by its framed version that mimics the move $F\Omega_1$.

1.8.3. Long knots. Recall that a long knot is a string link on one string. A long knot can be converted into a usual knot by choosing an orientation (say, upwards) and joining the top and the bottom points by an arc of a sufficiently big circle. It is easy to prove that this construction provides a one-to-one correspondence between the sets of equivalence classes of long knots and knots, and, therefore the two theories are isomorphic.

Some constructions on knots look more natural in the context of long knots. For example, the cut and paste procedure for the connected sum becomes a simple concatenation.

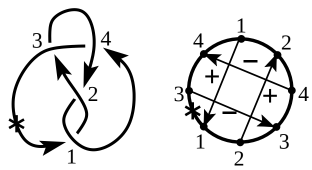
1.8.4. Gauss diagrams and virtual knots. Plane knot diagrams are convenient for presenting knots graphically, but for other purposes, such as coding knots in a computer-recognizable form, *Gauss diagrams* are suited better.

Definition. A Gauss diagram is an oriented circle with a distinguished set of distinct points divided into ordered pairs, each pair carrying a sign ± 1.

Graphically, an ordered pair of points on a circle can be represented by a chord with an arrow connecting them and pointing, say, to the second point. Gauss diagrams are considered up to orientation-preserving homeomorphisms of the circle. Sometimes, an additional basepoint is marked on the circle and the diagrams are considered up to homeomorphisms that keep the basepoint fixed. In this case, we speak of *based Gauss diagrams.*

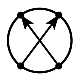
To a plane knot diagram one can associate a Gauss diagram as follows. Pairs of points on the circle correspond to the values of the parameter where the diagram has a self-intersection, each arrow points from the overcrossing to the undercrossing and its sign is equal to the local writhe at the crossing.

Here is an example of a plane knot diagram and the corresponding Gauss diagram:



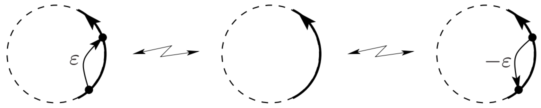
1.8.5. Exercise. What happens to a Gauss diagram, if (a) the knot is mirrored, (b) the knot is reversed?

A knot diagram can be uniquely reconstructed from the corresponding Gauss diagram. We call a Gauss diagram *realizable* if it comes from a knot. Not every Gauss diagram is realizable, the simplest example being

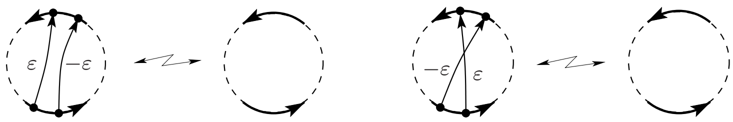
.

As we know, two oriented knot diagrams give the same knot type if and only if they are related by a sequence of oriented Reidemeister moves The corresponding moves translated into the language of Gauss diagrams look as follows:

$V\Omega_1$:



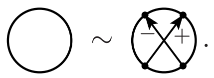
$V\Omega_2$:



$V\Omega_3$:

In fact, the two moves $V\Omega_3$ do not exhaust all the possibilities for representing the third Reidemeister move on Gauss diagrams. It can be shown, however, that all the other versions of the third move are combinations of the moves $V\Omega_2$ and $V\Omega_3$, see the exercises 24 – 26 on page 39 for examples and [**Öll**] for a proof.

These moves, of course, have a geometric meaning only for realizable diagrams. However, they make sense for all Gauss diagrams, whether realizable or not. In particular a realizable diagram may be equivalent to non-realizable one:



Definition. A *virtual knot* is a Gauss diagram considered up to the Reidemeister moves $V\Omega_1$, $V\Omega_2$, $V\Omega_3$. A *long*, or *based* virtual knot is a based Gauss diagram, considered up to Reidemeister moves that do not involve segments with the basepoint on them. Contrary to the case of classical knots, the theories of circular and long virtual knots differ.

It can be shown that the isotopy classes of knots form a subset of the set of virtual knots. In other words, if there is a chain of Reidemeister moves connecting two realizable Gauss diagrams, we can always modify it so that it goes only though realizable diagrams.

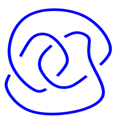
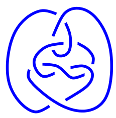
Virtual knots were introduced by L. Kauffman [**Ka5**]. Almost at the same time, they turned up in the work of M. Goussarov, M. Polyak, O. Viro [**GPV**]. There are various geometric interpretations of virtual knots. Many knot invariants are known to extend to invariants of virtual knots.

1.8.6. Knots in arbitrary manifolds. We have defined knots as embeddings of the circle into the Euclidean space $\mathbb{R}^3$. In this definition $\mathbb{R}^3$ can be replaced by the 3-sphere S^3, since the one-point compactification $\mathbb{R}^3 \to S^3$ establishes a one-to-one correspondence between the equivalence classes of knots in both manifolds. Going further and replacing $\mathbb{R}^3$ by an arbitrary 3-manifold M, we can arrive to a theory of knots in M which may well be different from the usual case of knots in $\mathbb{R}^3$; see, for instance, [**Kal, Va6**].

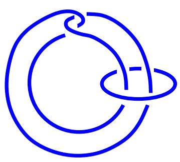
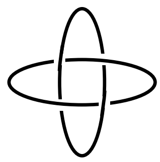
If the dimension of the manifold M is bigger than 3, then all knots in M that represent the same element of the fundamental group $\pi_1(M)$, are isotopic. It does not mean, however, that the theory of knots in M is trivial: the space of all embeddings $S^1 \to M$ may have non-trivial higher homology groups. These homology groups are certainly of interest in dimension 3 too; see [**Va6**]. Another way of doing knot theory in higher-dimensional manifolds is studying multidimensional knots, like embeddings $S^2 \to \mathbb{R}^4$, see, for example, [**Rol**]. An analogue of knot theory for 2-manifolds is Arnold's theory of immersed curves [**Ar3**].

Exercises

(1) Find the following knots in the knot table (page 26):

(a)  (b)  (c) 

(2) Can you find the following links in the picture on page 20?

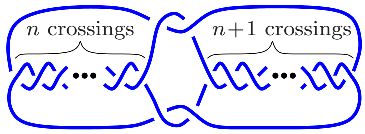
(3) Borromean rings (see page 20) have the property that after deleting any component the remaining two-component link becomes trivial. Links with such property are called *Brunnian*. Find a Brunnian link with 4 components.

(4) Table 1.5.2.1 shows 35 topological types of knots up to change of orientation and taking the mirror images. How many distinct knots do these represent?

(5) Find an isotopy that transforms the knot 6_3 into its mirror image $\overline{6_3}$.

(6) Repeat Perko's achievement: find an isotopy that transforms one of the knots of the Perko pair into another one.

(7) Let G_n be the *Goeritz diagram* [**Goer**] with $2n+5$ crossings, as in the figure below.

$G_n =$ 

(a) Show that G_n represents a trivial knot.

(b) Prove that for $n \geqslant 3$ in any sequence of the Reidemeister moves transforming G_n into the plane circle there is an intermediate knot diagram with more than $2n+5$ crossings.

(c) Find a sequence of 23 Reidemeister moves (containing the Ω_1 move 5 times, the Ω_2 move 7 times, and the Ω_3 move 11 times) transforming G_3 into the plane circle. See the picture of G_3 in 1.2.4 on page 20.

(8) Decompose the knot on the right into a connected sum of prime knots. 

(9) Show that by changing some crossings from overcrossing to undercrossing or vice versa, any knot diagram can be transformed into a diagram of the unknot.

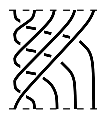
(10) (*C. Adams* [**AdC**]) Show that by changing some crossings from overcrossing to undercrossing or vice versa, any knot diagram can be made alternating.

(11) Represent the knots 4_1, 5_1, 5_2 as closed braids.

(12) Analogously to the braid closure, one can define the closure of a string link. Represent the Whitehead link and the Borromean rings from Section 1.2.5 (page 20) as closures of string links on 2 and 3 strands respectively.

(13) Find a sequence of Markov moves that transforms the closure of the braid $\sigma_1^2\sigma_2^3\sigma_1^4\sigma_2$ into the closure of the braid $\sigma_1^2\sigma_2\sigma_1^4\sigma_2^3$.

(14) Garside's *fundamental braid* $\Delta \in B_n$ is defined as $\Delta := (\sigma_1\sigma_2\dots\sigma_{n-1})(\sigma_1\sigma_2\dots\sigma_{n-2})\dots(\sigma_1\sigma_2)(\sigma_1)$.

$\Delta =$ 

(a) Prove that $\sigma_i\Delta = \Delta\sigma_{n-i}$ for every standard generator $\sigma_i \in B_n$.

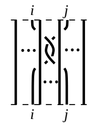
(b) Prove that $\Delta^2 = (\sigma_1\sigma_2\dots\sigma_{n-1})^n$.

(c) Check that Δ^2 belongs to the centre $Z(B_n)$ of the braid group.

(d) Show that any braid can be represented as a product of a certain power (possibly negative) of Δ and a *positive braid*, that is, a braid that contains only positive powers of standard generators σ_i.

In fact, for $n \geqslant 3$, the centre $Z(B_n)$ is the infinite cyclic group generated by Δ^2. The word and conjugacy problems in the braid group were solved by F. Garside [**Gar**]. The structure of positive braids that occur in the last statement was studied in [**Adya, ElMo**].

(15) (a) Prove that the sign of the permutation corresponding to a braid b is equal to the parity of the number of crossings of b, that is $(-1)^{\ell(b)}$, where $\ell(b)$ is the length of b as a word in generators $\sigma_1, \dots, \sigma_{n-1}$.

(b) Prove that the subgroup P_n of pure braids is generated by the braids A_{ij} linking the i-th and j-th strings with each other behind all other strings. $A_{ij} =$ 

(16) Let V be a vector space of dimension n with a distinguished basis $e_1, \dots, e_n$, and let Ξ_i be the counterclockwise $90°$ rotation in the plane $\langle e_i, e_{i+1}\rangle$: $\Xi_i(e_i) = e_{i+1}$, $\Xi_i(e_{i+1}) = -e_i$, $\Xi_i(e_j) = e_j$ for $j \neq i, i+1$. Prove that sending each elementary generator $\sigma_i \in B_n$ to Ξ_i we get a representation $B_n \to GL_n(\mathbb{R})$ of the braid group.

(17) **Burau representation.** Consider the free module over the ring of Laurent polynomials $\mathbb{Z}[x^{\pm 1}]$ with a basis $e_1, \dots, e_n$. The *Burau representation* $B_n \to GL_n(\mathbb{Z}[x^{\pm 1}])$ sends $\sigma_i \in B_n$ to the linear operator that transforms e_i into $(1-x)e_i + e_{i+1}$, and e_{i+1} into xe_i.

(a) Prove that it is indeed a representation of the braid group.

(b) The Burau representation is reducible. It splits into the trivial one-dimensional representation and an $(n-1)$-dimensional irreducible representation which is called the *reduced Burau representation.* Find a basis of the reduced Burau representation where the matrices have the form

$$\sigma_1 \mapsto \begin{pmatrix} -x & x & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \quad \sigma_i \mapsto \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \begin{matrix} 1 & 0 & 0 \\ 1 & -x & x \\ 0 & 0 & 1 \end{matrix} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad \sigma_{n-1} \mapsto \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 1 & -x \end{pmatrix}$$

Answer. $\{xe_1 - e_2, xe_2 - e_3, \dots, xe_{n-1} - e_n\}$

The Burau representation is faithful for $n \leqslant 3$ [**Bir1**], and not faithful for $n \geqslant 5$ [**Big1**]. The case $n = 4$ remains open.

(18) **Lawrence–Krammer–Bigelow representation.** Let V be a free $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ module of dimension $n(n-1)/2$ with a basis $e_{i,j}$ for $1 \leqslant i < j \leqslant n$. The *Lawrence–Krammer–Bigelow representation* can be defined via the action of $\sigma_k \in B_n$ on V:

$$\sigma_k(e_{i,j}) = \begin{cases} e_{i,j} & \text{if } k < i-1 \text{ or } k > j, \\ e_{i-1,j} + (1-q)e_{i,j} & \text{if } k = i-1, \\ tq(q-1)e_{i,i+1} + qe_{i+1,j} & \text{if } k = i < j-1, \\ tq^2 e_{i,j} & \text{if } k = i = j-1, \\ e_{i,j} + tq^{k-i}(q-1)^2 e_{k,k+1} & \text{if } i < k < j-1, \\ e_{i,j-1} + tq^{j-i}(q-1)e_{j-1,j} & \text{if } i < k = j-1, \\ (1-q)e_{i,j} + qe_{i,j+1} & \text{if } k = i-1. \end{cases}$$

Prove that this assignment determines a representation of the braid group. It was shown in [**Big2, Kram**] that this representation is faithful for any $n \geqslant 1$. Therefore the braid group is a linear group.

(19) Represent the knots 4_1, 5_1, 5_2 as products of simple tangles.

(20) Consider the following two knots given as products of simple tangles:

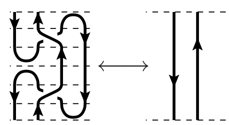
$$(\overleftarrow{\max}\otimes\overrightarrow{\max})\cdot(\mathrm{id}^*\otimes X_+\otimes\mathrm{id}^*)\cdot(\mathrm{id}^*\otimes X_+\otimes\mathrm{id}^*)\cdot(\mathrm{id}^*\otimes X_+\otimes\mathrm{id}^*)\cdot(\underrightarrow{\min}\otimes\underleftarrow{\min})$$

and

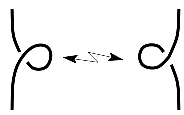
$$\overrightarrow{\max}\cdot(\mathrm{id}\otimes\overrightarrow{\max}\otimes\mathrm{id}^*)\cdot(X_+\otimes\mathrm{id}^*\otimes\mathrm{id}^*)\cdot(X_+\otimes\mathrm{id}^*\otimes\mathrm{id}^*)\cdot(X_+\otimes\mathrm{id}^*\otimes\mathrm{id}^*)\cdot(\mathrm{id}\otimes\underleftarrow{\min}\otimes\mathrm{id}^*)\cdot\underleftarrow{\min}$$

(a) Show that these two knots are equivalent.
(b) Indicate a sequence of the Turaev moves that transforms one product into another.
(c) Forget about the orientations and consider the corresponding unoriented tangles. Find a sequence of unoriented Turaev moves that transforms one product into another.

(21) Represent the oriented tangle move on the right as a sequence of oriented Turaev moves from page 32.



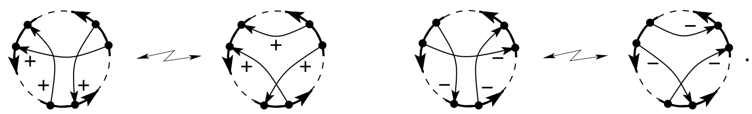
(22) **Whitney trick.** Show that the move $F\Omega_1$ in the framed Reidemeister Theorem 1.8.2 can be replaced by the move shown on the right.



(23) The group $\mathbb{Z}_2^{k+1}$ acts on oriented k-component links, changing the orientation of each component and taking the mirror image of the link. How many different links are there in the orbit of an oriented Whitehead link under this action?

(24) Show that each of the moves $V\Omega_3$ can be obtained as a combination of the moves $V\Omega_2$ with the moves $V\Omega_3'$ below:

$V\Omega_3'$:

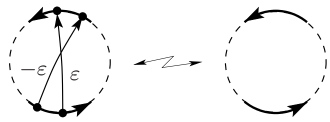


Conversely, show that the moves $V\Omega_3'$ can be obtained as combinations of the moves $V\Omega_2$ and $V\Omega_3$.

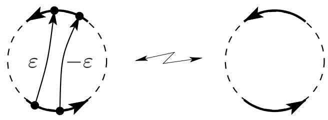
(25) Show that the following moves are equivalent modulo $V\Omega_2$.

This means that either one can be obtained as a combination of another one with the $V\Omega_2$ moves.

(26) (*O.-P. Östlund* [**Öll**]) Show that the second version of $V\Omega_2$:



is redundant. It can be obtained as a combination of the first version,



,

with the moves $V\Omega_1$ and $V\Omega_3$.

Chapter 2

Knot invariants

Knot invariants are functions of knots that do not change under isotopies. The study of knot invariants is at the core of knot theory; indeed, the isotopy class of a knot is, tautologically, a knot invariant.

2.1. Definition and first examples

Let $\mathcal{K}$ be the set of all equivalence classes of knots.

Definition. A *knot invariant with values in a set* S is a map from $\mathcal{K}$ to S.

In the same way one can speak of invariants of links, framed knots, etc.

2.1.1. Crossing number. Any knot can be represented by a plane diagram in infinitely many ways.

Definition. The *crossing number* $c(K)$ of a knot K is the minimal number of crossing points in a plane diagram of K.

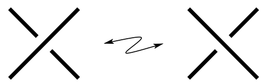
Exercise. Prove that if $c(K) \leqslant 2$, then the knot K is trivial.

It follows that the minimal number of crossings required to draw a diagram of a nontrivial knot is at least 3. A little later we shall see that the trefoil knot is indeed nontrivial.

Obviously, $c(K)$ is a knot invariant taking values in the set of nonnegative integers.

2.1.2. Unknotting number. Another integer-valued invariant of knots which admits a simple definition is the unknotting number.

Represent a knot by a plane diagram. The diagram can be transformed by plane isotopies, Reidemeister moves and crossing changes:



As we know, modifications of the first two kinds preserve the topological type of the knot, and only crossing switches can change it.

Definition. The *unknotting number* $u(K)$ of a knot K is the minimal number of crossing changes in a plane diagram of K that convert it to a trivial knot, provided that any number of plane isotopies and Reidemeister moves is also allowed.

Exercise. What is the unknotting number of the knots 3_1 and 8_3?

Finding the unknotting number, if it is greater than 1, is a difficult task; for example, the second question of the previous exercise was answered only in 1986 (by T. Kanenobu and H. Murakami).

2.1.3. Knot group. The *knot group* is the fundamental group of the complement to the knot in the ambient space: $\pi(K) = \pi_1(\mathbb{R}^3 \setminus K)$. The knot group is a very strong invariant. For example, a knot is trivial if and only if its group is infinite cyclic. More generally, two prime knots with isomorphic fundamental groups are isotopic. For a detailed discussion of knot groups see [**Lik**].

Exercise. Prove that

(1) the group of the trefoil is generated by two elements x, y with one relation $x^2 = y^3$;
(2) this group is isomorphic to the braid group B_3 (in terms of x, y find another pair of generators a, b that satisfy $aba = bab$).

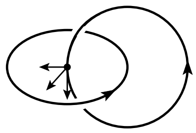
2.2. Linking number

The *linking number* is an example of a Vassiliev invariant of two-component links; it has an analog for framed knots, called *self-linking number*.

Intuitively, the linking number $lk(A, B)$ of two oriented spatial curves A and B is the number of times A winds around B. To give a precise definition, choose an oriented disk D_A immersed in space so that its oriented boundary is the curve A (this means that the ordered pair consisting of an outward-looking normal vector to A and the orienting tangent vector to A gives a positive basis in the tangent space to D_A). The linking number $lk(A, B)$ is then defined as the intersection number of D_A and B. To find the intersection number, if necessary, make a small perturbation of D_A so as to make it meet the curve B only at finitely many points of transversal intersection. At each intersection point, define the sign to be equal to ± 1

depending on the orientations of D_A and B at this point. More specifically, let (e_1, e_2) be a positive pair of tangent vectors to D_A, while e_3 a positively directed tangent vector to B at the intersection point; the sign is set to $+1$ if and only if the frame (e_1, e_2, e_3) defines a positive orientation of $\mathbb{R}^3$. Then the linking number $lk(A, B)$ is the sum of these signs over all intersection points $p \in D_A \cap B$. One can prove that the result does not depend on the choice of the surface D_A and that $lk(A, B) = lk(B, A)$.

Example. The two curves shown in the picture



have their linking number equal to -1.

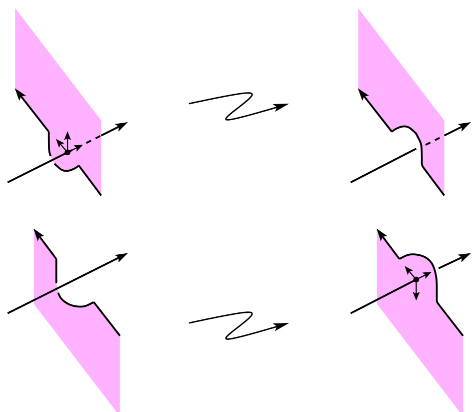
Given a plane diagram of a two-component link, there is a simple combinatorial formula for the linking number. Let I be the set of crossing points involving branches of both components A and B (crossing points involving branches of only one component are irrelevant here). Then I is the disjoint union of two subsets I_B^A (points where A passes over B) and I_A^B (where B passes over A).

2.2.1. Proposition.

$$lk(A, B) = \sum_{p \in I_B^A} w(p) = \sum_{p \in I_A^B} w(p) = \frac{1}{2} \sum_{p \in I} w(p)$$

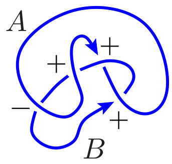
where $w(p)$ is the local writhe of the crossing point.

Proof. Crossing changes at all points $p \in I_A^B$ make the two components unlinked. Call the new curves A' and B', then $lk(A', B') = 0$. It is clear from the pictures below that each crossing switch changes the linking number by $-w$ where w is the local writhe:



Therefore, $lk(A,B) - \sum_{p \in I_A^B} w(p) = 0$, and the assertion follows. □

Example. For the two curves below both ways to compute the linking number give $+1$:



2.2.2. Integral formulae. There are various integral formulae for the linking number. The most famous formula was found by Gauss (see [**Spi**] for a proof).

Theorem. *Let A and B be two non-intersecting curves in $\mathbb{R}^3$, parameterized, respectively, by the smooth functions $\alpha, \beta : S^1 \to \mathbb{R}^3$. Then*

$$lk(A,B) = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{(\beta(v) - \alpha(u), du, dv)}{|\beta(v) - \alpha(u)|^3},$$

where the parentheses in the numerator stand for the mixed product of 3 vectors.

Geometrically, this formula computes the degree of the Gauss map from $A \times B = S^1 \times S^1$ to the 2-sphere S^2, that is, the number of times the normalized vector connecting a point on A to a point on B goes around the sphere.

A different integral formula for the linking number will be stated and proved in Chapter 8, see page 223. It represents the simplest term of the *Kontsevich integral*, which encodes all Vassiliev invariants.

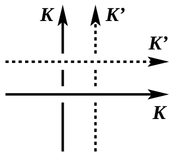
2.2.3. Self-linking. Let K be a framed knot and let K' be the knot obtained from K by a small shift in the direction of the framing.

Definition. The *self-linking number* of K is the linking number of K and K'.

Note, by the way, that the linking number is the same if K is shifted in the direction, *opposite* to the framing.

Proposition. *The self-linking number of a framed knot given by a diagram D with blackboard framing is equal to the total writhe of the diagram D.*

Proof. Indeed, in the case of blackboard framing, the only crossings of K with K' occur near the crossing points of K. The neighbourhood of each crossing point looks like



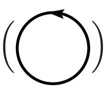
The local writhe of the crossing where K passes over K' is the same as the local writhe of the crossing point of the knot K with itself. Therefore, the claim follows from the combinatorial formula for the linking number (Proposition 2.2.1). $\square$

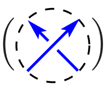
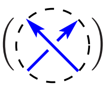
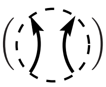
2.3. Conway polynomial

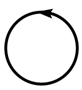
In what follows we shall usually consider invariants with values in a commutative ring. Of special importance in knot theory are *polynomial knot invariants* taking values in the rings of polynomials (or Laurent polynomials[1]) in one or several variables, usually with integer coefficients.

Historically, the first polynomial invariant for knots was the *Alexander polynomial* $A(K)$ introduced in 1928 [**Al**]. See [**CrF, Lik, Rol**] for a discussion of the beautiful topological theory related to the Alexander polynomial. In 1970 J. Conway [**Con**] found a simple recursive construction of a polynomial invariant $C(K)$ which differs from the Alexander polynomial only by a change of variable, namely, $A(K) = C(K) \mid_{t \mapsto x^{1/2} - x^{-1/2}}$. In this book, we only use Conway's normalization. Conway's definition, given in terms of plane diagrams, relies on crossing point resolutions that may take a knot diagram into a link diagram; therefore, we shall speak of links rather than knots.

2.3.1. Definition. The Conway polynomial C is an invariant of oriented links (and, in particular, an invariant of oriented knots) taking values in the ring $\mathbb{Z}[t]$ and defined by the two properties:

$C\Big($  $\Big) = 1,$

$C\Big($  $\Big) - C\Big($  $\Big) = tC\Big($  $\Big)$.

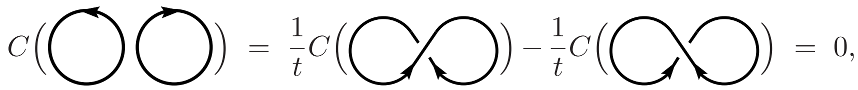
Here  stands for the unknot (trivial knot) while the three pictures in the second line stand for three diagrams that are identical everywhere except for the fragments shown. The second relation is referred to as *Conway's skein relation*. Skein relations are equations on the values of some functions

[1] A *Laurent polynomial* in x is a polynomial in x and x^{-1}.

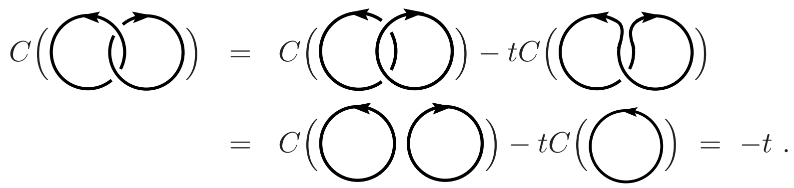
on knots (links, etc.) represented by diagrams that differ from each other by local changes near a crossing point. These relations often give a convenient way to work with knot invariants.

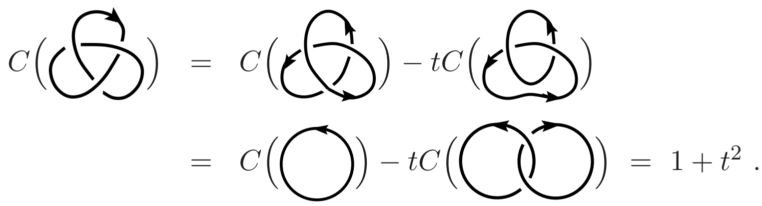
It is not quite trivial to prove the existence of an invariant satisfying this definition, but as soon as this fact is established, the computation of the Conway polynomial becomes fairly easy.

2.3.2. Example.

(i) 

because the two knots on the right are equivalent (both are trivial).

(ii) 

(iii) 

2.3.3. The values of the Conway polynomial on knots with up to 8 crossings are given in Table 2.3.3.1. Note that the Conway polynomial of the inverse knot K^* and the mirror knot $\overline{K}$ coincides with that of knot K.

For every n, the coefficient c_n of t^n in C is a numerical invariant of the knot.

2.4. Jones polynomial

The invention of the Jones polynomial [**Jo1**] in 1985 produced a genuine revolution in knot theory. The original construction of V. Jones was given in terms of state sums and von Neumann algebras. It was soon noted, however, that the Jones polynomial can be defined by skein relations, in the spirit of Conway's definition 2.3.1.

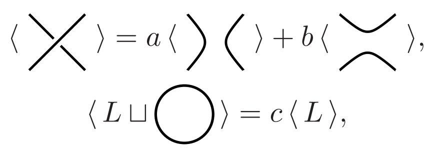
Instead of simply giving the corresponding formal equations, we explain, following L. Kauffman [**Ka6**], *how* this definition could be invented. As with the Conway polynomial, the construction given below requires that we

K	$C(K)$	K	$C(K)$	K	$C(K)$
3_1	$1+t^2$	7_6	$1+t^2-t^4$	8_{11}	$1-t^2-2t^4$
4_1	$1-t^2$	7_7	$1-t^2+t^4$	8_{12}	$1-3t^2+t^4$
5_1	$1+3t^2+t^4$	8_1	$1-3t^2$	8_{13}	$1+t^2+2t^4$
5_2	$1+2t^2$	8_2	$1-3t^4-t^6$	8_{14}	$1-2t^4$
6_1	$1-2t^2$	8_3	$1-4t^2$	8_{15}	$1+4t^2+3t^4$
6_2	$1-t^2-t^4$	8_4	$1-3t^2-2t^4$	8_{16}	$1+t^2+2t^4+t^6$
6_3	$1+t^2+t^4$	8_5	$1-t^2-3t^4-t^6$	8_{17}	$1-t^2-2t^4-t^6$
7_1	$1+6t^2+5t^4+t^6$	8_6	$1-2t^2-2t^4$	8_{18}	$1+t^2-t^4-t^6$
7_2	$1+3t^2$	8_7	$1+2t^2+3t^4+t^6$	8_{19}	$1+5t^2+5t^4+t^6$
7_3	$1+5t^2+2t^4$	8_8	$1+2t^2+2t^4$	8_{20}	$1+2t^2+t^4$
7_4	$1+4t^2$	8_9	$1-2t^2-3t^4-t^6$	8_{21}	$1-t^4$
7_5	$1+4t^2+2t^4$	8_{10}	$1+3t^2+3t^4+t^6$		

Table 2.3.3.1. Conway polynomials of knots with up to 8 crossings

consider invariants on the totality of all links, not only knots, because the transformations used may turn a knot diagram into a link diagram with several components.

Suppose that we are looking for an invariant of unoriented links, denoted by angular brackets, that has a prescribed behaviour with respect to the resolution of diagram crossings and the addition of a disjoint copy of the unknot:



where a, b and c are certain fixed coefficients.

For the bracket $\langle\,,\rangle$ to be a link invariant, it must be stable under the three Reidemeister moves Ω_1, Ω_2, Ω_3 (see Section 1.3).

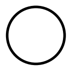
2.4.1. Exercise. Show that the bracket $\langle\,,\rangle$ is Ω_2-invariant if and only if $b=a^{-1}$ and $c=-a^2-a^{-2}$. Prove that Ω_2-invariance in this case implies Ω_3-invariance.

2.4.2. Exercise. Suppose that $b=a^{-1}$ and $c=-a^2-a^{-2}$. Check that the behaviour of the bracket with respect to the first Reidemeister move is described by the equations

$\langle$  $\rangle = -a^{-3} \langle$  $\rangle,$

$\langle$  $\rangle = -a^{3} \langle$  $\rangle.$

In the assumptions $b = a^{-1}$ and $c = -a^2 - a^{-2}$, the bracket polynomial $\langle L \rangle$ normalized by the initial condition

$\langle$  $\rangle = 1$

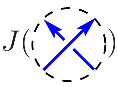
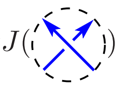
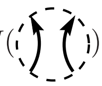
is referred to as the *Kauffman bracket* of L. We see that the Kauffman bracket changes only under the addition (or deletion) of a small loop, and this change depends on the local writhe of the corresponding crossing. It is easy, therefore, to write a formula for a quantity that would be invariant under all three Reidemeister moves:

$$J(L) = (-a)^{-3w} \langle L \rangle,$$

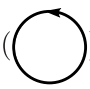
where w is the total writhe of the diagram (the difference between the number of positive and negative crossings).

The invariant $J(L)$ is a Laurent polynomial called the *Jones polynomial* (in *a-normalization*). The more standard *t-normalization* is obtained by the substitution $a = t^{-1/4}$. Note that the Jones polynomial is an invariant of an oriented link, although in its definition we use the Kauffman bracket which is determined by a diagram without orientation.

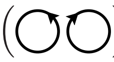
2.4.3. Exercise. Check that the Jones polynomial is uniquely determined by the skein relation

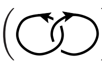
$t^{-1}J($  $) - tJ($  $) = (t^{1/2} - t^{-1/2})J($  $)$ (1)

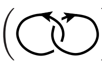
and the initial condition

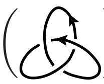
$J($  $) = 1.$ (2)

2.4.4. Example. Let us compute the value of the Jones polynomial on the left trefoil 3_1. The calculation requires several steps, each consisting of one application of the rule (1) and some applications of rule (2) and/or using the results of the previous steps. We leave the details to the reader.

(i) $J\Big($  $\Big) = -t^{1/2} - t^{-1/2}.$

(ii) $J\Big($  $\Big) = -t^{1/2} - t^{5/2}.$

(iii) $J\Big($  $\Big) = -t^{-5/2} - t^{-1/2}.$

(iv) $J\Big($  $\Big) = -t^{-4} + t^{-3} + t^{-1}$.

2.4.5. Exercise. Repeat the previous calculation for the right trefoil and prove that $J(\overline{3_1}) = t + t^3 - t^4$.

We see that the Jones polynomial J can tell apart two knots which the Conway polynomial C cannot. This does not mean, however, that J is stronger than C. There are pairs of knots, for example, $K_1 = 10_{71}$, $K_2 = 10_{104}$ such that $J(K_1) = J(K_2)$, but $C(K_1) \neq C(K_2)$ (see, for instance, [**Sto2**]).

2.4.6. The values of the Jones polynomial on standard knots with up to 8 crossings are given in Table 2.4.6.1. The Jones polynomial does not change when the knot is inverted (this is no longer true for links), see Exercise 26. The behaviour of the Jones polynomial under mirror reflection is described in Exercise 25.

2.5. Algebra of knot invariants

Knot invariants with values in a given commutative ring $\mathcal{R}$ form an algebra $\mathcal{I}$ over that ring with respect to usual pointwise operations on functions

$$(f+g)(K) = f(K) + g(K),$$
$$(fg)(K) = f(K)g(K).$$

Extending knot invariants by linearity to the whole algebra of knots we see that

$$\mathcal{I} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{K}, \mathcal{R}).$$

In particular, as an $\mathcal{R}$-module (or a vector space, if $\mathcal{R}$ is a field) $\mathcal{I}$ is dual to the algebra $\mathcal{RK} := \mathbb{Z}\mathcal{K} \otimes \mathcal{R}$, where $\mathbb{Z}\mathcal{K}$ is the algebra of knots introduced in Section 1.6. The products on $\mathcal{RK}$ and $\mathcal{I}$ are carried by this duality to *co-products* on the space of invariants and on the algebra $\mathcal{RK}$ of knots; later (in Chapter 4.3) we shall see that $\mathcal{RK}$ and $\mathcal{I}$ are, in fact, mutually dual *bialgebras*.

2.6. Quantum invariants

The subject of this section is not entirely elementary. However, we are not going to develop here a full theory of quantum groups and corresponding invariants, confining ourselves to some basic ideas which can be understood without going deep into complicated details. The reader will see that it is possible to use quantum invariants without even knowing what a quantum group is!

3_1	$-t^{-4}+t^{-3}+t^{-1}$
4_1	$t^{-2}-t^{-1}+1-t+t^2$
5_1	$-t^{-7}+t^{-6}-t^{-5}+t^{-4}+t^{-2}$
5_2	$-t^{-6}+t^{-5}-t^{-4}+2t^{-3}-t^{-2}+t^{-1}$
6_1	$t^{-4}-t^{-3}+t^{-2}-2t^{-1}+2-t+t^2$
6_2	$t^{-5}-2t^{-4}+2t^{-3}-2t^{-2}+2t^{-1}-1+t$
6_3	$-t^{-3}+2t^{-2}-2t^{-1}+3-2t+2t^2-t^3$
7_1	$-t^{-10}+t^{-9}-t^{-8}+t^{-7}-t^{-6}+t^{-5}+t^{-3}$
7_2	$-t^{-8}+t^{-7}-t^{-6}+2t^{-5}-2t^{-4}+2t^{-3}-t^{-2}+t^{-1}$
7_3	$t^2-t^3+2t^4-2t^5+3t^6-2t^7+t^8-t^9$
7_4	$t-2t^2+3t^3-2t^4+3t^5-2t^6+t^7-t^8$
7_5	$-t^{-9}+2t^{-8}-3t^{-7}+3t^{-6}-3t^{-5}+3t^{-4}-t^{-3}+t^{-2}$
7_6	$-t^{-6}+2t^{-5}-3t^{-4}+4t^{-3}-3t^{-2}+3t^{-1}-2+t$
7_7	$-t^{-3}+3t^{-2}-3t^{-1}+4-4t+3t^2-2t^3+t^4$
8_1	$t^{-6}-t^{-5}+t^{-4}-2t^{-3}+2t^{-2}-2t^{-1}+2-t+t^2$
8_2	$t^{-8}-2t^{-7}+2t^{-6}-3t^{-5}+3t^{-4}-2t^{-3}+2t^{-2}-t^{-1}+1$
8_3	$t^{-4}-t^{-3}+2t^{-2}-3t^{-1}+3-3t+2t^2-t^3+t^4$
8_4	$t^{-5}-2t^{-4}+3t^{-3}-3t^{-2}+3t^{-1}-3+2t-t^2+t^3$
8_5	$1-t+3t^2-3t^3+3t^4-4t^5+3t^6-2t^7+t^8$
8_6	$t^{-7}-2t^{-6}+3t^{-5}-4t^{-4}+4t^{-3}-4t^{-2}+3t^{-1}-1+t$
8_7	$-t^{-2}+2t^{-1}-2+4t-4t^2+4t^3-3t^4+2t^5-t^6$
8_8	$-t^{-3}+2t^{-2}-3t^{-1}+5-4t+4t^2-3t^3+2t^4-t^5$
8_9	$t^{-4}-2t^{-3}+3t^{-2}-4t^{-1}+5-4t+3t^2-2t^3+t^4$
8_{10}	$-t^{-2}+2t^{-1}-3+5t-4t^2+5t^3-4t^4+2t^5-t^6$
8_{11}	$t^{-7}-2t^{-6}+3t^{-5}-5t^{-4}+5t^{-3}-4t^{-2}+4t^{-1}-2+t$
8_{12}	$t^{-4}-2t^{-3}+4t^{-2}-5t^{-1}+5-5t+4t^2-2t^3+t^4$
8_{13}	$-t^{-3}+3t^{-2}-4t^{-1}+5-5t+5t^2-3t^3+2t^4-t^5$
8_{14}	$t^{-7}-3t^{-6}+4t^{-5}-5t^{-4}+6t^{-3}-5t^{-2}+4t^{-1}-2+t$
8_{15}	$t^{-10}-3t^{-9}+4t^{-8}-6t^{-7}+6t^{-6}-5t^{-5}+5t^{-4}-2t^{-3}+t^{-2}$
8_{16}	$-t^{-6}+3t^{-5}-5t^{-4}+6t^{-3}-6t^{-2}+6t^{-1}-4+3t-t^2$
8_{17}	$t^{-4}-3t^{-3}+5t^{-2}-6t^{-1}+7-6t+5t^2-3t^3+t^4$
8_{18}	$t^{-4}-4t^{-3}+6t^{-2}-7t^{-1}+9-7t+6t^2-4t^3+t^4$
8_{19}	$t^3+t^5-t^8$
8_{20}	$-t^{-5}+t^{-4}-t^{-3}+2t^{-2}-t^{-1}+2-t$
8_{21}	$t^{-7}-2t^{-6}+2t^{-5}-3t^{-4}+3t^{-3}-2t^{-2}+t^{-1}$

Table 2.4.6.1. Jones polynomials of knots with up to 8 crossings

2.6.1. The discovery of the Jones polynomial inspired many people to search for other skein relations compatible with Reidemeister moves and thus defining knot polynomials. This lead to the introduction of the HOMFLY ([**HOM, PT**]) and Kauffman's ([**Ka3, Ka4**]) polynomials. It soon became clear that all these polynomials are the first members of a vast family of knot invariants called *quantum invariants*.

The original idea of quantum invariants (in the case of 3-manifolds) was proposed by E. Witten in the famous paper [**Wit1**]. Witten's approach coming from physics was not completely justified from the mathematical viewpoint. The first mathematically impeccable definition of quantum invariants of links and 3-manifolds was given by Reshetikhin and Turaev [**RT1, Tur2**], who used in their construction the notion of *quantum groups* introduced shortly before that by V. Drinfeld in [**Dr4**] (see also [**Dr3**]) and M. Jimbo in [**Jimb**]. In fact, a *quantum group* is not a group at all. Instead, it is a family of algebras, more precisely, of *Hopf algebras* (see Appendix A.2.21), depending on a complex parameter q and satisfying certain axioms. The *quantum group* $U_q\mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$ is a remarkable deformation of the universal enveloping algebra (see Appendix A.1.7) of $\mathfrak{g}$ (corresponding to the value $q = 1$) in the class of Hopf algebras.

In this section, we show how the Jones polynomial J can be obtained by the techniques of quantum groups, following the approach of Reshetikhin and Turaev. It turns out that J coincides, up to normalization, with the quantum invariant corresponding to the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ in its standard two-dimensional representation (see Appendix A.1.1). Later in the book, we shall sometimes refer to the ideas illustrated in this section. For detailed expositions of quantum groups, we refer the interested reader to [**Jan, Kas, KRT**].

2.6.2. Let $\mathfrak{g}$ be a semisimple Lie algebra and let V be its finite dimensional representation. One can view V as a representation of the universal enveloping algebra $U(\mathfrak{g})$ (see Appendix, page 420). It is remarkable that this representation can also be deformed with parameter q to a representation of the quantum group $U_q\mathfrak{g}$. The vector space V remains the same, but the action now depends on q. For a generic value of q all irreducible representations of $U_q\mathfrak{g}$ can be obtained in this way. However, when q is a root of unity the representation theory is different and resembles the representation theory of $\mathfrak{g}$ in finite characteristic. It can be used to derive quantum invariants of 3-manifolds. For the purposes of knot theory it is enough to use generic values of q.

2.6.3. An important property of quantum groups is that every representation gives rise to a solution R of the *quantum Yang–Baxter equation*

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)$$

where R (the *R-matrix*) is an invertible linear operator $R : V \otimes V \to V \otimes V$, and both sides of the equation are understood as linear transformations $V \otimes V \otimes V \to V \otimes V \otimes V$.

Exercise. Given an R-matrix, construct a representation of the braid group B_n in the space $V^{\otimes n}$.

There is a procedure to construct an R-matrix associated with a representation of a Lie algebra. We are not going to describe it in general, confining ourselves just to one example: the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ and its standard two dimensional representation V (for $\mathfrak{sl}_N$ case see exercise (38) on page 68). In this case the associated R-matrix has the form

$$R: \begin{cases} e_1 \otimes e_1 & \mapsto \quad q^{1/4} e_1 \otimes e_1 \\ e_1 \otimes e_2 & \mapsto \quad q^{-1/4} e_2 \otimes e_1 \\ e_2 \otimes e_1 & \mapsto \quad q^{-1/4} e_1 \otimes e_2 + (q^{1/4} - q^{-3/4}) e_2 \otimes e_1 \\ e_2 \otimes e_2 & \mapsto \quad q^{1/4} e_2 \otimes e_2 \end{cases}$$

for an appropriate basis $\{e_1, e_2\}$ of the space V. The inverse of R (we shall need it later) is given by the formulae

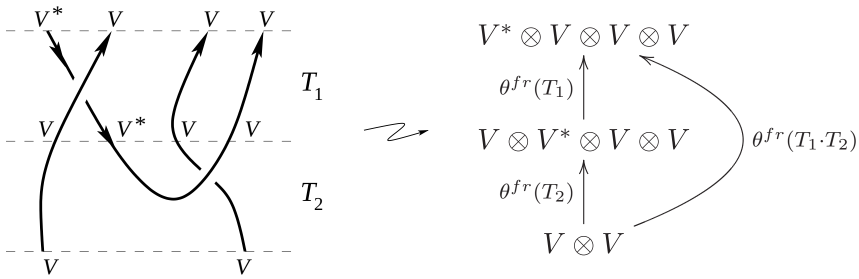
$$R^{-1}: \begin{cases} e_1 \otimes e_1 & \mapsto \quad q^{-1/4} e_1 \otimes e_1 \\ e_1 \otimes e_2 & \mapsto \quad q^{1/4} e_2 \otimes e_1 + (-q^{3/4} + q^{-1/4}) e_1 \otimes e_2 \\ e_2 \otimes e_1 & \mapsto \quad q^{1/4} e_1 \otimes e_2 \\ e_2 \otimes e_2 & \mapsto \quad q^{-1/4} e_2 \otimes e_2 \end{cases}$$

2.6.4. Exercise. Check that this operator R satisfies the quantum Yang-Baxter equation.

2.6.5. The general procedure of constructing quantum invariants is organized as follows (see details in [**Oht1**]). Consider a knot diagram in the plane and take a generic horizontal line. To each intersection point of the line with the diagram we assign either the representation space V or its dual V^* depending on whether the orientation of the knot at this intersection is directed upwards or downwards. Then take the tensor product of all such spaces over the whole horizontal line. If the knot diagram does not intersect the line then the corresponding vector space is the ground field $\mathbb{C}$.

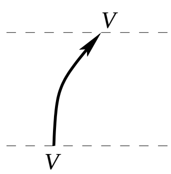
A portion of a knot diagram between two such horizontal lines represents a tangle T (see the general definition in Section 1.7). We assume that this tangle is framed by the blackboard framing. With T we associate a linear transformation $\theta^{fr}(T)$ from the vector space corresponding to the bottom of T to the vector space corresponding to the top of T. The following three properties hold for the linear transformation $\theta^{fr}(T)$:

- $\theta^{fr}(T)$ is an invariant of the isotopy class of the framed tangle T;
- $\theta^{fr}(T_1 \cdot T_2) = \theta^{fr}(T_1) \circ \theta^{fr}(T_2)$;
- $\theta^{fr}(T_1 \otimes T_2) = \theta^{fr}(T_1) \otimes \theta^{fr}(T_2)$.

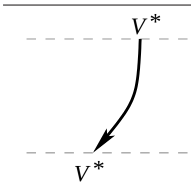


Now we can define a knot invariant $\theta^{fr}(K)$ regarding the knot K as a tangle between the two lines below and above K. In this case $\theta^{fr}(K)$ would be a linear transformation from $\mathbb{C}$ to $\mathbb{C}$, that is, multiplication by a number. Since our linear transformations depend on the parameter q, this number is actually a function of q.

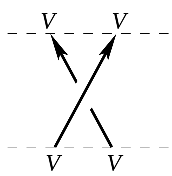
2.6.6. Because of the multiplicativity property $\theta^{fr}(T_1 \cdot T_2) = \theta^{fr}(T_1) \circ \theta^{fr}(T_2)$ it is enough to define $\theta^{fr}(T)$ only for elementary tangles T such as a crossing, a minimum or a maximum point. This is precisely where quantum groups come in. Given a quantum group $U_q\mathfrak{g}$ and its finite dimensional representation V, one can associate certain linear transformations with elementary tangles in a way consistent with the Turaev oriented moves from page 32. The R-matrix appears here as the linear transformation corresponding to a positive crossing, while R^{-1} corresponds to a negative crossing. Of course, for a trivial tangle consisting of a single string connecting the top and bottom, the corresponding linear operator should be the identity transformation. So we have the following correspondence valid for all quantum groups:



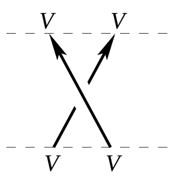
$$\begin{array}{c} V \\ \uparrow{\scriptstyle id_V} \\ V \end{array}$$



$$\begin{array}{c} V^* \\ \uparrow{\scriptstyle id_{V^*}} \\ V^* \end{array}$$

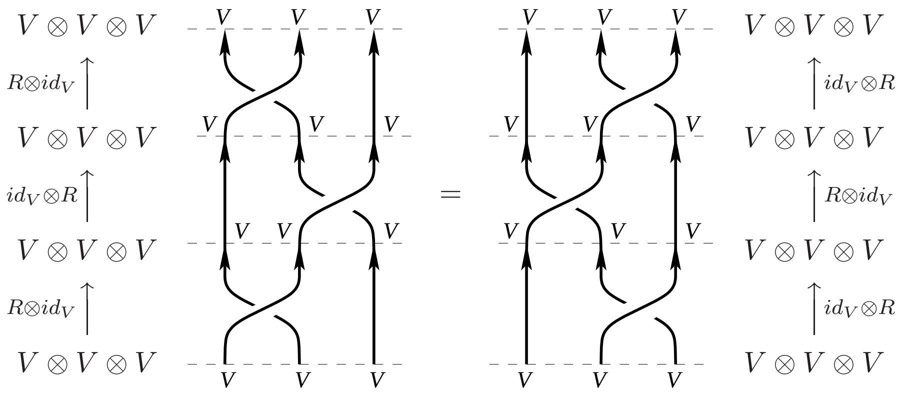


$$\begin{array}{c} V \otimes V \\ \uparrow{\scriptstyle R} \\ V \otimes V \end{array}$$



$$\begin{array}{c} V \otimes V \\ \uparrow{\scriptstyle R^{-1}} \\ V \otimes V \end{array}$$

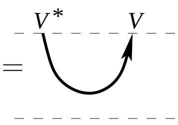
Using this we can easily check that the invariance of a quantum invariant under the third Reidemeister move is nothing else but the quantum Yang–Baxter equation:

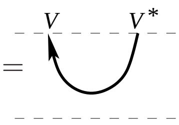


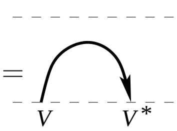
$$(R\otimes id_V)\circ(id_V\otimes R)\circ(R\otimes id_V) \quad = \quad (id_V\otimes R)\circ(R\otimes id_V)\circ(id_V\otimes R)$$

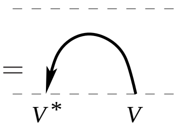
Similarly, the fact that we assigned mutually inverse operators (R and R^{-1}) to positive and negative crossings implies the invariance under the second Reidemeister move. (The first Reidemeister move is treated in Exercise 37a below.)

To complete the construction of our quantum invariant we should assign appropriate operators to the minimum and maximum points. These depend on all the data involved: the quantum group, the representation and the R-matrix. For the quantum group $U_q\mathfrak{sl}_2$, its standard two dimensional representation V and the R-matrix chosen in 2.6.3 these operators are:

$\overrightarrow{\min} =$ 	$V^*\otimes V$ $\uparrow$ $\mathbb{C}$	$q^{-1/2}e^1\otimes e_1 + q^{1/2}e^2\otimes e_2$ $\uparrow$ 1

$\overleftarrow{\min} =$ 	$V\otimes V^*$ $\uparrow$ $\mathbb{C}$	$e_1\otimes e^1 + e_2\otimes e^2$ $\uparrow$ 1

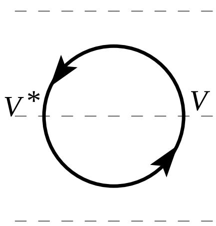
$\overrightarrow{\max} =$ 	$\mathbb{C}$ $\uparrow$ $V\otimes V^*$	$q^{1/2}$ $\uparrow$ $e_1\otimes e^1$	0 $\uparrow$ $e_1\otimes e^2$	0 $\uparrow$ $e_2\otimes e^1$	$q^{-1/2}$ $\uparrow$ $e_2\otimes e^2$

$\overleftarrow{\max} =$ 	$V^* \otimes V \longrightarrow \mathbb{C}$	$e^1 \otimes e_1 \longmapsto 1$	$e^1 \otimes e_2 \longmapsto 0$	$e^2 \otimes e_1 \longmapsto 0$	$e^2 \otimes e_2 \longmapsto 1$

where $\{e^1, e^2\}$ is the basis of V^* dual to the basis $\{e_1, e_2\}$ of the space V.

We leave to the reader the exercise to check that these operators are consistent with the oriented Turaev moves from page 32. See Exercise 38 for their generalization to $\mathfrak{sl}_N$.

2.6.7. Example. Let us compute the $\mathfrak{sl}_2$-quantum invariant of the unknot. Represent the unknot as a product of two tangles and compute the composition of the corresponding transformations

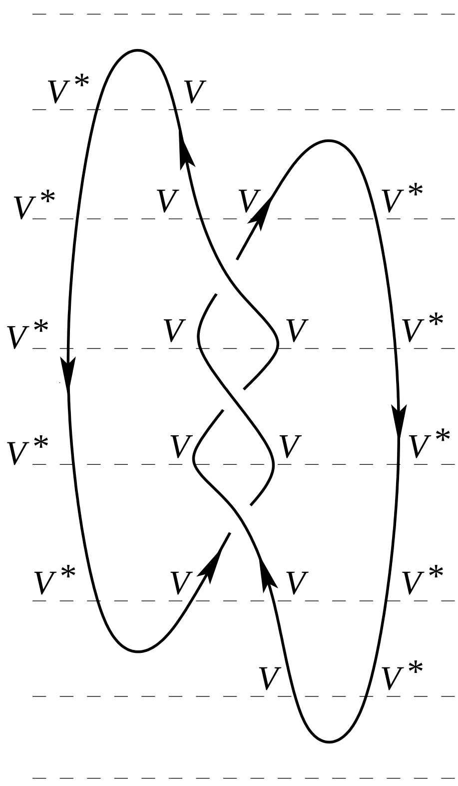


$$\mathbb{C} \longrightarrow V^* \otimes V \longrightarrow \mathbb{C}$$

$$1 \longmapsto \underbrace{\overbrace{q^{-1/2}e^1 \otimes e_1}^{} + \overbrace{q^{1/2}e^2 \otimes e_2}^{}}_{} \longmapsto \underbrace{q^{-1/2}}_{} + \underbrace{q^{1/2}}_{}$$

So $\theta^{fr}(\text{unknot}) = q^{1/2} + q^{-1/2}$. Therefore, in order to normalize our invariant so that its value on the unknot is equal to 1, we must divide $\theta^{fr}(\cdot)$ by $q^{1/2} + q^{-1/2}$. We denote the normalized invariant by $\widetilde{\theta}^{fr}(\cdot) = \frac{\theta^{fr}(\cdot)}{q^{1/2}+q^{-1/2}}$.

2.6.8. Example. Let us compute the quantum invariant for the left trefoil. Represent the diagram of the trefoil as follows.



$$
\begin{tikzcd}
\mathbb{C} \\
V^* \otimes V \arrow[u] \\
V^* \otimes V \otimes V \otimes V^* \arrow[u] \\
V^* \otimes V \otimes V \otimes V^* \arrow[u, "\mathrm{id}_{V^*}\otimes R^{-1}\otimes \mathrm{id}_{V^*}"'] \\
V^* \otimes V \otimes V \otimes V^* \arrow[u, "\mathrm{id}_{V^*}\otimes R^{-1}\otimes \mathrm{id}_{V^*}"'] \\
V^* \otimes V \otimes V \otimes V^* \arrow[u, "\mathrm{id}_{V^*}\otimes R^{-1}\otimes \mathrm{id}_{V^*}"'] \\
V \otimes V^* \arrow[u] \\
\mathbb{C} \arrow[u]
\end{tikzcd}
$$

Two maps at the bottom send $1 \in \mathbb{C}$ into the tensor

$$
\begin{aligned}
1 \quad \mapsto \quad & q^{-1/2} e^1 \otimes e_1 \otimes e_1 \otimes e^1 \quad + \quad q^{-1/2} e^1 \otimes e_1 \otimes e_2 \otimes e^2 \\
& \qquad + \quad q^{1/2} e^2 \otimes e_2 \otimes e_1 \otimes e^1 \quad + \quad q^{1/2} e^2 \otimes e_2 \otimes e_2 \otimes e^2 \ .
\end{aligned}
$$

Then applying R^{-3} to two tensor factors in the middle we get

$$
\begin{aligned}
& q^{-1/2} e^1 \otimes \Big(q^{-3/4} e_1 \otimes e_1 \Big) \otimes e^1 \\
& + q^{-1/2} e^1 \otimes \Big(\big(-q^{9/4} + q^{5/4} - q^{1/4} + q^{-3/4} \big) e_1 \otimes e_2 \\
& \qquad\qquad + \big(-q^{7/4} - q^{3/4} - q^{-1/4} \big) e_2 \otimes e_1 \Big) \otimes e^2 \\
& + q^{1/2} e^2 \otimes \Big(\big(q^{7/4} - q^{3/4} + q^{-1/4} \big) e_1 \otimes e_2 + \big(-q^{5/4} + q^{1/4} \big) e_2 \otimes e_1 \Big) \otimes e^1 \\
& + q^{1/2} e^2 \otimes \Big(q^{-3/4} e_2 \otimes e_2 \Big) \otimes e^2 \ .
\end{aligned}
$$

Finally, the two maps at the top contract the whole tensor into a number

$$\begin{aligned}\theta^{fr}(3_1) &= q^{-1/2}q^{-3/4}q^{1/2} + q^{-1/2}\big(-q^{9/4}+q^{5/4}-q^{1/4}+q^{-3/4}\big)q^{-1/2}\\ &\quad +q^{1/2}\big(-q^{5/4}+q^{1/4}\big)q^{1/2} + q^{1/2}q^{-3/4}q^{-1/2}\\ &= 2q^{-3/4}-q^{5/4}+q^{1/4}-q^{-3/4}+q^{-7/4}-q^{9/4}+q^{5/4}\\ &= q^{-7/4}+q^{-3/4}+q^{1/4}-q^{9/4}\end{aligned}$$

Dividing by the normalizing factor $q^{1/2}+q^{-1/2}$ we get

$$\frac{\theta^{fr}(3_1)}{q^{1/2}+q^{-1/2}} \quad = \quad q^{-5/4}+q^{3/4}-q^{7/4}\,.$$

The invariant $\theta^{fr}(K)$ remains unchanged under the second and third Reidemeister moves. However, it varies under the first Reidemeister move and thus depends on the framing. One can *deframe* it according to the formula

$$\theta(K) = q^{-\frac{c\cdot w(K)}{2}}\theta^{fr}(K)\,,$$

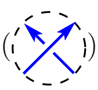
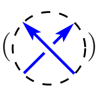
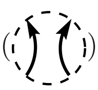
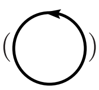
where $w(K)$ is the writhe of the knot diagram and c is the quadratic Casimir number (see Appendix A.1.1) defined by the Lie algebra $\mathfrak{g}$ and its representation. For $\mathfrak{sl}_2$ and the standard 2-dimensional representation $c = 3/2$. The writhe of the left trefoil in our example equals -3. Hence for the unframed normalized quantum invariant we have

$$\widetilde{\theta}(3_1) = \frac{\theta(3_1)}{q^{1/2}+q^{-1/2}} \quad = \quad q^{9/4}\Big(q^{-5/4}+q^{3/4}-q^{7/4}\Big) \quad = \quad q+q^3-q^4\,.$$

The substitution $q = t^{-1}$ gives the Jones polynomial $t^{-1}+t^{-3}-t^{-4}$.

2.7. Two-variable link polynomials

2.7.1. HOMFLY polynomial. The *HOMFLY polynomial* $P(L)$ is an unframed link invariant. It is defined as the Laurent polynomial in two variables a and z with integer coefficients satisfying the following skein relation and the initial condition:

$aP($$)$ $-$ $a^{-1}P($$)$ $=$ $zP($$)$; $P($$)$ $=$ 1.

The existence of such an invariant is a difficult theorem. It was established simultaneously and independently by five groups of authors [**HOM, PT**] (see also [**Lik**]). The HOMFLY polynomial is equivalent to the collection of quantum invariants associated with the Lie algebra $\mathfrak{sl}_N$ and its standard N-dimensional representation for all values of N (see Exercise 38 on page 68 for details).

3_1	$(2a^2-a^4)+a^2z^2$
4_1	$(a^{-2}-1+a^2)-z^2$
5_1	$(3a^4-2a^6)+(4a^4-a^6)z^2+a^4z^4$
5_2	$(a^2+a^4-a^6)+(a^2+a^4)z^2$
6_1	$(a^{-2}-a^2+a^4)+(-1-a^2)z^2$
6_2	$(2-2a^2+a^4)+(1-3a^2+a^4)z^2-a^2z^4$
6_3	$(-a^{-2}+3-a^2)+(-a^{-2}+3-a^2)z^2+z^4$
7_1	$(4a^6-3a^8)+(10a^6-4a^8)z^2+(6a^6-a^8)z^4+a^6z^6$
7_2	$(a^2+a^6-a^8)+(a^2+a^4+a^6)z^2$
7_3	$(a^{-4}+2a^{-6}-2a^{-8})+(3a^{-4}+3a^{-6}-a^{-8})z^2+(a^{-4}+a^{-6})z^4$
7_4	$(2a^{-4}-a^{-8})+(a^{-2}+2a^{-4}+a^{-6})z^2$
7_5	$(2a^4-a^8)+(3a^4+2a^6-a^8)z^2+(a^4+a^6)z^4$
7_6	$(1-a^2+2a^4-a^6)+(1-2a^2+2a^4)z^2-a^2z^4$
7_7	$(a^{-4}-2a^{-2}+2)+(-2a^{-2}+2-a^2)z^2+z^4$
8_1	$(a^{-2}-a^4+a^6)+(-1-a^2-a^4)z^2$
8_2	$(3a^2-3a^4+a^6)+(4a^2-7a^4+3a^6)z^2+(a^2-5a^4+a^6)z^4-a^4z^6$
8_3	$(a^{-4}-1+a^4)+(-a^{-2}-2-a^2)z^2$
8_4	$(a^4-2+2a^{-2})+(a^4-2a^2-3+a^{-2})z^2+(-a^2-1)z^4$
8_5	$(4a^{-2}-5a^{-4}+2a^{-6})+(4a^{-2}-8a^{-4}+3a^{-6})z^2$ $+(a^{-2}-5a^{-4}+a^{-6})z^4-a^{-4}z^6$
8_6	$(2-a^2-a^4+a^6)+(1-2a^2-2a^4+a^6)z^2+(-a^2-a^4)z^4$
8_7	$(-2a^{-4}+4a^{-2}-1)+(-3a^{-4}+8a^{-2}-3)z^2+(-a^{-4}+5a^{-2}-1)z^4$ $+a^{-2}z^6$
8_8	$(-a^{-4}+a^{-2}+2-a^2)+(-a^{-4}+2a^{-2}+2-a^2)z^2+(a^{-2}+1)z^4$
8_9	$(2a^{-2}-3+2a^2)+(3a^{-2}-8+3a^2)z^2+(a^{-2}-5+a^2)z^4-z^6$
8_{10}	$(-3a^{-4}+6a^{-2}-2)+(-3a^{-4}+9a^{-2}-3)z^2+(-a^{-4}+5a^{-2}-1)z^4$ $+a^{-2}z^6$
8_{11}	$(1+a^2-2a^4+a^6)+(1-a^2-2a^4+a^6)z^2+(-a^2-a^4)z^4$
8_{12}	$(a^{-4}-a^{-2}+1-a^2+a^4)+(-2a^{-2}+1-2a^2)z^2+z^4$
8_{13}	$(-a^{-4}+2a^{-2})+(-a^{-4}+2a^{-2}+1-a^2)z^2+(a^{-2}+1)z^4$
8_{14}	$1+(1-a^2-a^4+a^6)z^2+(-a^2-a^4)z^4$
8_{15}	$(a^4+3a^6-4a^8+a^{10})+(2a^4+5a^6-3a^8)z^2+(a^4+2a^6)z^4$
8_{16}	$(-a^4+2a^2)+(-2a^4+5a^2-2)z^2+(-a^4+4a^2-1)z^4+a^2z^6$
8_{17}	$(a^{-2}-1+a^2)+(2a^{-2}-5+2a^2)z^2+(a^{-2}-4+a^2)z^4-z^6$
8_{18}	$(-a^{-2}+3-a^2)+(a^{-2}-1+a^2)z^2+(a^{-2}-3+a^2)z^4-z^6$
8_{19}	$(5a^{-6}-5a^{-8}+a^{-10})+(10a^{-6}-5a^{-8})z^2+(6a^{-6}-a^{-8})z^4+a^{-6}z^6$
8_{20}	$(-2a^4+4a^2-1)+(-a^4+4a^2-1)z^2+a^2z^4$
8_{21}	$(3a^2-3a^4+a^6)+(2a^2-3a^4+a^6)z^2-a^4z^4$

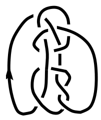
Table 2.7.1.1. HOMFLY polynomials of knots with up to 8 crossings

Important properties of the HOMFLY polynomial are contained in the following exercises.

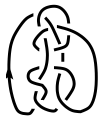
2.7.2. Exercise.

(1) Prove the uniqueness of such invariant. In other words, prove that the relation above are sufficient to compute the HOMFLY polynomial.

(2) Compute the HOMFLY polynomial for the knots 3_1, 4_1 and compare your results with those given in Table 2.7.1.1.

$C =$ 

(3) (*A. Sossinsky* [**Sos**]) Compare the HOMFLY polynomials of the Conway and Kinoshita-Terasaka knots on the right.

$KT =$ 

2.7.3. Exercise. Prove that the HOMFLY polynomial of a knot is preserved when the knot orientation is reversed.

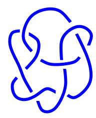
2.7.4. Exercise. (*W. B. R. Lickorish* [**Lik**]) Prove that

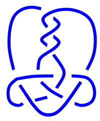
(1) $P(\overline{L}) = \overline{P(L)}$, where $\overline{L}$ is the mirror reflection of L and $\overline{P(L)}$ is the polynomial obtained from $P(L)$ by substituting a^{-1} for a;

(2) $P(K_1 \# K_2) = P(K_1) \cdot P(K_2)$;

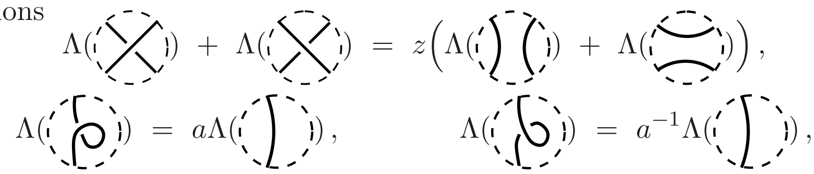
(3) $P(L_1 \sqcup L_2) = \dfrac{a - a^{-1}}{z} \cdot P(L_1) \cdot P(L_2)$, where $L_1 \sqcup L_2$ means the split union of links (that is, the union of L_1 and L_2 such that each of these two links is contained inside its own ball, and the two balls do not have common points);

(4) $P(8_8) = P(10_{129})$.
These knots can be distinguished by the two-variable Kauffman polynomial defined below.

$8_8 =$ 

$10_{129} =$ 

2.7.5. Two-variable Kauffman polynomial. In [**Ka4**], L. Kauffman found another invariant Laurent polynomial $F(L)$ in two variables a and z. Firstly, for a unoriented link diagram D we define a polynomial $\Lambda(D)$ which is invariant under Reidemeister moves Ω_2 and Ω_3 and satisfies the relations



and the initial condition $\Lambda(\bigcirc) = 1$.

Now, for any diagram D of an oriented link L we put

$$F(L) := a^{-w(D)}\Lambda(D).$$

It turns out that this polynomial is equivalent to the collection of the quantum invariants associated with the Lie algebra $\mathfrak{so}_N$ and its standard N-dimensional representation for all values of N (see [**Tur3**]).

3_1	$(-2a^2 - a^4) + (a^3 + a^5)z + (a^2 + a^4)z^2$
4_1	$(-a^{-2} - 1 - a^2) + (-a^{-1} - a)z + (a^{-2} + 2 + a^2)z^2 + (a^{-1} + a)z^3$
5_1	$(3a^4 + 2a^6) + (-2a^5 - a^7 + a^9)z + (-4a^4 - 3a^6 + a^8)z^2$ $+(a^5 + a^7)z^3 + (a^4 + a^6)z^4$
5_2	$(-a^2 + a^4 + a^6) + (-2a^5 - 2a^7)z + (a^2 - a^4 - 2a^6)z^2$ $+(a^3 + 2a^5 + a^7)z^3 + (a^4 + a^6)z^4$
6_1	$(-a^{-2} + a^2 + a^4) + (2a + 2a^3)z + (a^{-2} - 4a^2 - 3a^4)z^2$ $+(a^{-1} - 2a - 3a^3)z^3 + (1 + 2a^2 + a^4)z^4 + (a + a^3)z^5$
6_2	$(2 + 2a^2 + a^4) + (-a^3 - a^5)z + (-3 - 6a^2 - 2a^4 + a^6)z^2$ $+(-2a + 2a^5)z^3 + (1 + 3a^2 + 2a^4)z^4 + (a + a^3)z^5$
6_3	$(a^{-2} + 3 + a^2) + (-a^{-3} - 2a^{-1} - 2a - a^3)z + (-3a^{-2} - 6 - 3a^2)z^2$ $+(a^{-3} + a^{-1} + a + a^3)z^3 + (2a^{-2} + 4 + 2a^2)z^4 + (a^{-1} + a)z^5$
7_1	$(-4a^6 - 3a^8) + (3a^7 + a^9 - a^{11} + a^{13})z + (10a^6 + 7a^8 - 2a^{10} + a^{12})z^2$ $+(-4a^7 - 3a^9 + a^{11})z^3 + (-6a^6 - 5a^8 + a^{10})z^4 + (a^7 + a^9)z^5$ $+(a^6 + a^8)z^6$
7_2	$(-a^2 - a^6 - a^8) + (3a^7 + 3a^9)z + (a^2 + 3a^6 + 4a^8)z^2$ $+(a^3 - a^5 - 6a^7 - 4a^9)z^3 + (a^4 - 3a^6 - 4a^8)z^4 + (a^5 + 2a^7 + a^9)z^5$ $+(a^6 + a^8)z^6$
7_3	$(-2a^{-8} - 2a^{-6} + a^{-4}) + (-2a^{-1}1 + a^{-9} + 3a^{-7})z$ $+(-a^{-10} + 6a^{-8} + 4a^{-6} - 3a^{-4})z^2 + (a^{-11} - a^{-9} - 4a^{-7} - 2a^{-5})z^3$ $+(a^{-10} - 3a^{-8} - 3a^{-6} + a^{-4})z^4 + (a^{-9} + 2a^{-7} + a^{-5})z^5$ $+(a^{-8} + a^{-6})z^6$
7_4	$(-a^{-8} + 2a^{-4}) + (4a^{-9} + 4a^{-7})z + (2a^{-8} - 3a^{-6} - 4a^{-4} + a^{-2})z^2$ $+(-4a^{-9} - 8a^{-7} - 2a^{-5} + 2a^{-3})z^3 + (-3a^{-8} + 3a^{-4})z^4$ $+(a^{-9} + 3a^{-7} + 2a^{-5})z^5 + (a^{-8} + a^{-6})z^6$
7_5	$(2a^4 - a^8) + (-a^5 + a^7 + a^9 - a^{11})z + (-3a^4 + a^8 - 2a^{10})z^2$ $+(-a^5 - 4a^7 - 2a^9 + a^{11})z^3 + (a^4 - a^6 + 2a^{10})z^4$ $+(a^5 + 3a^7 + 2a^9)z^5 + (a^6 + a^8)z^6$
7_6	$(1 + a^2 + 2a^4 + a^6) + (a + 2a^3 - a^7)z + (-2 - 4a^2 - 4a^4 - 2a^6)z^2$ $+(-4a - 6a^3 - a^5 + a^7)z^3 + (1 + a^2 + 2a^4 + 2a^6)z^4$ $+(2a + 4a^3 + 2a^5)z^5 + (a^2 + a^4)z^6$
7_7	$(a^{-4} + 2a^{-2} + 2) + (2a^{-3} + 3a^{-1} + a)z + (-2a^{-4} - 6a^{-2} - 7 - 3a^2)z^2$ $+(-4a^{-3} - 8a^{-1} - 3a + a^3)z^3 + (a^{-4} + 2a^{-2} + 4 + 3a^2)z^4$ $+(2a^{-3} + 5a^{-1} + 3a)z^5 + (a^{-2} + 1)z^6$

Table 2.7.5.1. Kauffman polynomials of knots with up to 7 crossings

8_1	$(-a^{-2} - a^4 - a^6) + (-3a^3 - 3a^5)z + (a^{-2} + 7a^4 + 6a^6)z^2$
	$+(a^{-1} - a^{+}5a^3 + 7a^5)z^3 + (1 - 2a^2 - 8a^4 - 5a^6)z^4$
	$+(a - 4a^3 - 5a^5)z^5 + (a^2 + 2a^4 + a^6)z^6 + (a^3 + a^5)z^7$
8_2	$(-3a^2 - 3a^4 - a^6) + (a^3 + a^5 - a^7 - a^9)z$
	$+(7a^2 + 12a^4 + 3a^6 - a^8 + a^{10})z^2 + (3a^3 - a^5 - 2a^7 + 2a^9)z^3$
	$+(-5a^2 - 12a^4 - 5a^6 + 2a^8)z^4 + (-4a^3 - 2a^5 + 2a^7)z^5$
	$+(a^2 + 3a^4 + 2a^6)z^6 + (a^3 + a^5)z^7$
8_3	$(a^{-4} - 1 + a^4) + (-4a^{-1} - 4a)z + (-3a^{-4} + a^{-2} + 8 + a^2 - 3a^4)z^2$
	$+(-2a^{-3} + 8a^{-1} + 8a - 2a^3)z^3 + (a^{-4} - 2a^{-2} - 6 - 2a^2 + a^4)z^4$
	$+(a^{-3} - 4a^{-1} - 4a + a^3)z^5 + (a^{-2} + 2 + a^2)z^6 + (a^{-1} + a)z^7$
8_4	$(-2a^{-2} - 2 + a^4) + (-a^{-1} + a + 2a^3)z$
	$+(7a^{-2} + 10 - a^2 - 3a^4 + a^6)z^2 + (4a^{-1} - 3a - 5a^3 + 2a^5)z^3$
	$+(-5a^{-2} - 11 - 3a^2 + 3a^4)z^4 + (-4a^{-1} - a + 3a^3)z^5$
	$+(a^{-2} + 3 + 2a^2)z^6 + (a^{-1} + a)z^7$
8_5	$(-2a^{-6} - 5a^{-4} - 4a^{-2}) + (4a^{-7} + 7a^{-5} + 3a^{-3})z$
	$+(a^{-10} - 2a^{-8} + 4a^{-6} + 15a^{-4} + 8a^{-2})z^2 + (2a^{-9} - 8a^{-7} - 10a^{-5})z^3$
	$+(3a^{-8} - 7a^{-6} - 15a^{-4} - 5a^{-2})z^4 + (4a^{-7} + a^{-5} - 3a^{-3})z^5$
	$+(3a^{-6} + 4a^{-4} + a^{-2})z^6 + (a^{-5} + a^{-3})z^7$
8_6	$(2 + a^2 - a^4 - a^6) + (-a - 3a^3 - a^5 + a^7)z$
	$+(-3 - 2a^2 + 6a^4 + 3a^6 - 2a^8)z^2 + (-a + 5a^3 + 2a^5 - 4a^7)z^3$
	$+(1 - 6a^4 - 4a^6 + a^8)z^4 + (a - 2a^3 - a^5 + 2a^7)z^5$
	$+(a^2 + 3a^4 + 2a^6)z^6 + (a^3 + a^5)z^7$
8_7	$(-2a^{-4} - 4a^{-2} - 1) + (-a^{-7} + 2a^{-3} + 2a^{-1} + a)z$
	$+(-2a^{-6} + 4a^{-4} + 12a^{-2} + 6)z^2 + (a^{-7} - a^{-5} - 2a^{-3} - 3a^{-1} - 3a)z^3$
	$+(2a^{-6} - 3a^{-4} - 12a^{-2} - 7)z^4 + (2a^{-5} - a^{-1} + a)z^5$
	$+(2a^{-4} + 4a^{-2} + 2)z^6 + (a^{-3} + a^{-1})z^7$
8_8	$(-a^{-4} - a^{-2} + 2 + a^2) + (2a^{-5} + 3a^{-3} + a^{-1} - a - a^3)z$
	$+(4a^{-4} + 5a^{-2} - 1 - 2a^2)z^2 + (-3a^{-5} - 5a^{-3} - 3a^{-1} + a^3)z^3$
	$+(-6a^{-4} - 9a^{-2} - 1 + 2a^2)z^4 + (a^{-5} + a^{-1} + 2a)z^5$
	$+(2a^{-4} + 4a^{-2} + 2)z^6 + (a^{-3} + a^{-1})z^7$
8_9	$(-2a^{-2} - 3 - 2a^2) + (a^{-3} + a^{-1} + a + a^3)z$
	$+(-2a^{-4} + 4a^{-2} + 12 + 4a^2 - 2a^4)z^2 + (-4a^{-3} - a^{-1} - a - 4a^3)z^3$
	$+(a^{-4} - 4a^{-2} - 10 - 4a^2 + a^4)z^4 + (2a^{-3} + 2a^3)z^5$
	$+(2a^{-2} + 4 + 2a^2)z^6 + (a^{-1} + a)z^7$
8_{10}	$(-3a^{-4} - 6a^{-2} - 2) + (-a^{-7} + 2a^{-5} + 6a^{-3} + 5a^{-1} + 2a)z$
	$+(-a^{-6} + 6a^{-4} + 12a^{-2} + 5)z^2 + (a^{-7} - 3a^{-5} - 9a^{-3} - 8a^{-1} - 3a)z^3$
	$+(2a^{-6} - 5a^{-4} - 13a^{-2} - 6)z^4 + (3a^{-5} + 3a^{-3} + a^{-1} + a)z^5$
	$+(3a^{-4} + 5a^{-2} + 2)z^6 + (a^{-3} + a^{-1})z^7$

Table 2.7.5.1. Kauffman polynomials of knots with 8 crossings

8_{11}	$(1-a^2-2a^4-a^6)+(a^3+3a^5+2a^7)z+(-2+6a^4+2a^6-2a^8)z^2$
	$+(-3a-2a^3-3a^5-4a^7)z^3+(1-2a^2-7a^4-3a^6+a^8)z^4$
	$+(2a+a^3+a^5+2a^7)z^5+(2a^2+4a^4+2a^6)z^6+(a^3+a^5)z^7$
8_{12}	$(a^{-4}+a^{-2}+1+a^2+a^4)+(a^{-3}+a^3)z$
	$+(-2a^{-4}-2a^{-2}-2a^2-2a^4)z^2+(-3a^{-3}-3a^{-1}-3a-3a^3)z^3$
	$+(a^{-4}-a^{-2}-4-a^2+a^4)z^4+(2a^{-3}+2a^{-1}+2a+2a^3)z^5$
	$+(2a^{-2}+4+2a^2)z^6+(a^{-1}+a)z^7$
8_{13}	$(-a^{-4}-2a^{-2})+(2a^{-5}+4a^{-3}+3a^{-1}+a)z+(5a^{-4}+7a^{-2}-2a^2)z^2$
	$+(-3a^{-5}-7a^{-3}-9a^{-1}-4a+a^3)z^3+(-6a^{-4}-11a^{-2}-2+3a^2)z^4$
	$+(a^{-5}+a^{-3}+4a^{-1}+4a)z^5+(2a^{-4}+5a^{-2}+3)z^6+(a^{-3}+a^{-1})z^7$
8_{14}	$1+(a+3a^3+3a^5+a^7)z+(-2-a^2+3a^4+a^6-a^8)z^2$
	$+(-3a-6a^3-8a^5-5a^7)z^3+(1-a^2-7a^4-4a^6+a^8)z^4$
	$+(2a+3a^3+4a^5+3a^7)z^5+(2a^2+5a^4+3a^6)z^6+(a^3+a^5)z^7$
8_{15}	$(a^4-3a^6-4a^8-a^{10})+(6a^7+8a^9+2a^{11})z$
	$+(-2a^4+5a^6+8a^8-a^{12})z^2+(-2a^5-11a^7-14a^9-5a^{11})z^3$
	$+(a^4-5a^6-10a^8-3a^{10}+a^{12})z^4+(2a^5+5a^7+6a^9+3a^{11})z^5$
	$+(3a^6+6a^8+3a^{10})z^6+(a^7+a^9)z^7$
8_{16}	$(-2a^2-a^4)+(a^{-1}+3a^{+}4a^3+2a^5)z+(5+10a^2+4a^4-a^6)z^2$
	$+(-2a^{-1}-6a-10a^3-5a^5+a^7)z^3+(-8-18a^2-7a^4+3a^6)z^4$
	$+(a^{-1}-a+3a^3+5a^5)z^5+(3+8a^2+5a^4)z^6+(2a+2a^3)z^7$
8_{17}	$(-a^{-2}-1-a^2)+(a^{-3}+2a^{-1}+2a+a^3)z$
	$+(-a^{-4}+3a^{-2}+8+3a^2-a^4)z^2+(-4a^{-3}-6a^{-1}-6a-4a^3)z^3$
	$+(a^{-4}-6a^{-2}-14-6a^2+a^4)z^4+(3a^{-3}+2a^{-1}+2a+3a^3)z^5$
	$+(4a^{-2}+8+4a^2)z^6+(2a^{-1}+2a)z^7$
8_{18}	$(a^{-2}+3+a^2)+(a^{-1}+a)z+(3a^{-2}+6+3a^2)z^2$
	$+(-4a^{-3}-9a^{-1}-9a-4a^3)z^3+(a^{-4}-9a^{-2}-20-9a^2+a^4)z^4$
	$+(4a^{-3}+3a^{-1}+3a+4a^3)z^5+(6a^{-2}+12+6a^2)z^6+(3a^{-1}+3a)z^7$
8_{19}	$(-a^{-10}-5a^{-8}-5a^{-6})+(5a^{-9}+5a^{-7})z+(10a^{-8}+10a^{-6})z^2$
	$+(-5a^{-9}-5a^{-7})z^3+(-6a^{-8}-6a^{-6})z^4+(a^{-9}+a^{-7})z^5$
	$+(a^{-8}+a^{-6})z^6$
8_{20}	$(-1-4a^2-2a^4)+(a^{-1}+3a+5a^3+3a^5)z+(2+6a^2+4a^4)z^2$
	$+(-3a-7a^3-4a^5)z^3+(-4a^2-4a^4)z^4+(a+2a^3+a^5)z^5$
	$+(a^2+a^4)z^6$
8_{21}	$(-3a^2-3a^4-a^6)+(2a^3+4a^5+2a^7)z+(3a^2+5a^4-2a^8)z^2$
	$+(-a^3-6a^5-5a^7)z^3+(-2a^4-a^6+a^8)z^4+(a^3+3a^5+2a^7)z^5$
	$+(a^4+a^6)z^6$

Table 2.7.5.1. Kauffman polynomials of knots with 8 crossings (Continuation)

As in the previous section, we conclude with a series of exercises with additional information on the Kauffman polynomial.

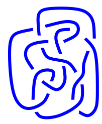
2.7.6. Exercise. Prove that the defining relations are sufficient to compute the Kauffman polynomial.

2.7.7. Exercise. Compute the Kauffman polynomial for the knots 3_1, 4_1 and compare the results with those given in the above table.

2.7.8. Exercise. Prove that the Kauffman polynomial of a knot is preserved when the knot orientation is reversed.

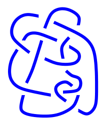
2.7.9. Exercise. (*W. B. R. Lickorish* [**Lik**]) Prove that

(1) $F(\overline{L}) = \overline{F(L)}$, where $\overline{L}$ is the mirror reflection of L, and $\overline{F(L)}$ is the polynomial obtained from $F(L)$ by substituting a^{-1} for a;

$11_{255} =$ 

(2) $F(K_1 \# K_2) = F(K_1) \cdot F(K_2)$;

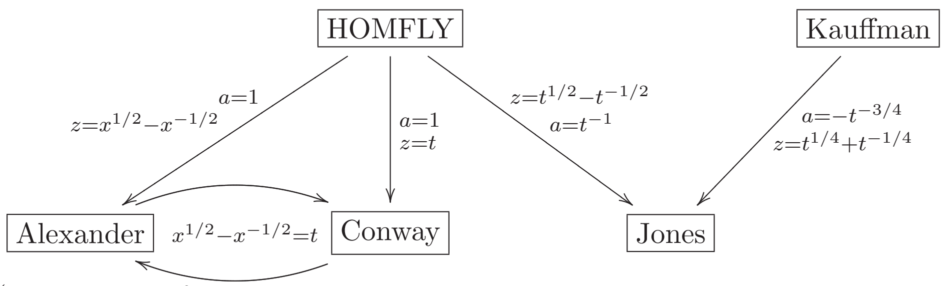
(3) $F(L_1 \sqcup L_2) = \Big((a+a^{-1})z^{-1}-1\Big) \cdot F(L_1) \cdot F(L_2)$, where $L_1 \sqcup L_2$ means the split union of links;

$11_{257} =$ 

(4) $F(11_{255}) = F(11_{257})$;
(these knots can be distinguished by the Conway and, hence, by the HOMFLY polynomial).

(5) $F(L^*) = a^{4lk(K,L-K)}F(L)$, where the link L^* is obtained from an oriented link L by reversing the orientation of a connected component K.

2.7.10. Comparative strength of polynomial invariants. Let us say that an invariant I_1 *dominates* an invariant I_2, if the equality $I_1(K_1) = I_1(K_2)$ for any two knots K_1 and K_2 implies the equality $I_2(K_1) = I_2(K_2)$. Denoting this relation by arrows, we have the following comparison chart:



(the absence of an arrow between the two invariants means that neither of them dominates the other).

Exercise. Find in this chapter all the facts sufficient to justify this chart.

Exercises

(1) **Bridge number.** The *bridge number* $b(K)$ of a knot K can be defined as the minimal number of local maxima of the projection of the knot onto a straight line, where the minimum is taken over all projections and over all closed curves in $\mathbb{R}^3$ representing the knot. Show that that

$$b(K_1 \# K_2) = b(K_1) + b(K_2) - 1 \ .$$

Knots of bridge number 2 are also called *rational knots*.

(2) Prove that the Conway and the Jones polynomials of a knot are preserved when the knot orientation is reversed.

(3) Compute the Conway and the Jones polynomials for the links from Section 1.2.5, page 20, with some orientations of their components.

(4) A link is called *split* if it is equivalent to a link which has some components in a ball while the other components are located outside of the ball. Prove that the Conway polynomial of a split link is trivial: $C(L) = 0$.

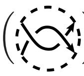
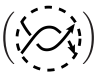
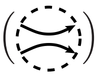
(5) For a split link $L_1 \sqcup L_2$ prove that

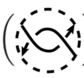
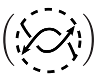
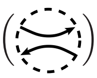
$$J(L_1 \sqcup L_2) = (-t^{1/2} - t^{-1/2}) \cdot J(L_1) \cdot J(L_2) \ .$$

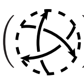
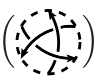
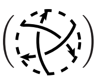
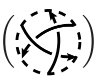
(6) Prove that $C(K_1 \# K_2) = C(K_1) \cdot C(K_2)$.

(7) Prove that $J(K_1 \# K_2) = J(K_1) \cdot J(K_2)$.

(8) (cf. *J. H. Conway* [**Con**]) Check that the Conway polynomial satisfies the following relations.

(a) C() $+ C$() $= (2+t^2)C$() ;

(b) C() $+ C$() $= 2C$() ;

(c) C() $+ C$() $= C$() $+ C$() .

(9) What effect has the reversal of the parametrization of one component on the Conway polynomial of a link?

(10) Compute the Conway polynomials of the Conway and the Kinoshita–Terasaka knots (see pages 59 and 257).

(11) Prove that for any knot K the Conway polynomial $C(K)$ is an even polynomial in t and its constant term is equal to 1:

$$C(K) = 1 + c_2(K)t^2 + c_4(K)t^4 + \dots$$

(12) Let L be a link with two components K_1 and K_2. Prove that the Conway polynomial $C(L)$ is an odd polynomial in t and its lowest coefficient is equal to the linking number $lk(K_1, K_2)$:

$$C(L) = lk(K_1, K_2)t + c_3(L)t^3 + c_5(L)t^5 + \dots$$

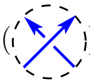
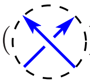
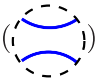
(13) Prove that for a link L with k components the Conway polynomial $C(L)$ is divisible by t^{k-1} and is odd or even depending on the parity of k:

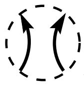
$$C(L) = c_{k-1}(L)t^{k-1} + c_{k+1}(L)t^{k+1} + c_{k+3}(L)t^{k+3} + \dots$$

(14) For a knot K, show that $C(K)\big|_{t=2i} \equiv 1$ or $5 \pmod 8$ depending of the parity of $c_2(K)$. The reduction of $c_2(K)$ modulo 2 is called the *Arf invariant* of K.

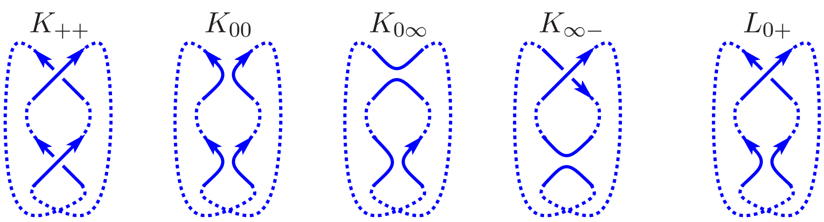
(15) Show that $J(L)\big|_{t=-1} = C(L)\big|_{t=2i}$ for any link L. This value is called the *determinant* of the link L. The previous problem then implies that the determinant of a knot is congruent to 1 modulo 4.
Hint. Choose $\sqrt{t}$ in such a way that $\sqrt{-1} = -i$.

(16) Check the following *switching formula* for the Jones polynomial.

$J($$) - tJ($$) = t^{3\lambda_0}(1-t)J($$)$,

where λ_0 is the linking number of two components of the link, , obtained by smoothing the crossing according the orientation. Note that the knot in the right hand side of the formula is unoriented. That is because such a smoothing destroys the orientation. Since the Jones polynomial does not distinguish orientation of a knot we may choose it arbitrarily.

(17) **Interlacing crossings formulae.** Suppose K_{++} is a knot diagram with two positive crossings which are *interlaced*. That means when we trace the knot we first past the first crossing, then the second, then again the first, and after that the second. Consider the following four knots and one link:



Check that the Jones polynomial satisfies the relation

$$J(K_{++}) = tJ(K_{00}) + t^{3\lambda_{0+}}\Big(J(K_{0\infty}) - tJ(K_{\infty-})\Big) ,$$

where λ_{0+} is the linking number of two components of the link L_{0+}.

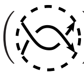
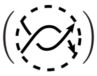
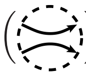
Check the similar relations for K_{+-} and K_{--}:

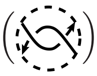
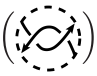
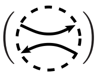
$$J(K_{+-}) = J(K_{00}) + t^{3\lambda_0-+1}\Big(J(K_{0\infty}) - J(K_{\infty+})\Big)\ ,$$

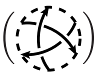
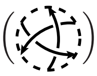
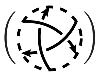
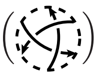
$$J(K_{--}) = t^{-1}J(K_{00}) + t^{3\lambda_0-}\Big(J(K_{0\infty}) - t^{-1}J(K_{\infty+})\Big)\ .$$

If a knot diagram does not contain interlacing crossings then it represents the unknot. Thus the three relations above allow to compute the Jones polynomial for knots recursively without referring to links.

(18) Show that the Jones polynomial satisfies the following relations.

(a) $t^{-2}J\Big($  $\Big) + t^2J\Big($  $\Big) \ = \ (t+t^{-1})J\Big($  $\Big)$;

(b) $tJ\Big($  $\Big) + t^{-1}J\Big($  $\Big) \ = \ (t+t^{-1})J\Big($  $\Big)$;

(c) $t^2J\Big($  $\Big) + t^{-2}J\Big($  $\Big) \ = \ t^{-2}J\Big($  $\Big) + t^2J\Big($  $\Big)$.

Compare these relations with those of Exercise 8 for the Conway polynomial.

(19) Prove that for a link L with an odd number of components, $J(L)$ is a polynomial in t and t^{-1}, and for a link L with an even number of components $J(L) = t^{1/2} \cdot$ (a polynomial in t and t^{-1}).

(20) Prove that for a link L with k components $J(L)\big|_{t=1} = (-2)^{k-1}$. In particular, $J(K)\big|_{t=1} = 1$ for a knot K.

(21) Prove that $\left.\dfrac{d(J(K))}{dt}\right|_{t=1} = 0$ for any knot K.

(22) Evaluate the Kauffman bracket $\langle L\rangle$ at $a = e^{\pi i/3}$, $b = a^{-1}$, $c = -a^2 - a^{-2}$. Deduce from here that $J(L)\big|_{t=e^{2\pi i/3}} = 1$.
Hint. $\sqrt{t} = a^{-2} = e^{4\pi i/3}$.

(23) Let L be a link with k components. For odd (resp. even) k let a_j ($j = 0, 1, 2,$ or 3) be the sum of the coefficients of $J(L)$ (resp. $J(L)/\sqrt{t}$, see problem 19) at t^s for all $s \equiv j \pmod 4$.
(a) For odd k, prove that $a_1 = a_3$.
(b) For even k, prove that $a_0 + a_1 = a_2 + a_3$.

(24) (*W. B. R. Lickorish* [**Lik**, Theorem 10.6]) Let $t = i$ with $t^{1/2} = e^{\pi i/4}$. Prove that for a knot K, $J(K)\big|_{t=i} = (-1)^{c_2(K)}$.

(25) For the mirror reflection $\overline{L}$ of a link L prove that $J(\overline{L})$ is obtained from $J(L)$ by substituting t^{-1} for t.

(26) For the link L^* obtained from an oriented link L by reversing the orientation of one of its components K, prove that $J(L^*) = t^{-3lk(K,L-K)}J(L)$.

(27)* Find a non-trivial knot K with $J(K) = 1$.

(28) (*L. Kauffman* [**Ka6**], *K. Murasugi* [**Mur1**], *M. Thistlethwaite* [**Th**]). Prove the for a reduced alternating knot diagram K (Section 1.3.3) the number of crossings is equal to $span(J(K))$, that is, to the difference beween the maximal and minimal degrees of t in the Jones polynomial $J(K)$. (This exercise is not particularly difficult, although it solves a one hundred years old conjecture of Tait. Anyway, the reader can find a simple solution in [**Tur1**].)

(29) Let L be a link with k components. Show that its HOMFLY polynomial $P(L)$ is an even function in each of the variables a and z if k is odd, and it is an odd function if k is even.

(30) For a link L with k components show that the lowest power of z in its HOMFLY polynomial is z^{-k+1}. In particular the HOMFLY polynomial $P(K)$ of a knot K is a genuine polynomial in z. This means that it does not contain terms with z raised to a negative power.

(31) For a knot K let $p_0(a) := P(K)|_{z=0}$ be the constant term of the HOMFLY polynomial. Show that its derivative at $a = 1$ equals zero.

(32) Let L be a link with two components K_1 and K_2. Consider $P(L)$ as a Laurent polynomial in z with coefficients in Laurent polynomials in a. Let $p_{-1}(a)$ and $p_1(a)$ be the coefficients at z^{-1} and z. Check that $p_{-1}\big|_{a=1} = 0$, $p'_{-1}\big|_{a=1} = 2$, $p''_{-1}\big|_{a=1} = -8lk(K_1, K_2) - 2$, and $p_1\big|_{a=1} = lk(K_1, K_2)$.

(33) (*W. B. R. Lickorish* [**Lik**]) Prove that for an oriented link L with k components

$$(J(L))^2\Big|_{t=-q^{-2}} = (-1)^{k-1}F(L)\Bigg|_{\substack{a=q^3\\ z=q+q^{-1}}},$$

where $J(L)$ is the Jones polynomial and $F(L)$ is the two-variable Kauffman polynomial from page 59.

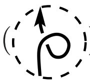
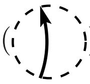
(34) Let L be a link with k components. Show that its two-variable Kauffman polynomial $F(L)$ is an even function of both variables a and z (that is, it consists of monomials $a^i z^j$ with i and j of the same parity) if k is odd, and it is an odd function (different parities of i and j) if k is even.

(35) Prove that the Kauffman polynomial $F(K)$ of a knot K is a genuine polynomial in z.

(36) For a knot K let $f_0(a) := F(K)|_{z=0}$ be the constant term of the Kauffman polynomial. Show that it is related to the constant term of the HOMFLY polynomial of K as $f_0(a) = p_0(\sqrt{-1} \cdot a)$.

(37) **Quantum $\mathfrak{sl}_2$-invariant.** Let $\theta(\cdot)$ and $\theta^{fr}(\cdot)$ be the quantum invariants constructed in Sections 2.6.3 and 2.6.6 for the Lie algebra $\mathfrak{sl}_2$ and its standard 2-dimensional representation.

(a) Prove the following dependence of $\theta^{fr}(\cdot)$ on the first Reidemeister move

$\theta^{fr}($  $) = q^{3/4}\theta^{fr}($  $)$.

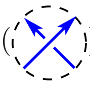
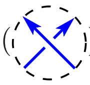
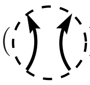
(b) Prove that $\theta(\cdot)$ remains unchanged under the first Reidemeister move.

(c) Compute the value $\theta(4_1)$.

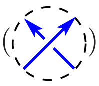
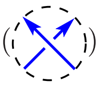
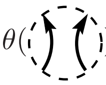
(d) Show that the R-matrix defined in page 52 satisfies the equation

$$q^{1/4}R - q^{-1/4}R^{-1} \quad = \quad (q^{1/2} - q^{-1/2})id_{V\otimes V} \,.$$

(e) Prove that $\theta^{fr}(\cdot)$ satisfies the skein relation

$q^{1/4}\theta^{fr}($  $) \; - \; q^{-1/4}\theta^{fr}($  $) \quad = \quad (q^{1/2} - q^{-1/2})\theta^{fr}($  $)$.

(f) Prove that $\theta(\cdot)$ satisfies the skein relation

$q\theta($  $) \; - \; q^{-1}\theta($  $) \quad = \quad (q^{1/2} - q^{-1/2})\theta($  $)$.

(g) For any link L with k components prove that

$$\theta^{fr}(L) = (-1)^k(q^{1/2} + q^{-1/2}) \cdot \langle L\rangle\Big|_{a=-q^{1/4}} ,$$

where $\langle\cdot\rangle$ is the Kauffman bracket defined on page 48.

(38) **Quantum $\mathfrak{sl}_N$ invariants.** Let V be an N dimensional vector space of the standard representation of the Lie algebra $\mathfrak{sl}_N$ with a basis $e_1, \ldots, e_N$. Consider the operator $R : V\otimes V \to V\otimes V$ given by the formulae

$$R(e_i\otimes e_j) = \begin{cases} q^{\frac{-1}{2N}} e_j\otimes e_i & \text{if} \quad i<j \\ q^{\frac{N-1}{2N}} e_i\otimes e_j & \text{if} \quad i=j \\ q^{\frac{-1}{2N}} e_j\otimes e_i + \left(q^{\frac{N-1}{2N}} - q^{\frac{-N-1}{2N}}\right) e_i\otimes e_j & \text{if} \quad i>j \end{cases}$$

which for $N = 2$ coincides with the operator from Section 2.6.3, page 52.

(a) Prove that it satisfies the quantum Yang–Baxter equation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23} \;,$$

where R_{ij} is the operator R acting on the i-th and j-the factors of $V\otimes V\otimes V$, that is, $R_{12} = R\otimes \mathrm{id}_V$ and $R_{23} = \mathrm{id}_V\otimes R$.

(b) Show that its inverse is given by the formulae

$$R^{-1}(e_i \otimes e_j) = \begin{cases} q^{\frac{1}{2N}} e_j \otimes e_i + \Big(-q^{\frac{N+1}{2N}} + q^{\frac{-N+1}{2N}}\Big) e_i \otimes e_j & \text{if} \quad i < j \\ q^{\frac{-N+1}{2N}} e_i \otimes e_j & \text{if} \quad i = j \\ q^{\frac{1}{2N}} e_j \otimes e_i & \text{if} \quad i > j \end{cases}$$

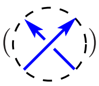
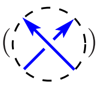
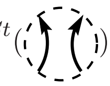
(c) Check that $q^{\frac{1}{2N}} R - q^{\frac{-1}{2N}} R^{-1} = (q^{1/2} - q^{-1/2})\mathrm{id}_{V \otimes V}$.

(d) Extending the assignments of operators for maximum/minimum tangles from page 54 we set:

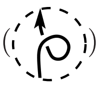
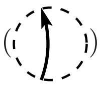
$$\begin{aligned}
&\underrightarrow{\min} : \mathbb{C} \to V^* \otimes V, && \underrightarrow{\min}(1) := \sum_{k=1}^{N} q^{\frac{-N-1}{2}+k} e^k \otimes e_k \ ; \\
&\underleftarrow{\min} : \mathbb{C} \to V \otimes V^*, && \underleftarrow{\min}(1) := \sum_{k=1}^{N} e^k \otimes e_k \ ; \\
&\overrightarrow{\max} : V \otimes V^* \to \mathbb{C}, && \overrightarrow{\max}\ (e_i \otimes e^j) := \begin{cases} 0 & \text{if } i \neq j \\ q^{\frac{N+1}{2}-i} & \text{if } i = j \end{cases} \ ; \\
&\overleftarrow{\max} : V^* \otimes V \to \mathbb{C}, && \overleftarrow{\max}\ (e^i \otimes e_j) := \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \ .
\end{aligned}$$

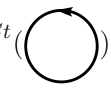
Prove that all these operators are consistent in the sense that their appropriate combinations are consistent with the oriented Turaev moves from page 32. Thus we get a link invariant $\theta^{fr,St}_{\mathfrak{sl}_N}$.

(e) Show the $\theta^{fr,St}_{\mathfrak{sl}_N}$ satisfies the following skein relation

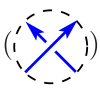
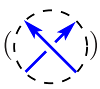
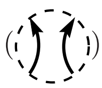
$q^{\frac{1}{2N}} \theta^{fr,St}_{\mathfrak{sl}_N}($  $)$ $-$ $q^{-\frac{1}{2N}} \theta^{fr,St}_{\mathfrak{sl}_N}($  $)$ $=$ $(q^{1/2} - q^{-1/2})\theta^{fr,St}_{\mathfrak{sl}_N}($  $)$

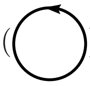
and the following framing and initial conditions

$\theta^{fr,St}_{\mathfrak{sl}_N}($  $)$ $=$ $q^{\frac{N-1/N}{2}} \theta^{fr,St}_{\mathfrak{sl}_N}($  $)$

$\theta^{fr,St}_{\mathfrak{sl}_N}($  $)$ $=$ $\dfrac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}}$.

(f) The quadratic Casimir number for the standard $\mathfrak{sl}_N$ representation is equal to $N - 1/N$. Therefore, the deframing of this invariant gives $\theta^{St}_{\mathfrak{sl}_N} := q^{-\frac{N-1/N}{2} \cdot w} \theta^{fr,St}_{\mathfrak{sl}_N}$ which satisfies

$q^{N/2} \theta^{St}_{\mathfrak{sl}_N}($  $)$ $-$ $q^{-N/2} \theta^{St}_{\mathfrak{sl}_N}($  $)$ $=$ $(q^{1/2} - q^{-1/2})\theta^{St}_{\mathfrak{sl}_N}($  $)$;

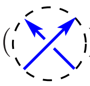
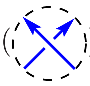
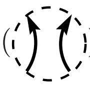
$$\theta^{St}_{\mathfrak{sl}_N}\Big(\ $$  $$\Big) = \frac{q^{N/2}-q^{-N/2}}{q^{1/2}-q^{-1/2}}\,.$$

Check that this invariant is essentially a specialization of the HOMFLY polynomial,

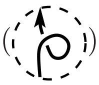
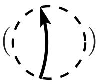
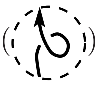
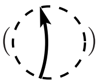
$$\theta^{St}_{\mathfrak{sl}_N}(L) = \frac{q^{N/2}-q^{-N/2}}{q^{1/2}-q^{-1/2}} P(L)\Bigg|_{\substack{a=q^{N/2}\\ z=q^{1/2}-q^{-1/2}}}\,.$$

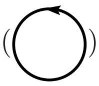
Prove that the set of invariants $\{\theta^{St}_{\mathfrak{sl}_N}\}$ for all values of N is equivalent to the HOMFLY polynomial. Thus $\{\theta^{fr,St}_{\mathfrak{sl}_N}\}$ may be considered as a framed version of the HOMFLY polynomial.

(39) A different framed version of the HOMFLY polynomial is defined in [**Ka7**, page 54]: $P^{fr}(L) := a^{w(L)}P(L)$. Show that P^{fr} satisfies the following skein relation

$P^{fr}\Big($  $\Big) - P^{fr}\Big($  $\Big) = zP^{fr}\Big($  $\Big)$

and the following framing and initial conditions

$P^{fr}\Big($  $\Big) = aP^{fr}\Big($  $\Big)$, $P^{fr}\Big($  $\Big) = a^{-1}P^{fr}\Big($  $\Big)$

$P^{fr}\Big($  $\Big) = 1\,.$

Chapter 3

Finite type invariants

In this chapter we introduce the main protagonist of this book — the *finite type*, or *Vassiliev* knot invariants.

First we define the Vassiliev skein relation and extend, with its help, arbitrary knot invariants to knots with double points. A Vassiliev invariant of order at most n is then defined as a knot invariant which vanishes identically on knots with more than n double points.

After that, we introduce a combinatorial object of great importance: the chord diagrams. Chord diagrams serve as a means to describe the symbols (highest parts) of the Vassiliev invariants.

Then we prove that classical invariant polynomials are all, in a sense, of finite type, explain a simple method of calculating the values of Vassiliev invariants on any given knot, and give a table of basis Vassiliev invariants up to degree 5.

Finally, we show how Vassiliev invariants can be defined for framed knots and for arbitrary tangles.

3.1. Definition of Vassiliev invariants

3.1.1. The original definition of finite type knot invariants was just an application of the general machinery developed by V.Vassiliev to study *complements of discriminants* in spaces of maps.

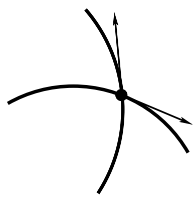
The discriminants in question are subspaces of maps with singularities of some kind. In particular, consider the space of all smooth maps of the circle into $\mathbb{R}^3$. Inside this space, define the discriminant as the subspace formed by maps that fail to be embeddings, such as curves with self-intersections, cusps etc. Then the complement to this discriminant can be considered as *the space of knots*. The connected components of the space of knots are precisely the isotopy classes of knots; knot invariants are locally constant functions on the space of knots.

Vassiliev's machinery produces a spectral sequence that may (or may not, nobody knows it yet) converge to the cohomology of the space of knots. The zero-dimensional classes produced by this spectral sequence correspond to knot invariants which are now known as Vassiliev invariants.

This approach is indispensable if one wants to understand the higher cohomology of the space of knots. However, if we are only after the zero-dimensional classes, that is, knot invariants, the definitions can be greatly simplified. In this chapter we follow the easy path that requires no knowledge of algebraic topology whatsoever. For the reader who is not intimidated by spectral sequences we outline Vassiliev's construction in Chapter **??**.

3.1.2. Singular knots and the Vassiliev skein relation. A *singular knot* is a smooth map $S^1 \to \mathbb{R}^3$ that fails to be an embedding. We shall only consider singular knots with the simplest singularities, namely transversal self-intersections, or *double points*.

Definition. Let f be a map of a one-dimensional manifold to $\mathbb{R}^3$. A point $p \in \mathrm{im}(f) \subset \mathbb{R}^3$ is a *double point* of f if $f^{-1}(p)$ consists of two points t_1 and t_2 and the two tangent vectors $f'(t_1)$ and $f'(t_2)$ are linearly independent. Geometrically, this means that in a neighbourhood of the point p the curve f has two branches with non-collinear tangents.

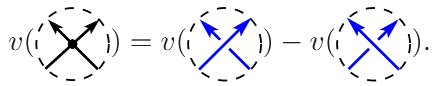


A double point

Remark. In fact, we gave a definition of a *simple double point*. We omit the word "simple" since these are the only double points we shall see.

Any knot invariant can be extended to knots with double points by means of the *Vassiliev skein relation*:

(3.1.2.1)



Here v is the knot invariant with values in some abelian group, the left-hand side is the value of v on a singular knot K (shown in a neighbourhood of a double point) and the right-hand side is the difference of the values of v on (possibly singular) knots obtained from K by replacing the double point with a positive and a negative crossing respectively. The process of applying the skein relation is also referred to as *resolving a double point.* It is clearly independent of the plane projection of the singular knot.

Using the Vassiliev skein relation recursively, we can extend any knot invariant to knots with an arbitrary number of double points. There are many ways to do this, since we can choose to resolve double points in an arbitrary order. However, the result is independent of any choice. Indeed, the calculation of the value of v on a singular knot K with n double points is in all cases reduced to the *complete resolution* of the knot K which yields an alternating sum

$$v(K) = \sum_{\varepsilon_1=\pm1,\ldots,\varepsilon_n=\pm1} (-1)^{|\varepsilon|} v(K_{\varepsilon_1,\ldots,\varepsilon_n}), \tag{3.1.2.2}$$

where $|\varepsilon|$ is the number of -1's in the sequence $\varepsilon_1, \ldots, \varepsilon_n$, and $K_{\varepsilon_1,\ldots,\varepsilon_n}$ is the knot obtained from K by a positive or negative resolution of the double points according to the sign of ε_i for the point number i.

3.1.3. Definition. (V. Vassiliev [**Va1**]). A knot invariant is said to be a *Vassiliev invariant* (or a *finite type invariant*) of order (or degree) $\leqslant n$ if its extension vanishes on all singular knots with more than n double points. A Vassiliev invariant is said to be of order (degree) n if it is of order $\leqslant n$ but not of order $\leqslant n-1$.

In general, a Vassiliev invariant may take values in an arbitrary abelian group. In practice, however, all our invariants will take values in commutative rings and it will be convenient to make this assumption from now on.

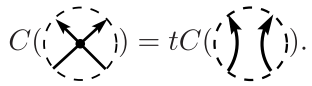
Notation. We shall denote by $\mathcal{V}_n$ the set of Vassiliev invariants of order $\leqslant n$ with values in a ring $\mathcal{R}$. Whenever necessary, we shall write $\mathcal{V}_n^{\mathcal{R}}$ to indicate the range of the invariants explicitly. It follows from the definition that, for each n, the set $\mathcal{V}_n$ is an $\mathcal{R}$-module. Moreover, $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$, so we have an increasing filtration

$$\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_n \subseteq \cdots \subseteq \mathcal{V} := \bigcup_{n=0}^{\infty} \mathcal{V}_n \ .$$

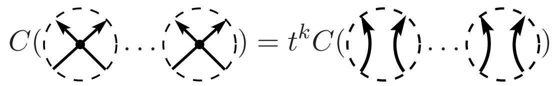
We shall further discuss this definition in the next section. First, let us see that there are indeed many (in fact, infinitely many) independent Vassiliev invariants.

3.1.4. Example. ([**BN0**]). The n-th coefficient of the Conway polynomial is a Vassiliev invariant of order $\leqslant n$.

Indeed, the definition of the Conway polynomial, together with the Vassiliev skein relation, implies that



Applying this relation several times, we get



for a singular knot with k double points. If $k \geqslant n+1$, then the coefficient at t^n in this polynomial is zero.

3.2. Algebra of Vassiliev invariants

3.2.1. The singular knot filtration. Consider the "tautological knot invariant" $\mathcal{K} \to \mathbb{Z}\mathcal{K}$ which sends a knot to itself. Applying the Vassiliev skein relation, we extend it to knots with double points; a knot with n double points is then sent to an alternating sum of 2^n genuine knots.

Let $\mathcal{K}_n$ be the submodule of $\mathbb{Z}\mathcal{K}$ spanned by the images of knots with n double points.

Exercise. Prove that $\mathcal{K}_n$ is an ideal of $\mathbb{Z}\mathcal{K}$.

A knot with $n+1$ double points gives rise to a difference of two knots with n double points in $\mathbb{Z}\mathcal{K}$; hence, we have the descending *singular knot filtration*

$$\mathbb{Z}\mathcal{K} = \mathcal{K}_0 \supseteq \mathcal{K}_1 \supseteq \dots \mathcal{K}_n \supseteq \dots$$

The definition of Vassiliev invariants can now be re-stated in the following terms:

Definition. Let $\mathcal{R}$ be a commutative ring. A Vassiliev invariant of order $\leqslant n$ is a linear function $\mathbb{Z}\mathcal{K} \to \mathcal{R}$ which vanishes on $\mathcal{K}_{n+1}$.

According to this definition, the module of $\mathcal{R}$-valued Vassiliev invariants of order $\leqslant n$ is naturally isomorphic to the space of linear functions $\mathbb{Z}\mathcal{K}/\mathcal{K}_n \to \mathcal{R}$. So, in a certain sense, the study of the Vassiliev invariants is equivalent to studying the filtration $\mathcal{K}_n$. In the next several chapters we shall mostly speak about invariants, rather than the filtration on the algebra of knots. Nevertheless, the latter approach, developed by Goussarov [**G2**] is important and we cannot skip it here altogether.

Definition. Two knots K_1 and K_2 are *n-equivalent* if they cannot be distinguished by Vassiliev invariants of degree n and smaller. A knot that is n-equivalent to the trivial knot is called *n-trivial*.

In other words, K_1 and K_2 are n-equivalent if and only if $K_1 - K_2 \in \mathcal{K}_{n+1}$.

Definition. Let $\Gamma_n\mathcal{K}$ be the set of n-trivial knots. The *Goussarov filtration* on $\mathcal{K}$ is the descending filtration

$$\mathcal{K} = \Gamma_1\mathcal{K} \supseteq \Gamma_2\mathcal{K} \supseteq \ldots \supseteq \Gamma_n\mathcal{K} \supseteq \ldots$$

The sets $\Gamma_n\mathcal{K}$ are, in fact, abelian monoids under the connected sum of knots (this follows from the fact that each $\mathcal{K}_n$ is a subalgebra of $\mathbb{Z}\mathcal{K}$). Goussarov proved that the monoid quotient $\mathcal{K}/\Gamma_n\mathcal{K}$ is an abelian group. We shall consider n-equivalence in greater detail in Chapter 12.

3.2.2. Vassiliev invariants as polynomials. A useful way to think of Vassiliev invariants is as follows. Let v be an invariant of singular knots with n double points and $\nabla(v)$ be the extension of v to singular knots with $n+1$ double points using the Vassiliev skein relation. We can consider ∇ as an operator between the corresponding spaces of invariants. Now, a function $v : \mathcal{K} \to \mathcal{R}$ is a Vassiliev invariant of degree $\leqslant n$, if it satisfies the difference equation $\nabla^{n+1}(v) = 0$. This can be seen as an analogy between Vassiliev invariants as a subspace of all knot invariants and polynomials as a subspace of all smooth functions on a real line: the role of differentiation is played by the operator ∇.

It is well known that continuous functions on a real line can be approximated by polynomials. The main open problem of the theory of finite type invariants is to find an analogue of this statement in the knot-theoretic context, namely, to understand to what extent an arbitrary numerical knot invariant can be approximated by Vassiliev invariants. More on this in Section 3.2.4.

3.2.3. The filtration on the algebra of Vassiliev invariants. The set of all Vassiliev invariants forms a commutative filtered algebra with respect to the usual (pointwise) multiplication of functions.

Theorem. *The product of two Vassiliev invariants of degrees $\leqslant p$ and $\leqslant q$ is a Vassiliev invariant of degree $\leqslant p+q$.*

Proof. Let f and g be two invariants with values in a ring $\mathcal{R}$, of degrees p and q respectively. Consider a singular knot K with $n = p+q+1$ double points. The complete resolution of K via the Vassiliev skein relation gives

$$(fg)(K) = \sum_{\varepsilon_1=\pm1,\ldots,\varepsilon_n=\pm1} (-1)^{|\varepsilon|} f(K_{\varepsilon_1,\ldots,\varepsilon_n}) g(K_{\varepsilon_1,\ldots,\varepsilon_n})$$

in the notations of (3.1.2.2) of 3.1.2. The alternating sum on the right-hand side is taken over all points of an n-dimensional binary cube

$$Q_n = \{\varepsilon_1, \dots, \varepsilon_n \,|\, \varepsilon_i = \pm 1\}.$$

In general, given a function v on Q_n and a subset $S \subseteq Q_n$ the *alternating sum* of v over S is defined as $\sum_{\varepsilon \in S} (-1)^{|\varepsilon|} v(\varepsilon)$.

If we set

$$f(\varepsilon_1, ..., \varepsilon_n) = f(K_{\varepsilon_1, \dots, \varepsilon_n})$$

and define $g(\varepsilon_1, ..., \varepsilon_n)$ similarly, we can think of f and g as functions on Q_n. The fact that f is of degree p means that the alternating sum of f on each $(p+1)$-face of Q_n is zero. Similarly, on each $(q+1)$-face of Q_n the alternating sum of g vanishes.

Lemma. *Let f, g be functions on Q_n, where $n = p+q+1$. If the alternating sums of f over any $(p+1)$-face, and of g over any $(q+1)$-face of Q_n are zero, so is the alternating sum of the product fg over the entire cube Q_n.*

Proof of the lemma. Use induction on n. For $n = 1$ we have $p = q = 0$ and

$$\begin{aligned}(fg)(1) - (fg)(-1) &= f(1)g(1) - f(-1)g(-1) + f(-1)g(1) - f(-1)g(1) \\ &= \big(f(1) - f(-1)\big)g(1) + f(-1)\big(g(1) - g(-1)\big) = 0.\end{aligned}$$

Denote by $\mathcal{F}_n$ the space of functions $Q_n \to \mathcal{R}$ with $\mathcal{F}_0 = \mathcal{R}$. We have operators

$$\rho_-, \rho_+ : \mathcal{F}_n \to \mathcal{F}_{n-1}$$

which take a function v to its restrictions to the $(n-1)$-dimensional faces $\varepsilon_1 = -1$ and $\varepsilon_1 = 1$ of Q_n:

$$\rho_-(v)(\varepsilon_2, \dots, \varepsilon_n) = v(-1, \varepsilon_2, \dots, \varepsilon_n)$$

and

$$\rho_+(v)(\varepsilon_2, \dots, \varepsilon_n) = v(1, \varepsilon_2, \dots, \varepsilon_n).$$

Let

$$\delta = \rho_+ - \rho_-.$$

Observe that if the alternating sum of v over any r-face of Q_n is zero, so is the alternating sum of $\delta(v)$ over any $(r-1)$-face of Q_{n-1}.

A direct check shows that the operator δ satisfies the following Leibniz rule:

$$\delta(fg) = \rho_+(f) \cdot \delta(g) + \delta(f) \cdot \rho_-(g).$$

The alternating sum of $\delta(g)$ over any q-face of Q_{n-1} vanishes, and so does the alternating sum of $\delta(f)$ over any p-face. Hence, by the induction assumption, the alternating sum of $\delta(fg)$ over the whole Q_{n-1} is zero. However, by the definition of δ, it coincides with the alternating sum of fg over Q_n. $\square$

Remark. The existence of the filtration on the algebra of Vassiliev invariants can be thought of as a manifestation of their polynomial character. Indeed, a polynomial of degree $\leqslant n$ in one variable can be defined as a function whose $n+1$st derivative is identically zero. Then the fact that a product of polynomials of degrees $\leqslant p$ and $\leqslant q$ has degree $\leqslant p+q$ can be proved by induction using the Leibniz formula. In our argument on Vassiliev invariants we have used the very same logic. A further discussion of the Leibniz formula for the finite type invariants can be found in [**Wil4**].

3.2.4. Approximation by Vassiliev invariants. There are knot invariants, of which we shall see many examples, which are not of finite type, but, nevertheless, can be approximated by Vassiliev invariants in a certain sense.

Definition. The *closure* of the space of Vassiliev invariants consists of all knot invariants u such that if $u(K_1) \neq u(K_2)$ for some knots K_1 and K_2, then there is a Vassiliev knot invariant v with $v(K_1) \neq v(K_2)$.

With this terminology the main problem of the theory of finite type invariants can be stated as follows: is it true that the closure of the space of (complex-valued) Vassiliev invariants coincides with the space of all (complex-valued) knot invariants? This is equivalent to asking whether the Vassiliev invariants can distinguish arbitrary knots.

An important class of invariants in the closure of the space of Vassiliev invariants are the *polynomial* and the *power series Vassiliev invariants*. A polynomial Vassiliev invariant is an element of the vector space

$$\mathcal{V}_\bullet = \bigoplus_{n=0}^{\infty} \mathcal{V}_n.$$

Since the product of two invariants of degrees m and n has degree at most $m+n$, the space $\mathcal{V}_\bullet$ is, in fact. a commutative graded algebra. The power series Vassiliev invariants are, by definition, the elements of its *graded completion* $\widehat{\mathcal{V}}_\bullet$ (see Appendix A.2.16, page 427).

The Conway polynomial $C(\cdot)$ is an example of a power series invariant. Observe that even though for any knot K the value $C(K)$ is a polynomial, the Conway polynomial $C(\cdot)$ is not a polynomial invariant according to the definition of this paragraph.

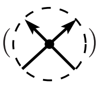
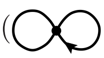
3.3. Vassiliev invariants of degrees 0, 1 and 2

3.3.1. Proposition. $\mathcal{V}_0 = \{\text{const}\}$, $\dim \mathcal{V}_0 = 1$.

Proof. Let $f \in \mathcal{V}_0$. By definition, the value of (the extension of) f on any singular knot with one double point is 0. Pick an arbitrary knot K. Any diagram of K can be turned into a diagram of the trivial knot K_0 by crossing changes done one at a time. By assumption, the jump of f at every crossing change is 0, therefore, $f(K) = f(K_0)$. Thus f is constant. □

3.3.2. Proposition. $\mathcal{V}_1 = \mathcal{V}_0$.

Proof. A singular knot with one double point is divided by the double point into two closed curves. An argument similar to the last proof shows that the value of v on any knot with one double point is equal to its value on the "figure infinity" singular knot and, hence, to 0:

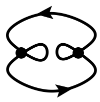
(3.3.2.1) $v($  $) = v($  $) = 0$

Therefore, $\mathcal{V}_1 = \mathcal{V}_0$. □

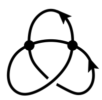
The first non-trivial Vassiliev invariant appears in degree 2: it is the second coefficient c_2 of the Conway polynomial, also known as the *Casson invariant*.

3.3.3. Proposition. $\dim \mathcal{V}_2 = 2$.

Proof. Let us explain why the argument of the proof of Propositions 3.3.1 and 3.3.2 does not work in this case. Take a knot with two double points and try to transform it into some fixed knot with two double points using smooth deformations and crossing changes. It is easy to see that any knot with two double points can be reduced to one of the following two basic knots:



Basic knot K_1



Basic knot K_2

— but these two knots cannot be obtained one from the other! The essential difference between them is in the order of the double points on the curve.

Let us label the double points of K_1 and K_2, say, by 1 and 2. When traveling along the first knot, K_1, the two double points are encountered in the order 1122 (or 1221, 2211, 2112 if you start from a different initial point). For the knot K_2 the sequence is 1212 (equivalent to 2121). The two sequences 1122 and 1212 are different even if cyclic permutations are allowed.

Now take an arbitrary singular knot K with two double points. If the cyclic order of these points is 1122, then we can transform the knot to K_1, passing in the process of deformation through some singular knots with three double points; if the order is 1212, we can reduce K in the same way to the second basic knot K_2.

The above argument shows that, to any $\mathcal{R}$-valued order 2 Vassiliev invariant there corresponds a function on the set of two elements $\{K_1, K_2\}$ with values in $\mathcal{R}$. We thus obtain a linear map $\mathcal{V}_2 \to \mathcal{R}^2$. The kernel of this map is equal to $\mathcal{V}_1$: indeed, the fact that a given invariant $f \in \mathcal{V}_2$ satisfies $f(K_1) = f(K_2) = 0$ means that it vanishes on *any* singular knot with 2 double points, which is by definition equivalent to saying that $f \in \mathcal{V}_1$.

This proves that $\dim \mathcal{V}_2 \leqslant 2$. In fact, $\dim \mathcal{V}_2 = 2$, since the second coefficient c_2 of the Conway polynomial is not constant (see Table 2.3.3). $\square$

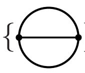
3.4. Chord diagrams

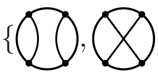
Now let us give a formal definition of the combinatorial structure which is implicit in the proof of Proposition 3.3.3.

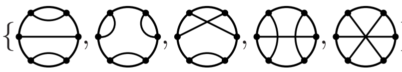
Definition. A *chord diagram* of order n (or degree n) is an oriented circle with a distinguished set of n disjoint pairs of distinct points, considered up to orientation preserving diffeomorphisms of the circle. The set of all chord diagrams of order n will be denoted by $\mathbf{A}_n$.

We shall usually omit the orientation of the circle in pictures of chord diagrams, assuming that it is oriented counterclockwise.

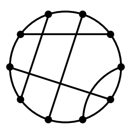
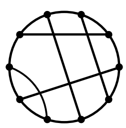
Examples.

$\mathbf{A}_1 = \{$  $\}$,

$\mathbf{A}_2 = \{$  $\}$,

$\mathbf{A}_3 = \{$  $\}$.

Remark. Chord diagrams that differ by a mirror reflection are, in general, different:

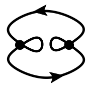
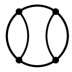
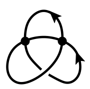
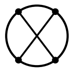
 $\neq$ 

This observation reflects the fact that we are studying *oriented* knots.

3.4.1. The chord diagram of a singular knot. Chord diagrams are used to code certain information about singular knots.

Definition. The chord diagram $\sigma(K) \in \mathbf{A}_n$ of a singular knot with n double points is obtained by marking on the parameterizing circle n pairs of points whose images are the n double points of the knot.

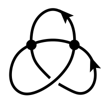
Examples.

$\sigma\Big($  $\Big) =$  $, \qquad \sigma\Big($  $\Big) =$  .

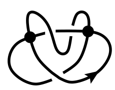
3.4.2. Proposition. (V. Vassiliev [**Va1**]). *The value of a Vassiliev invariant v of order $\leqslant n$ on a knot K with n double points depends only on the chord diagram of K:*

$$\sigma(K_1) = \sigma(K_2) \Rightarrow v(K_1) = v(K_2).$$

Proof. Suppose that $\sigma(K_1) = \sigma(K_2)$. Then there is a one-to-one correspondence between the chords of both chord diagrams, and, hence, between the double points of K_1 and K_2. Place K_1, K_2 in $\mathbb{R}^3$ so that the corresponding double points coincide together with both branches of the knot in the vicinity of each double point.



`Knot` K_1



`Knot` K_2

Now we can deform K_1 into K_2 in such a way that some small neighbourhoods of the double points do not move. We can assume that the only new singularities created in the process of this deformation are a finite number of double points, all at distinct values of the deformation parameter. By the Vassiliev skein relation, in each of these events the value of v does not change, and this implies that $v(K_1) = v(K_2)$. $\square$

Proposition 3.4.2 shows that there is a well defined map $\alpha_n : \mathcal{V}_n \to \mathcal{R}\mathbf{A}_n$ (the $\mathcal{R}$-module of $\mathcal{R}$-valued functions on the set $\mathbf{A}_n$):

$$\alpha_n(v)(D) = v(K),$$

where K is an arbitrary knot with $\sigma(K) = D$.

We want to understand the size and the structure of the space $\mathcal{V}_n$, so it would be of use to have a description of the kernel and the image of α_n.

The description of the kernel follows immediately from the definitions: $\ker \alpha_n = \mathcal{V}_{n-1}$. Therefore, we obtain an injective homomorphism

$$\overline{\alpha}_n : \mathcal{V}_n/\mathcal{V}_{n-1} \to \mathcal{R}\mathbf{A}_n. \qquad (3.4.2.1)$$

The problem of describing the image of α_n is much more difficult. The answer to it will be given in Theorem 4.2.1 on page 100.

Since there is only a finite number of diagrams of each order, Proposition 3.4.2 implies the following

3.4.3. Corollary. *The module of $\mathcal{R}$-valued Vassiliev invariants of degree at most n is finitely generated over $\mathcal{R}$.*

Since the map α_n discards the order $(n-1)$ part of a Vassiliev invariant v, we can, by analogy with differential operators, call the function $\alpha_n(v)$ on chord diagrams the *symbol* of the Vassiliev invariant v:

$$\mathrm{symb}(v) = \alpha_n(v),$$

where n is the order of v.

Example. The symbol of the Casson invariant is equal to 0 on the chord diagram with two parallel chords, and to 1 on the chord diagram with two interecting chords.

3.4.4. Remark. It may be instructive to state all the above in the dual setting of the singular knot filtration. The argument in the proof of Proposition 3.4.2 essentially says that $\mathbf{A}_n$ is the set of singular knots with n double points modulo isotopies and crossing changes. In terms of the singular knot filtration, we have shown that if two knots with n double points have the same chord diagram, then their difference lies in $\mathcal{K}_{n+1} \subset \mathbb{Z}\mathcal{K}$. Since $\mathcal{K}_n$ is spanned by the complete resolutions of knots with n double points, we have a surjective map

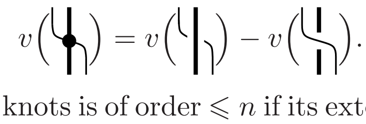
$$\mathbb{Z}\mathbf{A}_n \to \mathcal{K}_n/\mathcal{K}_{n+1}.$$

The kernel of this map , after tensoring with the rational numbers, is spanned by the so-called *4T* and *1T relations*, defined in the next chapter. This is the content of Theorem 4.2.1.

3.5. Invariants of framed knots

A *singular framing* on a closed curve immersed in $\mathbb{R}^3$ is a smooth normal vector field on the curve which has a finite number of simple zeroes. A *singular framed knot* is a knot with simple double points in $\mathbb{R}^3$ equipped with a singular framing whose set of zeroes is disjoint from the set of double points.

Invariants of framed knots are extended to singular framed knots by means of the Vassiliev skein relation; for double points it has the same form as before, and for the zeroes of the singular framing it can be drawn as



An invariant of framed knots is of order $\leqslant n$ if its extension vanishes on knots with more than n singularities (double points or zeroes of the framing).

Let us denote the space of invariants of order $\leqslant n$ by $\mathcal{V}_n^{fr}$. There is a natural inclusion $i : \mathcal{V}_n \to \mathcal{V}_n^{fr}$ defined by setting $i(f)(K) = f(K')$ where

K is a framed knot, and K' is the same knot without framing. It turns out that this is a proper inclusion for all $n \geqslant 1$.

Let us determine the framed Vassiliev invariants of small degree. Any invariant of degree zero is, in fact, an unframed knot invariant and, hence, is constant. Indeed, increasing the framing by one can be thought of as passing a singularity of the framing, and this does not change the value of a degree zero invariant.

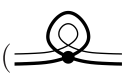
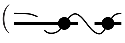
3.5.1. Exercise. (1) Prove that $\dim \mathcal{V}_1^{fr} = 2$, and that $\mathcal{V}_1^{fr}$ is spanned by the constants and the self-linking number.
(2) Find the dimension and a basis of the vector space $\mathcal{V}_2^{fr}$.

3.5.2. Exercise. Let v be a framed Vassiliev invariant degree n, and K — an unframed knot. Let $v(K, k)$ be the value of v on K equipped with a framing with self-linking number k. Show that $v(K, k)$ is a polynomial in k of degree at most n.

3.5.3. Chord diagrams for framed knots. We have seen that chord diagrams on n chords can be thought of as singular knots with n double points modulo isotopies and crossing changes. Following the same logic, we should define a chord diagram for framed knots as an equivalence class of framed singular knots with n singularities modulo isotopies, crossing changes and additions of zeroes of the framing. In this way, the value of a degree n Vassiliev invariant on a singular framed knot with n singularities will only depend on the chord diagram of the knot.

As a combinatorial object, a framed chord diagram of degree n can be defined as a usual chord diagram of degree $n-k$ together with k dots marked on the circle. The chords correspond to the double points of a singular knot and the dots represent the zeroes of the framing.

In the sequel we shall not make any use of diagrams with dots, for the following reason. If $\mathcal{R}$ is a ring where 2 is invertible, a zero of the framing on a knot with n singularities can be replaced, modulo knots with $n+1$ singularities, by “half of a double point”:

$v($  $) = \frac{1}{2}v($  $) - \frac{1}{2}v($  $)$

for any invariant v. In particular, if we replace a dot with a chord whose endpoints are next to each other on some diagram, the symbol of any Vassiliev invariant on this diagram is simply multiplied by 2.

On the other hand, the fact that we can use the same chord diagrams for both framed and unframed knots does not imply that the corresponding theories of Vassiliev invariants are the same. In particular, we shall see that the symbol of any invariant of unframed knots vanishes on a diagram which has a chord that has no intersections with other chords. This does not hold for an arbitrary framed invariant.

Example. The symbol of the self-linking number is the function equal to 1 on the chord diagram with one chord.

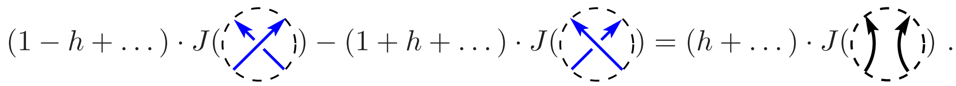
3.6. Classical knot polynomials as Vassiliev invariants

In Example 3.1.4, we have seen that the coefficients of the Conway polynomial are Vassiliev invariants. The Conway polynomial, taken as a whole, is not, of course, a finite type invariant, but it is an infinite linear combination of such; in other words, it is a power series Vassiliev invariant. This property holds for all classical knot polynomials — but only after a suitable substitution.

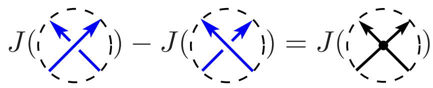
3.6.1. Modify the Jones polynomial of a knot K substituting $t = e^h$ and then expanding it into a formal power series in h. Let $j_n(K)$ be the coefficient of h^n in this expansion.

Theorem ([**BL, BN1**]). *The coefficient $j_n(K)$ is a Vassiliev invariant of order $\leqslant n$.*

Proof. Plugging $t = e^h = 1 + h + \dots$ into the skein relation from Section 2.4.3 we get



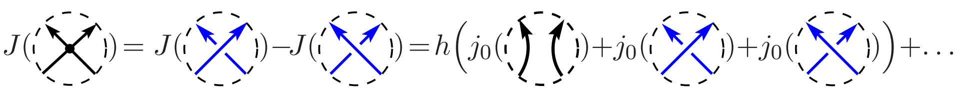
We see that the difference

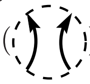


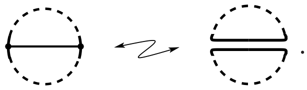
is congruent to 0 modulo h. Therefore, the Jones polynomial of a singular knot with k double points is divisible by h^k. In particular, for $k \geqslant n+1$ the coefficient of h^n equals zero. $\square$

Below we shall give an explicit description of the symbols of the finite type invariants j_n; the similar description for the Conway polynomial is left as an exercise (no. 16 at the end of the chapter, page 95).

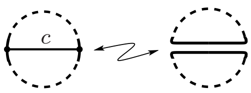
3.6.2. Symbol of the Jones invariant $j_n(K)$. To find the symbol of $j_n(K)$, we must compute the coefficient of h^n in the Jones polynomial $J(K_n)$ of a singular knot K_n with n double points in terms of its chord diagram $\sigma(K_n)$. Since

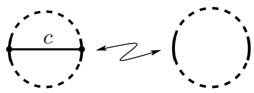


the contribution of a double point of K_n to the coefficient $j_n(K_n)$ is the sum of the values of $j_0(\cdot)$ on the three links in the parentheses above. The values of $j_0(\cdot)$ for the last two links are equal, since, according to Exercise 4 to this chapter, to $j_0(L) = (-2)^{\#(\text{components of } L)-1}$. So it does not depend on the specific way L is knotted and linked and we can freely change the under/over-crossings of L. On the level of chord diagrams these two terms mean that we just forget about the chord corresponding to this double point. The first term, $j_0($$)$, corresponds to the smoothing of the double point according to the orientation of our knot (link). On the level of chord diagrams this corresponds to the doubling of a chord:



This leads to the following procedure of computing the value of the symbol of $j_n(D)$ on a chord diagram D. Introduce *a state* s for D as an arbitrary function on the set chords of D with values in the set $\{1, 2\}$. With each state s we associate an immersed plane curve obtained from D by resolving (either doubling or deleting) all its chords according to s:



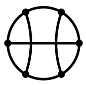
, if $s(c) = 1$;


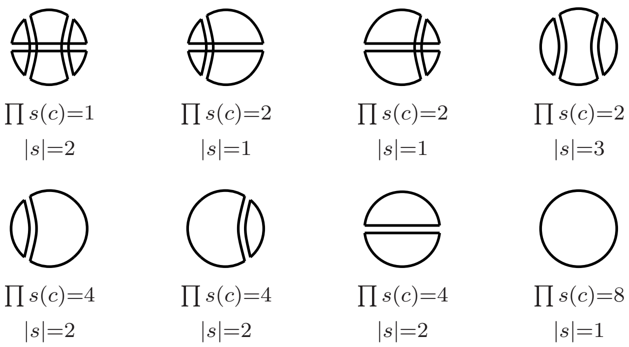
, if $s(c) = 2$.

Let $|s|$ denote the number of components of the curve obtained in this way. Then

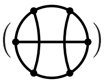
$$\operatorname{symb}(j_n)(D) = \sum_s \Big(\prod_c s(c)\Big)\ (-2)^{|s|-1}\ ,$$

where the product is taken over all n chords of D, and the sum is taken over all 2^n states for D.

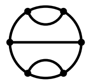
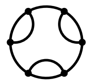
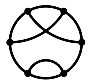
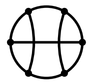
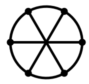
For example, to compute the value of the symbol of j_3 on the chord diagram  we must consider 8 states:



Therefore,

symb(j_3)() $= -2+2+2+2(-2)^2+4(-2)+4(-2)+4(-2)+8=-6$

Similarly one can compute the values of symb(j_3) on all chord diagrams with three chords. Here is the result:

D					
symb(j_3)(D)	0	0	0	-6	-12

This function on chord diagrams, as well as the whole Jones polynomial, is closely related to the Lie algebra $\mathfrak{sl}_2$ and its standard 2-dimensional representation. We shall return to this subject several times in the sequel (see Sections 6.1.3, 6.1.7 etc).

3.6.3. According to Exercise 25 (page 66), for the mirror reflection $\overline{K}$ of a knot K the power series expansion of $J(\overline{K})$ can be obtained from the series $J(K)$ by substituting $-h$ for h. This means that $j_{2k}(\overline{K}) = j_{2k}(K)$ and $j_{2k+1}(\overline{K}) = -j_{2k+1}(K)$.

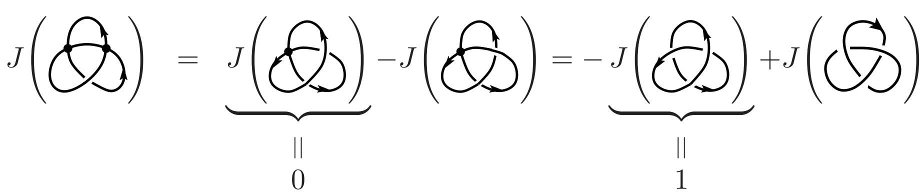
3.6.4. Table 3.6.4.1 displays the first five terms of the power series expansion of the Jones polynomial after the substitution $t = e^h$.

3_1	1	$-3h^2$	$+6h^3$	$-(29/4)h^4$	$+(13/2)h^5$	$+\dots$
4_1	1	$+3h^2$		$+(5/4)h^4$		$+\dots$
5_1	1	$-9h^2$	$+30h^3$	$-(243/4)h^4$	$+(185/2)h^5$	$+\dots$
5_2	1	$-6h^2$	$+18h^3$	$-(65/2)h^4$	$+(87/2)h^5$	$+\dots$
6_1	1	$+6h^2$	$-6h^3$	$+(17/2)h^4$	$-(13/2)h^5$	$+\dots$
6_2	1	$+3h^2$	$-6h^3$	$+(41/4)h^4$	$-(25/2)h^5$	$+\dots$
6_3	1	$-3h^2$		$-(17/4)h^4$		$+\dots$
7_1	1	$-18h^2$	$+84h^3$	$-(477/2)h^4$	$+511h^5$	$+\dots$
7_2	1	$-9h^2$	$+36h^3$	$-(351/4)h^4$	$+159h^5$	$+\dots$
7_3	1	$-15h^2$	$-66h^3$	$-(697/4)h^4$	$-(683/2)h^5$	$+\dots$
7_4	1	$-12h^2$	$-48h^3$	$-113h^4$	$-196h^5$	$+\dots$
7_5	1	$-12h^2$	$+48h^3$	$-119h^4$	$+226h^5$	$+\dots$
7_6	1	$-3h^2$	$+12h^3$	$-(89/4)h^4$	$+31h^5$	$+\dots$
7_7	1	$+3h^2$	$+6h^3$	$+(17/4)h^4$	$+(13/2)h^5$	$+\dots$
8_1	1	$+9h^2$	$-18h^3$	$+(135/4)h^4$	$-(87/2)h^5$	$+\dots$

8_2	1		$-6h^3$	$+27h^4$	$-(133/2)h^5$	$+\dots$
8_3	1	$+12h^2$		$+17h^4$		$+\dots$
8_4	1	$+9h^2$	$-6h^3$	$+(63/4)h^4$	$-(25/2)h^5$	$+\dots$
8_5	1	$+3h^2$	$+18h^3$	$+(209/4)h^4$	$+(207/2)h^5$	$+\dots$
8_6	1	$+6h^2$	$-18h^3$	$+(77/2)h^4$	$-(123/2)h^5$	$+\dots$
8_7	1	$-6h^2$	$-12h^3$	$-(47/2)h^4$	$-31h^5$	$+\dots$
8_8	1	$-6h^2$	$-6h^3$	$-(29/2)h^4$	$-(25/2)h^5$	$+\dots$
8_9	1	$+6h^2$		$+(23/2)h^4$		$+\dots$
8_{10}	1	$-9h^2$	$-18h^3$	$-(123/4)h^4$	$-(75/2)h^5$	$+\dots$
8_{11}	1	$+3h^2$	$-12h^3$	$+(125/4)h^4$	$-55h^5$	$+\dots$
8_{12}	1	$+9h^2$		$+(51/4)h^4$		$+\dots$
8_{13}	1	$-3h^2$	$-6h^3$	$-(53/4)h^4$	$-(25/2)h^5$	$+\dots$
8_{14}	1			$+6h^4$	$-18h^5$	$+\dots$
8_{15}	1	$-12h^2$	$+42h^3$	$-80h^4$	$+(187/2)h^5$	$+\dots$
8_{16}	1	$-3h^2$	$+6h^3$	$-(53/4)h^4$	$+(37/2)h^5$	$+\dots$
8_{17}	1	$+3h^2$		$+(29/4)h^4$		$+\dots$
8_{18}	1	$-3h^2$		$+(7/4)h^4$		$+\dots$
8_{19}	1	$-15h^2$	$-60h^3$	$-(565/4)h^4$	$-245h^5$	$+\dots$
8_{20}	1	$-6h^2$	$+12h^3$	$-(35/2)h^4$	$+19h^5$	$+\dots$
8_{21}	1		$-6h^3$	$+21h^4$	$-(85/2)h^5$	$+\dots$

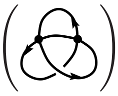
Table 3.6.4.1: Taylor expansion of the modified Jones polynomial

3.6.5. Example. In the following examples the h-expansion of the Jones polynomial starts with a power of h equal to the number of double points in a singular knot, in compliance with Theorem 3.6.1.

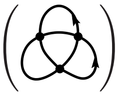
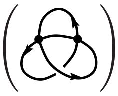
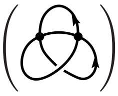


$$= \quad -1 + J(3_1) = -3h^2 + 6h^3 - \tfrac{29}{4}h^4 + \tfrac{13}{2}h^5 + \dots$$

Similarly,

$J\Big($  $\Big) = J(\overline{3_1}) - 1 = -3h^2 - 6h^3 - \frac{29}{4}h^4 - \frac{13}{2}h^5 + \dots$

Thus we have

$J\Big($  $\Big) = J\Big($  $\Big) - J\Big($  $\Big) = -12h^3 - 13h^5 + \dots$

3.6.6. J. Birman and X.-S. Lin proved in [**Bir2, BL**] that all quantum invariants produce Vassiliev invariants in the same way as the Jones polynomial. More precisely, let $\theta(K)$ be the quantum invariant constructed as in Section 2.6. It is a polynomial in q and q^{-1}. Now let us make a substitution $q = e^h$ and consider the coefficient $\theta_n(K)$ of h^n in the Taylor expansion of $\theta(K)$.

Theorem ([**BL, BN1**])**.** *The coefficient $\theta_n(K)$ is a Vassiliev invariant of order $\leqslant n$.*

Proof. The argument is similar to that used in Theorem 3.6.1: it is based on the fact that an R-matrix R and its inverse R^{-1} are congruent modulo h. □

3.6.7. The Casson invariant. The second coefficient of the Conway polynomial, or the *Casson invariant*, can be computed directly from any knot diagram by counting (with signs) pairs of crossings of certain type.

Namely, fix a based Gauss diagram G of a knot K, with an arbitrary basepoint, and consider all pairs of arrows of G that form a subdiagram of the following form:

(3.6.7.1) 

The Casson invariant $a_2(K)$ is defined as the number of such pairs of arrows with $\varepsilon_1\varepsilon_2 = 1$ minus the number of pairs of this form with $\varepsilon_1\varepsilon_2 = -1$.

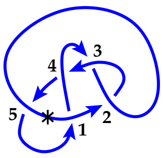
Theorem. *The Casson invariant coincides with the second coefficient of the Conway polynomial c_2.*

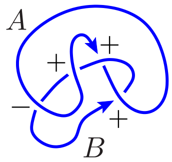
Proof. We shall prove that the Casson invariant as defined above, is a Vassiliev invariant of degree 2. It can be checked directly that it vanishes on the unknot and is equal to 1 on the left trefoil. Since the same holds for the invariant c_2 and $\dim \mathcal{V}_2 = 2$, the assertion of the theorem will follow.

First, let us verify that a_2 does not depend on the location of the basepoint on the Gauss diagram. It is enough to prove that whenever the basepoint is moved over the endpoint of one arrow, the value of a_2 remains the same.

Let c be an arrow of some Gauss diagram. For another arrow c' of the same Gauss diagram with the sign $\varepsilon(c')$, the *flow* of c' through c is equal to $\varepsilon(c')$ if c' intersects c, and is equal to 0 otherwise. The *flow to the right* through c is the sum of the flows through c of all arrows c' such that c' and c, in this order, form a positive basis of $\mathbb{R}^2$. The *flow to the left* is defined as the sum of the flows of all c' such that c', c form a negative basis. The *total flow* through the arrow c is the difference of the right and the left flows through c.

Now, let us observe that if a Gauss diagram is realizable, then the total flow through each of its arrows is equal to zero. Indeed, let us cut and reconnect the branches of the knot represented by the Gauss diagram in the vicinity of the crossing point that corresponds to the arrow c. What we get is a two-component link:





It is easy to see that the two ways of computing the linking number of the two components A and B (see Section 2.2) are equal to the right and the left flow through c respectively. Since the linking number is an invariant, the difference of the flows is 0.

Now, let us see what happens when the basepoint is moved over an endpoint of an arrow c. If this endpoint corresponds to an overcrossing, this means that the arrow c does not appear in any subdiagram of the form (3.6.7.1) and, hence, the value of a_2 remains unchanged. If the basepoint of the diagram is moved over an undercrossing, the value of a_2 changes by the amount that is equal to the number of all subdiagrams of G involving c, counted with signs. Taking the signs into the account, we see that this amount is equal to the total flow through the chord c in G, that is, zero.

Let us now verify that a_2 is invariant under the Reidemeister moves. This is clear for the move $V\Omega_1$, since an arrow with adjacent endpoints cannot participate in a subdiagram of the form (3.6.7.1).

The move $V\Omega_2$ involves two arrows; denote them by c_1 and c_2. Choose the basepoint "far" from the endpoints of c_1 and c_2, namely, in such a way that it belongs neither to the interval between the sources of c_1 and c_2, nor to the interval between the targets of these arrows. (Since a_2 does not depend on the location of the basepoint, there is no loss of generality in this choice.) Then the contribution to a_2 of any pair that contains the arrow c_1 cancels with the corresponding contribution for c_2.

The moves of type 3 involve three arrows. If we choose a basepoint far from all of these endpoints, only one of the three distinguished arrows can participate in a subdiagram of the from (3.6.7.1). It is then clear that exchanging the endpoints of the three arrows as in the move $V\Omega_3$ does not affect the value of a_2.

It remains to show that a_2 has degree 2. Consider a knot with 3 double points. Resolving the double point we obtain an alternating sum of eight knots whose Gauss diagrams are the same except for the directions and signs of 3 arrows. Any subdiagram of the form (3.6.7.1) fails to contain at least one of these three arrows. It is, therefore clear that for each instance that the Gauss diagram of one of the eight knots contains the diagram (3.6.7.1) as a subdiagram, there is another occurrence of (3.6.7.1) in another of the eight knots, counted in a_2 with the opposite sign. □

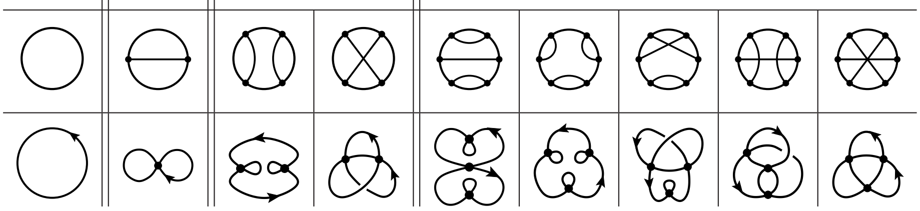
Remark. This method of calculating c_2 (invented by Polyak and Viro [**PV1, PV2**]) is an example of a *Gauss diagram formula*. See Chapter 13 for details and for more examples.

3.7. Actuality tables

In general, the amount of information needed to describe a knot invariant v is infinite, since v is a function on an infinite domain — the set of isotopy classes of knots. However, Vassiliev invariants require only a finite amount of information for their description. We already mentioned the analogy between Vassiliev invariants and polynomials. A polynomial of degree n can be described, for example, using the Lagrange interpolation formula, by its values in $n+1$ particular points. In a similar way a given Vassiliev invariant can be described by its values on a finitely many knots. These values are organized in the *actuality table* (see [**Va1, BL, Bir2**]).

3.7.1. Basic knots and actuality tables. To construct the actuality table we must choose a representative (*basic*) singular knot for every chord diagram. A possible choice of basic knots up to degree 3 is shown in the table.

$\mathbf{A}_0$	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$



The actuality table for a particular invariant v of order $\leqslant n$ consists of the set of its values on the set of all basic knots with at most n double points. The knowledge of this set is sufficient for calculating v for any knot.

Indeed, any knot K can be transformed into any other knot, in particular, into the basic knot with no singularities (in the table above this is the unknot), by means of crossing changes and isotopies. The difference of two knots that participate in a crossing change is a knot with a double point, hence in $\mathbb{Z}\mathcal{K}$ the knot K can be written as a sum of the basic non-singular knot and several knots with one double point. In turn, each knot with one double point can be transformed, by crossing changes and isotopies, into the basic singular knot *with the same chord diagram*, and can be written, as a result, as a sum of a basic knot with one double point and several knots with two double points. This process can be iterated until we obtain a representation of the knot K as a sum of basic knots with at most n double points and several knots with $n+1$ double points. Now, since v is of order $\leqslant n$, it vanishes on the knots with $n+1$ double points, so $v(K)$ can be written as a sum of the values of v on the basic knots with at most n singularities.

By Proposition 3.4.2, the values of v on the knots with precisely n double points depend only on their chord diagrams. For smaller number of double points, the values of v in the actuality table depend not only on chord diagrams, but also on the basic knots. Of course, the values in the actuality table cannot be arbitrary. They satisfy certain relations which we shall discuss later (see Section 4.1). The simplest of these relations, however, is easy to spot from the examples: the value of any invariant on a diagram with a chord that has no intersections with other chords is zero.

3.7.2. Example. The second coefficient c_2 of the Conway polynomial (Section 3.1.2) is a Vassiliev invariant of order $\leqslant 2$. Here is an actuality table for it.

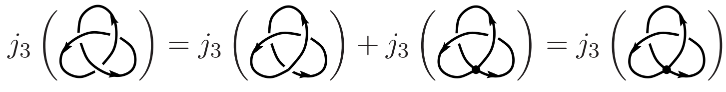
$$c_2 : \qquad 0 \,\|\, 0 \,\|\, 0 \,|\, 1$$

The order of the values in this table corresponds to the order of basic knots in the table on page 90.

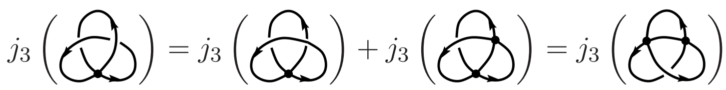
3.7.3. Example. A Vassiliev invariant of order 3 is given by the third coefficient j_3 of the Taylor expansion of Jones polynomial (Section 2.4). The actuality table for j_3 looks as follows.

$$j_3 : \qquad 0 \,\|\, 0 \,\|\, 0 \mid 6 \,\|\, 0 \mid 0 \mid 0 \mid -6 \mid -12$$

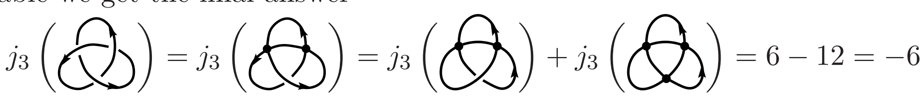
3.7.4. To illustrate the general procedure of computing the value of a Vassiliev invariant on a particular knot by means of actuality tables let us compute the value of j_3 on the right-hand trefoil. The right-hand trefoil is an ordinary knot, without singular points, so we have to deform it (using crossing changes) to our basic knot without double points, that is, the unknot. This can be done by one crossing change, and by the Vassiliev skein relation we have



because $j_3(\mathit{unknot}) = 0$ in the actuality table. Now the knot with one double point we got is not quite the one from our basic knots. We can deform it to a basic knot changing the upper right crossing.



Here we used the fact that *any* invariant vanishes on the basic knot with a single double point. The knot with two double points on the right-hand side of the equation still differs by one crossing from the basic knot with two double points. This means that we have to do one more crossing change. Combining these equations together and using the values from the actuality table we get the final answer



3.7.5. The first ten Vassiliev invariants. Using actuality tables, one can find the values of the Vassiliev invariants of low degree. Table 3.7.5.1 uses a certain basis in the space of Vassiliev invariants up to degree 5. It represents an abridged version of the table compiled by T. Stanford [**Sta1**], where the values of invariants up to degree 6 are given on all knots up to 10 crossings.

Some of the entries in Table 3.7.5.1 are different from [**Sta1**], this is due to the fact that, for some non-amphicheiral knots, Stanford uses mirror reflections of the Rolfsen's knots shown in Table 1.5.2.1.

		v_0	v_2	v_3	v_{41}	v_{42}	v_2^2	v_{51}	v_{52}	v_{53}	v_2v_3
0_1	$++$	1	0	0	0	0	0	0	0	0	0
3_1	$-+$	1	1	-1	1	-3	1	-3	1	-2	-1

4_1	$++$	1	-1	0	-2	3	1	0	0	0	0
5_1	$-+$	1	3	-5	1	-6	9	-12	4	-8	-15
5_2	$-+$	1	2	-3	1	-5	4	-7	3	-5	-6
6_1	$-+$	1	-2	1	-5	5	4	4	-1	2	-2
6_2	$-+$	1	-1	1	-3	1	1	3	-1	1	-1
6_3	$++$	1	1	0	2	-2	1	0	0	0	0
7_1	$-+$	1	6	-14	-4	-3	36	-21	7	-14	-84
7_2	$-+$	1	3	-6	0	-5	9	-9	6	-7	-18
7_3	$-+$	1	5	11	-3	-6	25	16	-8	13	55
7_4	$-+$	1	4	8	-2	-8	16	10	-8	10	32
7_5	$-+$	1	4	-8	0	-5	16	-14	6	-9	-32
7_6	$-+$	1	1	-2	0	-3	1	-2	3	-2	-2
7_7	$-+$	1	-1	-1	-1	4	1	0	2	0	1
8_1	$-+$	1	-3	3	-9	5	9	12	-3	5	-9
8_2	$-+$	1	0	1	-3	-6	0	2	0	-3	0
8_3	$++$	1	-4	0	-14	8	16	0	0	0	0
8_4	$-+$	1	-3	1	-11	4	9	0	-2	-1	-3
8_5	$-+$	1	-1	-3	-5	-5	1	-5	3	2	3
8_6	$-+$	1	-2	3	-7	0	4	9	-3	2	-6
8_7	$-+$	1	2	2	4	-2	4	7	-1	3	4
8_8	$-+$	1	2	1	3	-4	4	2	-1	1	2
8_9	$++$	1	-2	0	-8	1	4	0	0	0	0
8_{10}	$-+$	1	3	3	3	-6	9	5	-3	3	9
8_{11}	$-+$	1	-1	2	-4	-2	1	8	-1	2	-2
8_{12}	$++$	1	-3	0	-8	8	9	0	0	0	0
8_{13}	$-+$	1	1	1	3	0	1	6	0	3	1
8_{14}	$-+$	1	0	0	-2	-3	0	-2	0	-3	0
8_{15}	$-+$	1	4	-7	1	-7	16	-16	5	-10	-28
8_{16}	$-+$	1	1	-1	3	0	1	2	2	2	-1
8_{17}	$+-$	1	-1	0	-4	0	1	0	0	0	0
8_{18}	$++$	1	1	0	0	-5	1	0	0	0	0
8_{19}	$-+$	1	5	10	0	-5	25	18	-6	10	50
8_{20}	$-+$	1	2	-2	2	-5	4	-1	3	-1	-4
8_{21}	$-+$	1	0	1	-1	-3	0	1	-1	-1	0

Table 3.7.5.1: Vassiliev invariants of order $\leqslant 5$

The two signs after the knot number refer to their symmetry properties: a plus in the first position means that the knot is amphicheiral, a plus in the second position means that the knot is invertible.

3.8. Vassiliev invariants of tangles

Knots are tangles whose skeleton is a circle. A theory of Vassiliev invariants, similar to the theory for knots, can be constructed for isotopy classes of tangles with any given skeleton $\boldsymbol{X}$.

Indeed, similarly to the case of knots, one can introduce *tangles with double points*, with the only extra assumption that the double points lie in the interior of the tangle box. Then, any invariant of tangles can be extended to tangles with double points with the help of the Vassiliev skein relation. An invariant of tangles is a Vassiliev invariant of degree $\leqslant n$ if it vanishes on all tangles with more that n double points.

We stress that we define Vassiliev invariants separately for each skeleton $\boldsymbol{X}$. Nevertheless, there are relations among invariants of tangles with different skeleta.

Example. Assume that the isotopy classes of tangles with the skeleta $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ can be multiplied. Given a tangle T with skeleton $\boldsymbol{X}_1$ and a Vassiliev invariant v of tangles with skeleton $\boldsymbol{X}_1\boldsymbol{X}_2$, we can define an invariant of tangles on $\boldsymbol{X}_2$ of the same order as v by composing a tangle with T and applying v.

Example. In the above example the product of tangles can be replaced by their tensor product. (Of course, the condition that $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ can be multiplied is no longer necessary here.)

In particular, the Vassiliev invariants of tangles whose skeleton has one component, can be identified with the Vassiliev invariants of knots.

Example. Assume that $\boldsymbol{X}'$ is obtained from $\boldsymbol{X}$ by dropping one or several components. Then any Vassiliev invariant v' of tangles with skeleton $\boldsymbol{X}'$ gives rise to an invariant v of tangles on $\boldsymbol{X}$ of the same order; to compute v drop the components of the tangle that are not in $\boldsymbol{X}'$ and apply v'.

This example immediately produces a lot of tangle invariants of finite type: namely, those coming from knots. The simplest example of a Vassiliev invariant that does not come from knots is the linking number of two components of a tangle. So far, we have defined the linking number only for pairs of closed curves. If one or both of the components are not closed, we can use the constructions above to close them up in some fixed way.

Lemma. *The linking number of two components of a tangle is a Vassiliev invariant of order 1.*

Proof. Consider a two-component link with one double point. This double point can be of two types: either it is a self-intersection point of a single component, or it is an intersection of two different components. Using the

Vassiliev skein relation and the formula 2.2.1, we see that in the first case the linking number vanishes, while in the second case it is equal to 1. It follows that for a two-component link with two double points the linking number is always zero. □

Among the invariants for all classes of tangles, the string link invariants have attracted most attention. Two particular classes of string link invariants are the knot invariants (recall that string links on one strand are in one-to-one correspondence with knots) and the invariants of pure braids. We shall treat the Vassiliev invariants of pure braids in detail in Chapter 12.

Exercises

(1) Using the actuality tables, compute the value of j_3 on the left-hand trefoil.

(2) Choose the basic knots with four double points and construct the actuality tables for the fourth coefficients c_4 and j_4 of the Conway and Jones polynomials.

(3) Prove that $j_0(K) = 1$ and $j_1(K) = 0$ for any knot K.

(4) Show that the value of j_0 on a link with k components is equal to $(-2)^{k-1}$.

(5) For a link L with two components K_1 and K_2 prove that $j_1(L) = -3 \cdot lk(K_1, K_2)$. In other words,
$$J(L) = -2 - 3 \cdot lk(K_1, K_2) \cdot h + j_2(L) \cdot h^2 + j_3(L) \cdot h^3 + \dots \ .$$

(6) Prove that for any knot K the integer $j_3(K)$ is divisible by 6.

(7) For a knot K, find the relation between the second coefficients $c_2(K)$ and $j_2(K)$ of the Conway and Jones polynomials.

(8) Prove that $v(3_1\#3_1) = 2v(3_1) - v(0)$, where 0 is the trivial knot, for any Vassiliev invariant $v \in \mathcal{V}_3$.

(9) Prove that for a knot K the n-th derivative at 1 of the Jones polynomial
$$\left.\frac{d^n(J(K))}{dt^n}\right|_{t=1}$$
is a Vassiliev invariant of order $\leqslant n$. Find the relation between these invariants and $j_1, \dots, j_n$ for small values of n.

(10) Express the coefficients c_2, c_4, j_2, j_3, j_4, j_5 of the Conway and Jones polynomials in terms of the basis Vassiliev invariants from Table 3.7.5.1.

(11) Find the symbols of the Vassiliev invariants from Table 3.7.5.1.

(12) Express the invariants of Table 3.7.5.1 through the coefficients of the Conway and the Jones polynomials.

(13) Find the actuality tables for some of the Vassiliev invariants appearing in Table 3.7.5.1.

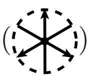
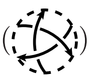
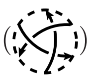
(14) Explain the correlation between the first sign and the zeroes in the last four columns of Table 3.7.5.1.

(15) Check that Vassiliev invariants up to order 4 are enough to distinguish, up to orientation, all knots with at most 8 crossings from Table 1.5.2.1 on page 26.

(16) Prove that the symbol of the coefficient c_n of the Conway polynomial can be calculated as follows. Double every chord of a given chord diagram D as in Section 3.6.2, and let $|D|$ be equal to the number of components of the obtained curve. Then

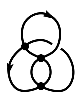
$$\mathrm{symb}(c_n)(D) = \begin{cases} 1, & \text{if } |D| = 1 \\ 0, & \text{otherwise} \end{cases} .$$

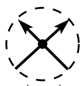
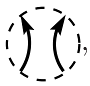
(17) Prove that there is a well-defined extension of knot invariants to singular knots with a non-degenerate *triple* point according to the rule

$f($  $) = f($  $) - f($  $)$.

Is it true that, according to this extension, a Vassiliev invariant of degree 2 is equal to 0 on any knot with a triple point?

Is it possible to use the same method to define an extension of knot invariants to knots with self-intersections of multiplicity higher than 3?

(18) Following Example 3.6.5, find the power series expansion of the modified Jones polynomial of the singular knot  .

(19) Prove the following relation between the Casson knot invariant c_2, extended to singular knots, and the linking number of two curves. Let K be a knot with one double point. Smoothing the double point by the rule  $\mapsto$ , one obtains a 2-component link L. Then $lk(L) = c_2(K)$.

(20) Is there a prime knot K such that $j_4(K) = 0$?

(21) **Vassiliev invariants from the HOMFLY polynomial.** For a link L make a substitution $a = e^h$ in the HOMFLY polynomial $P(L)$ and take the Taylor expansion in h. The result will be a Laurent polynomial in z and a power series in h. Let $p_{k,l}(L)$ be its coefficient at $h^k z^l$.

(a) Show that for any link L the total degree $k + l$ is not negative.

(b) If l is odd, then $p_{k,l} = 0$.
(c) Prove that $p_{k,l}(L)$ is a Vassiliev invariant of order $\leqslant k + l$.
(d) Describe the symbol of $p_{k,l}(L)$.

Chapter 4

Chord diagrams

A chord diagram encodes the order of double points along a singular knot. We saw in the last chapter that a Vassiliev invariant of order n gives rise to a function on chord diagrams with n chords. Here we shall describe the conditions, called one-term and four-term relations, that a function on chord diagrams should satisfy in order to come from a Vassiliev invariant. We shall see that the vector space spanned by chord diagrams modulo these relations has the structure of a Hopf algebra. This Hopf algebra turns out to be dual to the algebra of the Vassiliev invariants.

4.1. Four- and one-term relations

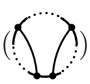
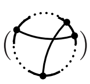
Recall that $\mathcal{R}$ denotes a commutative ring and $\mathcal{V}_n$ is the space of $\mathcal{R}$-valued Vassiliev invariants of order $\leqslant n$. Some of our results will only hold when $\mathcal{R}$ is a field of characteristic 0; sometimes we shall take $\mathcal{R} = \mathbb{C}$. On page 80 in Section 3.1.2 we constructed a linear inclusion (the symbol of an invariant)

$$\overline{\alpha}_n : \mathcal{V}_n/\mathcal{V}_{n-1} \to \mathcal{R}\mathbf{A}_n,$$

where $\mathcal{R}\mathbf{A}_n$ is the space of $\mathcal{R}$-valued functions on the set $\mathbf{A}_n$ of chord diagrams of order n.

To describe the image of $\overline{\alpha}_n$, we need the following definition.

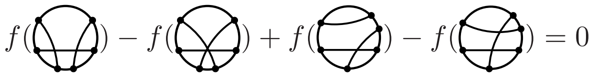
4.1.1. Definition. A function $f \in \mathcal{R}\mathbf{A}_n$ is said to satisfy the *4-term (or 4T) relations* if the alternating sum of the values of f is zero on the following quadruples of diagrams:

(4.1.1.1) $f($$) - f($$) + f($$) - f($$) = 0.$

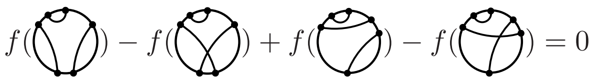
In this case f is also called a *(framed) weight system* of order n.

Here it is assumed that the diagrams in the pictures may have other chords with endpoints on the dotted arcs, while all the endpoints of the chords on the solid portions of the circle are explicitly shown. For example, this means that in the first and second diagrams the two bottom points are adjacent. The chords omitted from the pictures should be the same in all the four cases.

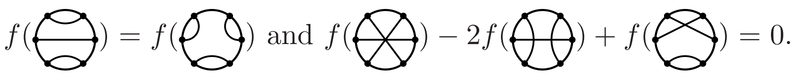
Example. Let us find all 4-term relations for chord diagrams of order 3. We must add one chord in one and the same way to all the four terms of Equation (4.1.1.1). Since there are 3 dotted arcs, there are 6 different ways to do that, in particular,



and



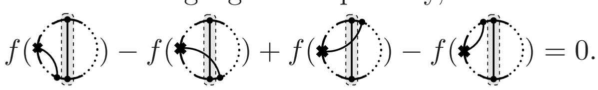
Some of the diagrams in these equations are equal, and the relations can be simplified as



The reader is invited to check that the remaining four 4-term relations (we wrote only 2 out of 6) are either trivial or coincide with one of these two.

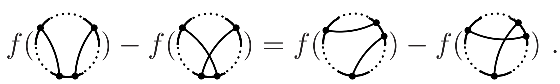
It is often useful to look at a 4T relation from the following point of view. We can think that one of the two chords that participate in equation (4.1.1.1) is fixed, and the other is moving. One of the ends of the moving chord is also fixed, while the other end travels around the fixed chord stopping at the four locations adjacent to its endpoints. The resulting four diagrams are then summed up with alternating signs. Graphically,

(4.1.1.2)



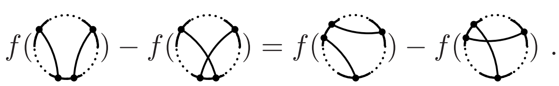
where the fixed end of the moving chord is marked by ✖.

Another way of writing the 4T relation, which will be useful in Section 5.1, is to split the four terms into two pairs:



Because of the obvious symmetry, this can be completed as follows:

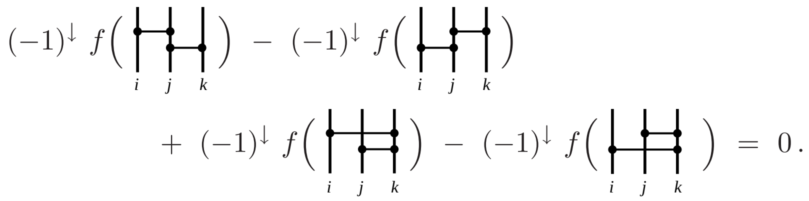
(4.1.1.3)



Note that for each order n the choice of a specific 4-term relation depends on the following data:

- a diagram of order $n-1$,
- a distinguished chord of this diagram ("fixed chord"), and
- a distinguished arc on the circle of this diagram (where the fixed endpoint of the "moving chord" is placed).

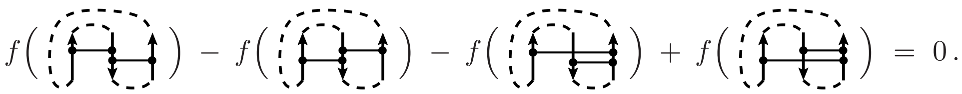
There are 3 fragments of the circle that participate in a 4-term relation, namely, those that are shown by solid lines in the equations above. If these 3 fragments are drawn as 3 vertical line segments, then the 4-term relation can be restated as follows:

(4.1.1.4) 

where $\downarrow$ stands for the number of endpoints of the chords in which the orientation of the strands is directed downwards. This form of a 4T relation is called a *horizontal 4T relation* . It first appeared, in a different context, in the work by T. Kohno [**Koh2**].

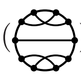
4.1.2. Exercise. Choose some orientations of the three fragments of the circle, add the portions necessary to close it up and check that the last form of the 4-term relation carries over into the ordinary four-term relation.

Here is an example:



We shall see in the next section that the four-term relations are always satisfied by the symbols of Vassiliev invariants, both in the usual and in the framed case. For the framed knots, there are no other relations; in the unframed case, there is another set of relations, called *one-term*, or *framing independence* relations.

4.1.3. Definition. An *isolated chord* is a chord that does not intersect any other chord of the diagram. A function $f \in \mathcal{R}\mathbf{A}_n$ is said to satisfy the *1-term relations* if it vanishes on every chord diagram with an isolated chord. An *unframed weight system* of order n is a weight system that satisfies the 1-term relations.

Here is an example of a 1T relation: $f($$) = 0$.

4.1.4. Notation. We denote by $\mathcal{W}_n^{fr}$ the subspace of $\mathcal{R}\mathbf{A}_n$ consisting of all (framed) weight systems of order n and by $\mathcal{W}_n \subset \mathcal{W}_n^{fr}$ the space of all unframed weight systems of order n.

4.2. The Fundamental Theorem

In Section 3.4 we showed that the symbol of an invariant gives an injective map $\overline{\alpha}_n : \mathcal{V}_n/\mathcal{V}_{n-1} \to \mathcal{R}\mathbf{A}_n$. The Fundamental Theorem on Vassiliev invariants describes its image.

4.2.1. Theorem (Vassiliev–Kontsevich)**.** *For $\mathcal{R} = \mathbb{C}$ the map $\overline{\alpha}_n$ identifies $\mathcal{V}_n/\mathcal{V}_{n-1}$ with the subspace of unframed weight systems $\mathcal{W}_n \subset \mathcal{R}\mathbf{A}_n$. In other words, the space of unframed weight systems is isomorphic to the graded space associated with the filtered space of Vassiliev invariants,*

$$\mathcal{W} = \bigoplus_{n=0}^{\infty} \mathcal{W}_n \;\cong\; \bigoplus_{n=0}^{\infty} \mathcal{V}_n/\mathcal{V}_{n+1}\ .$$

The theorem consists of two parts:

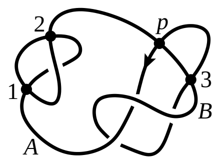
- (V. Vassiliev) The symbol of every Vassiliev invariant is an unframed weight system.
- (M. Kontsevich) Every unframed weight system is the symbol of a certain Vassiliev invariant.

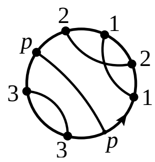
We shall now prove the first (easy) part of the theorem. The second (difficult) part will be proved later (in Section 8.8) using the Kontsevich integral.

The first part of the theorem consists of two assertions, and we prove them one by one.

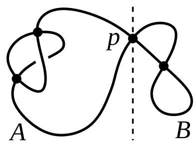
4.2.2. First assertion: *any function $f \in \mathcal{R}\mathbf{A}_n$ coming from an invariant $v \in \mathcal{V}_n$ satisfies the 1-term relations.*

Proof. Let K be a singular knot whose chord diagram contains an isolated chord. The double point p that corresponds to the isolated chord divides the knot into two parts: A and B.





The fact that the chord is isolated means that A and B do not have common double points. There may, however, be crossings involving branches from both parts. By crossing changes, we can untangle part A from part B thus obtaining a singular knot K' with the same chord diagram as K and with the property that the two parts lie on either side of some plane in $\mathbb{R}^3$ that passes through the double point p :

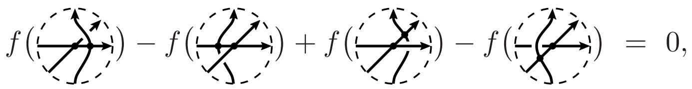


Here it is obvious that the two resolutions of the double point p give equivalent singular knots, therefore $v(K) = v(K') = v(K'_+) - v(K'_-) = 0$. □

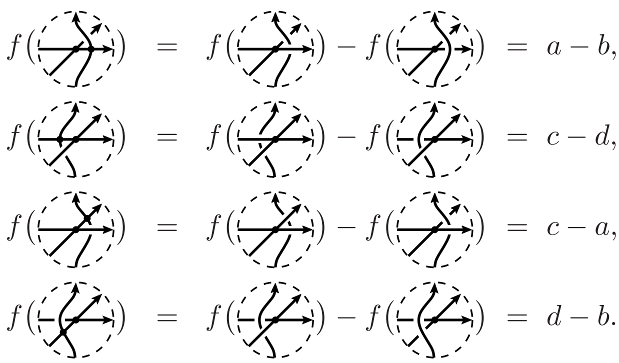
4.2.3. Second assertion: *any function $f \in \mathcal{R}\mathbf{A}_n$ coming from an invariant $v \in \mathcal{V}_n$ satisfies the 4-term relations.*

Proof. We shall use the following lemma.

Lemma (4-term relation for knots)**.** *Any Vassiliev invariant satisfies*

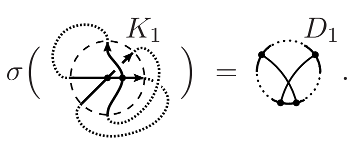


Proof. By the Vassiliev skein relation,

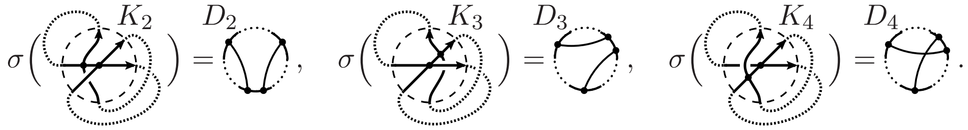


The alternating sum of these expressions is $(a-b)-(c-d)+(c-a)-(d-b)=0$, and the lemma is proved. □

Now, denote by $D_1, \ldots, D_4$ the four diagrams in a 4T relation. In order to prove the 4-term relation for the symbols of Vassiliev invariants, let us choose for the first diagram D_1 an arbitrary singular knot K_1 such that $\sigma(K_1) = D_1$:



Then the three remaining knots K_2, K_3, K_4 that participate in the 4-term relation for knots, correspond to the three remaining chord diagrams of the 4-term relation for chord diagrams, and the claim follows from the lemma.



□

4.2.4. The case of framed knots. As in the case of usual knots, for the invariants of framed knots we can define a linear map $\mathcal{V}_n^{fr}/\mathcal{V}_{n-1}^{fr} \to \mathcal{R}\mathbf{A}_n$. This map satisfies the 4T relations, but does *not* satisfy the 1T relation, since the two knots differing by a crossing change (see the proof of the first assertion in 4.2.2), are not equivalent as framed knots (the two framings differ by 2). The Fundamental Theorem also holds, in fact, for framed knots: we have the equality

$$\mathcal{V}_n^{fr}/\mathcal{V}_{n-1}^{fr} = \mathcal{W}_n^{fr};$$

it can be proved using the Kontsevich integral for framed knots (see Section 9.6).

This explains why the 1-term relation for the Vassiliev invariants of (unframed) knots is also called the *framing independence relation* .

4.2.5. We see that, in a sense, the 4T relations are more fundamental than the 1T relations. Therefore, in the sequel we shall mainly study combinatorial structures involving the 4T relations only. In any case, 1T relations can be added at all times, either by considering an appropriate subspace or an appropriate quotient space (see Section 4.4.5). This is especially easy to do in terms of the primitive elements (see page 113): the problem reduces to simply leaving out one primitive generator.

4.3. Bialgebras of knots and knot invariants

Prerequisites on bialgebras can be found in the Appendix (see page 420). In this section it will be assumed that $\mathcal{R} = \mathbb{F}$, a field of characteristic zero.

In Section 2.5 we noted that the algebra of knot invariants $\mathcal{I}$, as a vector space, is dual to the algebra of knots $\mathbb{F}\mathcal{K} = \mathbb{Z}\mathcal{K} \otimes \mathbb{F}$.

Now, using the construction of dual bialgebras (Section A.2.8), we can define the coproducts in $\mathbb{F}\mathcal{K}$ and $\mathcal{I}$, using the products in $\mathcal{I}$ and $\mathbb{F}\mathcal{K}$, respectively. The explicit formulae are:

$$\delta(K) = K \otimes K$$

for a knot K (then extended by linearity to the entire space $\mathbb{F}\mathcal{K}$) and

$$\delta(f)(K_1 \otimes K_2) = f(K_1 \# K_2)$$

for an invariant f and any pair of knots K_1 and K_2.

4.3.1. Exercise. Define the counits and check the compatibility of the product and coproduct in each of the two algebras.

Let us find the *primitive* and the *group-like* elements in the algebras $\mathbb{F}\mathcal{K}$ and $\mathcal{I}$ (see definitions in Appendix A.2.6 on page 422). As concerns the algebra of knots $\mathbb{F}\mathcal{K}$, both structures are quite poor: it follows from the definitions that $\mathcal{P}(\mathbb{F}\mathcal{K}) = 0$, while $\mathcal{G}(\mathbb{F}\mathcal{K})$ consists of only one element — the trivial knot. (Non-trivial knots are semigroup-like, but not group-like!)

The situation is quite different for the algebra of invariants. As a consequence of Proposition A.2.12 we obtain a description of primitive and group-like knot invariants (in particular, Vassiliev invariants): these are nothing but the *additive* and the *multiplicative* invariants, respectively, that is, the invariants satisfying the relations

$$f(K_1 \# K_2) = f(K_1) + f(K_2),$$
$$f(K_1 \# K_2) = f(K_1) f(K_2),$$

respectively, for any two knots K_1 and K_2.

It follows that the sets $\mathcal{P}(\mathcal{I})$ and $\mathcal{G}(\mathcal{I})$ are rather big; they are related with each other by the log-exp correspondence.

Examples. 1. The genus $g(K)$ of the knot K is a primitive invariant. Its exponent $2^{g(K)}$ is a group-like invariant. These invariants are not of finite type.

2. According to Exercises (6) and (7) to Chapter 2, both the Conway and the Jones polynomials are group-like knot invariants. They both belong to the closure of the space of finite type invariants, although neither is of finite type in the proper sense of the word. Taking the logarithm of either $C(K)$ or $J(K)$, one obtains primitive power series Vassiliev invariants. For example, the coefficient c_2 (the Casson invariant) is primitive.

4.3.2. Exercise. Find a finite linear combination of coefficients j_n of the Jones polynomial that gives a primitive Vassiliev invariant.

4.3.3. Exercise. Prove that the only group-like Vassiliev invariant is the constant 1.

The *singular knot filtration* $\mathcal{K}_n$ on $\mathbb{F}\mathcal{K}$ is obtained from the singular knot filtration on $\mathbb{Z}\mathcal{K}$ simply by tensoring it with the field $\mathbb{F}$.

4.3.4. Theorem. *The bialgebra of knots $\mathbb{F}\mathcal{K}$ supplied with the singular knot filtration (page 74) is a d-filtered bialgebra (page 427), that is, a bialgebra with a decreasing filtration.*

Proof. There are two assertions to prove:

(1) If $x \in \mathcal{K}_m$ and $y \in \mathcal{K}_n$, then $xy \in \mathcal{K}_{m+n}$,
(2) If $x \in \mathcal{K}_n$, then $\delta(x) \in \sum\limits_{p+q=n} \mathcal{K}_p \otimes \mathcal{K}_q$.

The first assertion was proved in Chapter 3.

To prove (2), first let us introduce some additional notation.

Let K be a knot given by a plane diagram with $\geqslant n$ crossings out of which exactly n are distinguished and numbered. Consider the set $\hat{K}$ of 2^n knots that may differ from K by crossing changes at the distinguished points and the vector space $X_K \subset \mathbb{F}\mathcal{K}$ spanned by $\hat{K}$. The group $\mathbb{Z}_2^n$ acts on the set $\hat{K}$; the action of i-th generator s_i consists in the flip of under/overcrossing at the distinguished point number i. We thus obtain a set of n commuting linear operators $s_i : X_K \to X_K$. Set $\sigma_i = 1 - s_i$. In these terms, a typical generator x of $\mathcal{K}_n$ can be written as $x = (\sigma_1 \circ \cdots \circ \sigma_n)(K)$. To evaluate $\delta(x)$, we must find the commutator relations between the operators δ and σ_i.

4.3.5. Lemma.

$$\delta \circ \sigma_i = (\sigma_i \otimes \mathrm{id} + s_i \otimes \sigma_i) \circ \delta,$$

where both the left-hand side and the right-hand side are understood as linear operators from X_K into $X_K \otimes X_K$.

Proof. Just check that the values of both operators on an arbitrary element of the set $\hat{K}$ are equal. □

A successive application of the lemma yields:

$$\begin{aligned}\delta \circ \sigma_1 \circ \cdots \circ \sigma_n &= \left(\prod_{i=1}^{n} (\sigma_i \otimes \mathrm{id} + s_i \otimes \sigma_i) \right) \circ \delta \\ &= \Big(\sum_{I \subset \{1,\dots,n\}} \prod_{i \in I} \sigma_i \prod_{i \notin I} s_i \otimes \prod_{i \notin I} \sigma_i \Big) \circ \delta \,.\end{aligned}$$

Therefore, an element $x = (\sigma_1 \circ \cdots \circ \sigma_n)(K)$ satisfies

$$\delta(x) = \sum_{I \subset \{1,\dots,n\}} \Big(\prod_{i \in I} \sigma_i \prod_{i \notin I} s_i\Big)(K) \otimes \Big(\prod_{i \notin I} \sigma_i\Big)(K)$$

which obviously belongs to $\sum_{p+q=n} \mathbb{Z}\mathcal{K}_p \otimes \mathbb{Z}\mathcal{K}_q$. □

4.3.6. Proposition. *The algebra of knot invariants $\mathcal{I}$ is an i-filtered bialgebra (page 427), that is, a bialgebra with an increasing filtration; in particular,* *this implies that the product of Vassiliev invariants of degrees $\leqslant p$ and $\leqslant q$ is a Vassiliev invariant of degree no greater than $p+q$.*

This theorem has been proved in Section 3.2.3. Also, in the setting of Hopf algebras it is a direct corollary of Theorems 4.3.4 and A.2.20 (see Appendix).

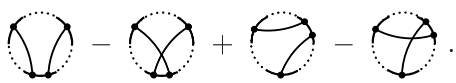
The algebra of Vassiliev invariants $\mathcal{V}$ is a subalgebra of $\mathcal{I}$; in the terminology of Section A.2.13 it is nothing but the reduced part of $\mathcal{I}$ by singular knot filtration.

4.3.7. Exercise. Prove that singular knot filtration on the algebra of knots is infinite and, moreover, that the dimension of the quotient $\mathbb{F}\mathcal{K}/\cap_{n=0}^{\infty}\mathcal{K}_n$ is infinite. As a consequence, show that the filtration of the algebra of Vassiliev invariants by degree is infinite and $\dim\mathcal{V}=\infty$.

4.4. Bialgebra of chord diagrams

4.4.1. The vector space of chord diagrams. A dual way to define the weight systems is to introduce the 1- and 4-term relations directly in the vector space spanned by chord diagrams.

4.4.2. Definition. The space $\mathcal{A}_n^{fr}$ of chord diagrams of order n is the vector space generated by the set $\mathbf{A}_n$ (all diagrams of order n) modulo the subspace spanned by all 4-term linear combinations

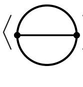


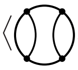
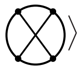
The space $\mathcal{A}_n$ of *unframed* chord diagrams of order n is the quotient of $\mathcal{A}_n^{fr}$ by the subspace spanned by all diagrams with an isolated chord.

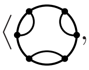
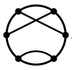
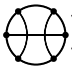
In these terms, the space of framed weight systems $\mathcal{W}_n^{fr}$ is dual to the space of framed chord diagrams $\mathcal{A}_n^{fr}$, and the space of unframed weight systems $\mathcal{W}_n$ — to that of unframed chord diagrams $\mathcal{A}_n$:

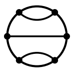
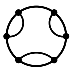
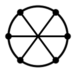
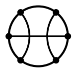
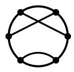
$$\begin{aligned}\mathcal{W}_n &= \mathrm{Hom}(\mathcal{A}_n,\mathcal{R}),\\ \mathcal{W}_n^{fr} &= \mathrm{Hom}(\mathcal{A}_n^{fr},\mathcal{R}).\end{aligned}$$

Below, we list the dimensions and some bases of the spaces $\mathcal{A}_n^{fr}$ for $n=1$, 2 and 3:

$\mathcal{A}_1^{fr}=\langle$  $\rangle$, $\dim\mathcal{A}_1^{fr}=1$.

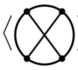
$\mathcal{A}_2^{fr} = \langle$,$\rangle$, $\dim \mathcal{A}_2^{fr} = 2$, since the only 4-term relation involving chord diagrams of order 2 is trivial.

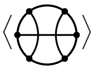
$\mathcal{A}_3^{fr} = \langle$,,$\rangle$, $\dim \mathcal{A}_3^{fr} = 3$, since $\mathbf{A}_3$ consists of 5 elements, and there are 2 independent 4-term relations (see page 98):

 =  and  − 2 +  = 0.

Taking into account the 1-term relations, we get the following result for the spaces of unframed chord diagrams of small orders:

$\mathcal{A}_1 = 0$, $\dim \mathcal{A}_1 = 0$.

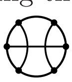
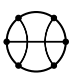
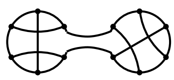
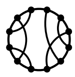
$\mathcal{A}_2 = \langle$$\rangle$, $\dim \mathcal{A}_2 = 1$.

$\mathcal{A}_3 = \langle$$\rangle$, $\dim \mathcal{A}_3 = 1$.

The result of similar calculations for order 4 diagrams is presented in Table 4.4.2.1. In this case $\dim \mathcal{A}_4^{fr} = 6$; the set $\{d_3^4, d_6^4, d_7^4, d_{15}^4, d_{17}^4, d_{18}^4\}$ is used in the table as a basis. The table is obtained by running Bar-Natan's computer program available at [**BN5**]. The numerical notation for chord diagrams like [12314324] is easy to understand: one writes the numbers on the circle in the positive direction and connects equal numbers by chords. Of all possible codes we choose the lexicographically minimal one.

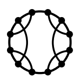
4.4.3. Multiplication of chord diagrams. Now we are ready to define the structure of an algebra in the vector space $\mathcal{A}^{fr} = \bigoplus_{k\geqslant 0} \mathcal{A}_k^{fr}$ of chord diagrams.

Definition. The *product* of two chord diagrams D_1 and D_2 is defined by cutting and glueing the two circles as shown:

 ·  =  = 

This map is then extended by linearity to

$$\mu : \mathcal{A}_m^{fr} \otimes \mathcal{A}_n^{fr} \to \mathcal{A}_{m+n}^{fr}.$$

Note that the product of diagrams depends on the choice of the points where the diagrams are cut: in the example above we could equally well cut the circles in other places and get a different result:  .

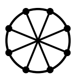
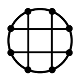
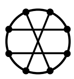
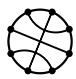
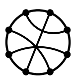
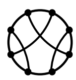
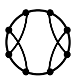
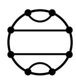
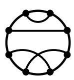
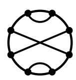
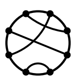
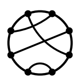
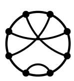
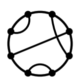
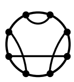
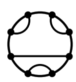
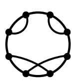
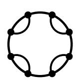
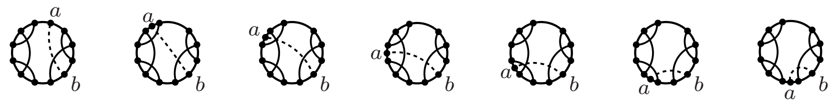
CD	Code and expansion	CD	Code and expansion
	$d_1^4 = [12341234]$ $= d_3^4 + 2d_6^4 - d_7^4 - 2d_{15}^4 + d_{17}^4$		$d_2^4 = [12314324]$ $= d_3^4 - d_6^4 + d_7^4$
	$d_3^4 = [12314234]$ $= d_3^4$		$d_4^4 = [12134243]$ $= d_6^4 - d_7^4 + d_{15}^4$
	$d_5^4 = [12134234]$ $= 2d_6^4 - d_7^4$		$d_6^4 = [12132434]$ $= d_6^4$
	$d_7^4 = [12123434]$ $= d_7^4$		$d_8^4 = [11234432]$ $= d_{18}^4$
	$d_9^4 = [11234342]$ $= d_{17}^4$		$d_{10}^4 = [11234423]$ $= d_{17}^4$
	$d_{11}^4 = [11234324]$ $= d_{15}^4$		$d_{12}^4 = [11234243]$ $= d_{15}^4$
	$d_{13}^4 = [11234234]$ $= 2d_{15}^4 - d_{17}^4$		$d_{14}^4 = [11232443]$ $= d_{17}^4$
	$d_{15}^4 = [11232434]$ $= d_{15}^4$		$d_{16}^4 = [11223443]$ $= d_{18}^4$
	$d_{17}^4 = [11223434]$ $= d_{17}^4$		$d_{18}^4 = [11223344]$ $= d_{18}^4$

Table 4.4.2.1. Chord diagrams of order 4

Lemma. *The product is well-defined modulo 4T relations.*

Proof. We shall show that the product of two diagrams is well-defined; it follows immediately that this is also true for linear combinations of diagrams. It is enough to prove that if one of the two diagrams, say D_2, is turned inside the product diagram by one "click" with respect to D_1, then the result is the same modulo 4T relations.

Note that such rotation is equivalent to the following transformation. Pick a chord in D_2 with endpoints a and b such that a is adjacent to D_1. Then, fixing the endpoint b, move a through the diagram D_1. In this process we obtain $2n+1$ diagrams P_0, P_1, ..., P_{2n}, where n is the order of D_1, and we must prove that $P_0 \equiv P_{2n} \mod 4T$. Now, it is not hard to see that the difference $P_0 - P_{2n}$ is, in fact, equal to the sum of all n four-term relations which are obtained by fixing the endpoint b and all chords of D_1, one by one. For example, if we consider the two products shown above and use the following notation:



P_0 P_1 P_2 P_3 P_4 P_5 P_6

then we must take the sum of the three linear combinations

$$P_0 - P_1 + P_2 - P_3,$$
$$P_1 - P_2 + P_4 - P_5,$$
$$P_3 - P_4 + P_5 - P_6,$$

and the result is exactly $P_0 - P_6$.

□

4.4.4. Comultiplication of chord diagrams. The *coproduct* in the algebra $\mathcal{A}^{fr}$

$$\delta : \mathcal{A}_n^{fr} \to \bigoplus_{k+l=n} \mathcal{A}_k^{fr} \otimes \mathcal{A}_l^{fr}$$

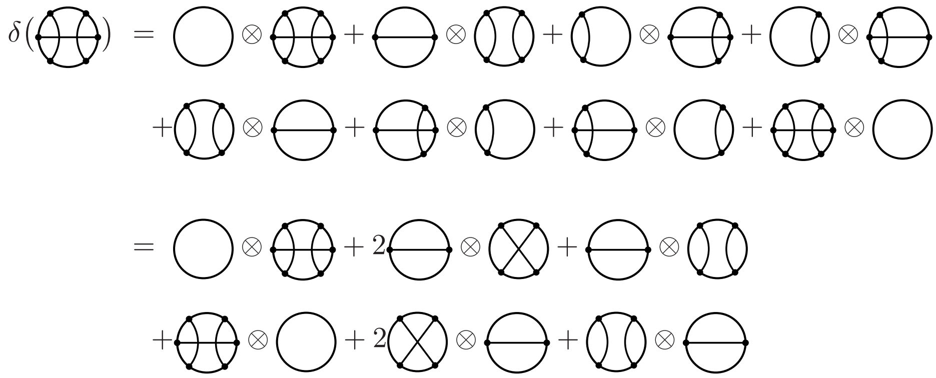
is defined as follows. For a diagram $D \in \mathcal{A}_n^{fr}$ we put

$$\delta(D) := \sum_{J \subseteq [D]} D_J \otimes D_{\overline{J}},$$

the summation taken over all subsets J of the set of chords of D. Here D_J is the diagram consisting of the chords that belong to J and $\overline{J} = [D] \setminus J$ is the complementary subset of chords. To the entire space $\mathcal{A}^{fr}$ the operator δ is extended by linearity.

If D is a diagram of order n, the total number of summands in the right-hand side of the definition is 2^n.

Example.



Lemma. *The coproduct δ is well-defined modulo 4T relations.*

Proof. Let $D_1 - D_2 + D_3 - D_4 = 0$ be a 4T relation. We must show that the sum $\delta(D_1) - \delta(D_2) + \delta(D_3) - \delta(D_4)$ can be written as a combination of 4T relations. Recall that a specific four-term relation is determined by the choice of a moving chord m and a fixed chord a. Now, take one and the same splitting $A \cup B$ of the set of chords in the diagrams D_i, the same for each i, and denote by A_i, B_i the resulting chord diagrams giving the contributions $A_i \otimes B_i$ to $\delta(D_i)$, $i = 1, 2, 3, 4$. Suppose that the moving chord m belongs to the subset A. Then $B_1 = B_2 = B_3 = B_4$ and $A_1 \otimes B_1 - A_2 \otimes B_2 + A_3 \otimes B_3 - A_4 \otimes B_4 = (A_1 - A_2 + A_3 - A_4) \otimes B_1$. If the fixed chord a belongs to A, then the $A_1 - A_2 + A_3 - A_4$ is a four-term combination; otherwise it is easy to see that $A_1 = A_2$ and $A_3 = A_4$ for an appropriate numbering. The case when $m \in B$ is treated similarly. $\square$

The *unit* and the *counit* in $\mathcal{A}^{fr}$ are defined as follows:

$$\iota : \mathcal{R} \to \mathcal{A}^{fr} \quad , \quad \iota(x) = x \bigcirc \, ,$$

$$\varepsilon : \mathcal{A}^{fr} \to \mathcal{R} \quad , \quad \varepsilon\big(x \bigcirc + \; ...\big) = x \; .$$

Exercise. Check the axioms of a bialgebra for $\mathcal{A}^{fr}$ and verify that it is commutative, cocommutative and connected.

4.4.5. Deframing the chord diagrams. The space of unframed chord diagrams $\mathcal{A}$ was defined as the quotient of the space $\mathcal{A}^{fr}$ by the subspace spanned by all diagrams with an isolated chord. In terms of the multiplication in $\mathcal{A}^{fr}$, this subspace can be described as the ideal of $\mathcal{A}^{fr}$ generated by Θ, the chord diagram with one chord, so that we can write:

$$\mathcal{A} = \mathcal{A}^{fr}/(\Theta).$$

It turns out that there is a simple explicit formula for a linear operator $p : \mathcal{A}^{fr} \to \mathcal{A}^{fr}$ whose kernel is the ideal (Θ). Namely, define $p_n : \mathcal{A}_n^{fr} \to \mathcal{A}_n^{fr}$ by

$$p_n(D) := \sum_{J \subseteq [D]} (-\Theta)^{n-|J|} \cdot D_J \; ,$$

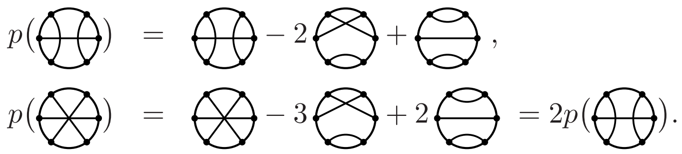
where, as earlier, $[D]$ stands for the set of chords in the diagram D and D_J means the subdiagram of D with only the chords from J left. The sum of p_n over all n is the operator $p : \mathcal{A}^{fr} \to \mathcal{A}^{fr}$.

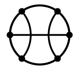
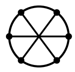
4.4.6. Exercise. Check that

(1) p is a homomorphism of algebras,
(2) $p(\Theta) = 0$ and hence p takes the entire ideal (Θ) into 0.
(3) p is a projector, that is, $p^2 = p$.
(4) the kernel of p is exactly (Θ).

We see, therefore, that the quotient map $\bar{p}: \mathcal{A}^{fr}/(\Theta) \to \mathcal{A}^{fr}$ is the isomorphism of $\mathcal{A}$ onto its image and we have a direct decomposition $\mathcal{A}^{fr} = \bar{p}(\mathcal{A}) \oplus (\Theta)$. Note that the first summand here is different from the subspace spanned merely by all diagrams without isolated chords!

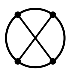
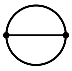
For example, $p(\mathcal{A}_3^{fr})$ is spanned by the two vectors

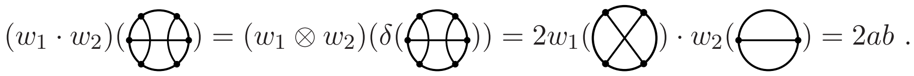


while the subspace generated by the elements  and  is 2-dimensional and has a nonzero intersection with the ideal (Θ).

4.5. Bialgebra of weight systems

According to 4.4.2 the vector space $\mathcal{W}^{fr}$ is dual to the space $\mathcal{A}^{fr}$. Since now $\mathcal{A}^{fr}$ is equipped with the structure of a Hopf algebra, the general construction of Section A.2.24 supplies the space $\mathcal{W}^{fr}$ with the same structure. In particular, weight systems can be multiplied: $(w_1 \cdot w_2)(D) := (w_1 \otimes w_2)(\delta(D))$ and comultiplied: $(\delta(w))(D_1 \otimes D_2) := w(D_1 \cdot D_2)$. The *unit* of $\mathcal{W}^{fr}$ is the weight system $\mathbf{I}_0$ which takes value 1 on the chord diagram without chords and vanishes elsewhere. The *counit* sends a weight system to its value on on the chord diagram without chords.

For example, if w_1 is a weight system which takes value a on the chord diagram , and zero value on all other chord diagrams, and w_1 takes value b on  and vanishes elsewhere, then



4.5.1. Proposition. *The symbol* $\mathrm{symb}: \mathcal{V}^{fr} \to \mathcal{W}^{fr}$ *commutes with multiplication and comultiplication.*

Proof of the proposition. Analyzing the proof of Theorem 3.2.3 one can conclude that for any two Vassiliev invariants of orders $\leqslant p$ and $\leqslant q$ the symbol of their product is equal to the product of their symbols. This implies that the map symb respects the multiplication. Now we prove that $\mathrm{symb}(\delta(v)) = \delta(\mathrm{symb}(v))$ for a Vassiliev invariant v of order $\leqslant n$. Let us apply both parts of this equality to the tensor product of two chord diagrams

D_1 and D_2 with the number of chords p and q respectively where $p+q=n$. We have

$$\mathrm{symb}(\delta(v))\big(D_1 \otimes D_2\big) = \delta(v)\big(K^{D_1} \otimes K^{D_2}\big) = v\big(K^{D_1} \# K^{D_2}\big) \ ,$$

where the singular knots K^{D_1} and K^{D_2} represent chord diagrams D_1 and D_2. But the singular knot $K^{D_1} \# K^{D_2}$ represents the chord diagram $D_1 \cdot D_2$. Since the total number of chords in $D_1 \cdot D_2$ is equal to n, the value of v on the corresponding singular knot would be equal to the value of its symbol on the chord diagram:

$$v\big(K^{D_1} \# K^{D_2}\big) = \mathrm{symb}(v)\big(D_1 \cdot D_2\big) = \delta(\mathrm{symb}(v))\big(D_1 \otimes D_2\big) \ .$$

□

Remark. The map symb : $\mathcal{V}^{fr} \to \mathcal{W}^{fr}$ is not a bialgebra homomorphism because it does not respect the addition. Indeed, the sum of two invariants $v_1 + v_2$ of different orders p and q with, say $p > q$ has the order p. That means $\mathrm{symb}(v_1 + v_2) = \mathrm{symb}(v_1) \neq \mathrm{symb}(v_1) + \mathrm{symb}(v_2)$.

However, we can extend the map symb to power series Vassiliev invariants by sending the invariant $\prod v_i \in \widehat{\mathcal{V}}_{\bullet}^{fr}$ to the element $\sum \mathrm{symb}(v_i)$ of the graded completion $\widehat{\mathcal{W}}^{fr}$. Then the above Proposition implies that the map symb : $\widehat{\mathcal{V}}_{\bullet}^{fr} \to \widehat{\mathcal{W}}^{fr}$ is a graded bialgebra homomorphism.

4.5.2. We call a weight system w *multiplicative* if for any two chord diagrams D_1 and D_2 we have

$$w(D_1 \cdot D_2) = w(D_1) w(D_2).$$

This is the same as to say that w is a semigroup-like element in the bialgebra of weight systems (see Appendix A.2.6). Note that a multiplicative weight system always takes value 1 on the chord diagram with no chords. The unit $\mathbf{I}_0$ is the only group-like element of the bialgebra $\mathcal{W}^{fr}$ (compare with Exercise 4.3.3 on page 103). However, the graded completion $\widehat{\mathcal{W}}^{fr}$ contains many interesting group-like elements.

Corollary of Proposition 4.5.1. *Suppose that*

$$v = \prod_{n=0}^{\infty} v_n \in \widehat{\mathcal{V}}_{\bullet}^{fr}$$

is multiplicative. Then its symbol is also multiplicative.

Indeed any homomorphism of bialgebras sends group-like elements to group-like elements.

4.5.3. A weight system that belongs to a homogeneous component $\mathcal{W}_n^{fr}$ of the space $\mathcal{W}^{fr}$ is said to be *homogeneous of degree* n. Let $w \in \widehat{\mathcal{W}^{fr}}$ be an element with homogeneous components $w_i \in \mathcal{W}_i^{fr}$ such that $w_0 = 0$. Then the exponential of w can be defined as the Taylor series

$$\exp(w) = \sum_{k=0}^{\infty} \frac{w^k}{k!}.$$

This formula makes sense because only a finite number of operations is required for the evaluation of each homogeneous component of this sum. One can easily check that the weight systems $\exp(w)$ and $\exp(-w)$ are inverse to each other:

$$\exp(w) \cdot \exp(-w) = \mathbf{I}_0.$$

By definition, a *primitive* weight system w satisfies

$$w(D_1 \cdot D_2) = \mathbf{I}_0(D_1) \cdot w(D_2) + w(D_1) \cdot \mathbf{I}_0(D_2).$$

(In particular, a primitive weight system is always zero on a product of two nontrivial diagrams $D_1 \cdot D_2$.) The exponential $\exp(w)$ of a primitive weight system w is multiplicative (group-like). Note that it always belongs to the completion $\widehat{\mathcal{W}}^{fr}$, even if w belongs to $\mathcal{W}^{fr}$.

A simple example of a homogeneous weight system of degree n is provided by the function on the set of chord diagrams which is equal to 1 on any diagram of degree n and to 0 on chord diagrams of all other degrees. This function clearly satisfies the four-term relations. Let us denote this weight system by $\mathbf{I}_n$.

4.5.4. Lemma. $\mathbf{I}_n \cdot \mathbf{I}_m = \binom{m+n}{n} \mathbf{I}_{n+m}$.

This directly follows from the definition of the multiplication for weight systems.

4.5.5. Corollary. (i) $\dfrac{\mathbf{I}_1^n}{n!} = \mathbf{I}_n$;

(ii) *If we set* $\mathbf{I} = \sum_{n=0}^{\infty} \mathbf{I}_n$ *(that is,* $\mathbf{I}$ *is the weight system that is equal to 1 on every chord diagram), then*

$$\exp(\mathbf{I}_1) = \mathbf{I}.$$

Strictly speaking, $\mathbf{I}$ is not an element of $\mathcal{W}^{fr} = \oplus_n \mathcal{W}_n^{fr}$ but of the graded completion $\widehat{\mathcal{W}}^{fr}$. Note that $\mathbf{I}$ is not the unit of $\widehat{\mathcal{W}}^{fr}$. Its unit, as well as the unit of $\mathcal{W}$ itself, is represented by the element $\mathbf{I}_0$.

4.5.6. Deframing the weight systems. Since $\mathcal{A} = \mathcal{A}^{fr}/(\Theta)$ is a quotient of $\mathcal{A}^{fr}$, the corresponding dual spaces are embedded one into another, $\mathcal{W} \subset \mathcal{W}^{fr}$. The elements of $\mathcal{W}$ take zero values on all chord diagrams with an isolated chord. In Section 4.1 they were called *unframed weight systems*. The deframing procedure for chord diagrams (Section 4.4.5) leads to a deframing procedure for weight systems. By duality, the projector $p : \mathcal{A}^{fr} \to \mathcal{A}^{fr}$ gives rise to a projector $p^* : \mathcal{W}^{fr} \to \mathcal{W}^{fr}$ whose value on an element $w \in \mathcal{W}_n^{fr}$ is defined by

$$w'(D) = p^*(w)(D) := w(p(D)) = \sum_{J \subseteq [D]} w\left((-\Theta)^{n-|J|} \cdot D_J\right) .$$

Obviously, $w'(D) = 0$ for any w and any chord diagram D with an isolated chord. Hence the operator $p^* : w \mapsto w'$ is a projection of the space $\widehat{\mathcal{W}}^{fr}$ onto its subspace $\widehat{\mathcal{W}}$ consisting of unframed weight systems.

The deframing operator looks especially nice for multiplicative weight systems.

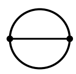
4.5.7. Exercise. Prove that for any number $\theta \in \mathbb{F}$ the exponent $e^{\theta \mathbf{I}_1} \in \widehat{W}$ is a multiplicative weight system.

4.5.8. Lemma. *Let $\theta = w(\Theta)$ for a multiplicative weight system w. Then its deframing is $w' = e^{-\theta \mathbf{I}_1} \cdot w$.*

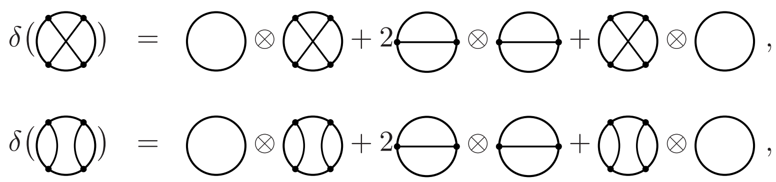
We leave the proof of this lemma to the reader as an exercise. The lemma, together with the previous exercise, implies that the deframing of a multiplicative weight system is again multiplicative.

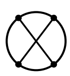
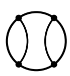
4.6. Primitive elements in $\mathcal{A}^{fr}$

The algebra of chord diagrams $\mathcal{A}^{fr}$ is commutative, cocommutative and connected. Therefore, by the Milnor-Moore Theorem A.2.25, any element of $\mathcal{A}^{fr}$ is uniquely represented as a polynomial in basis primitive elements. Let us denote the nth homogeneous component of the primitive subspace by $\mathcal{P}_n = \mathcal{A}_n^{fr} \cap \mathcal{P}(\mathcal{A}^{fr})$ and find an explicit description of $\mathcal{P}_n$ for small n.

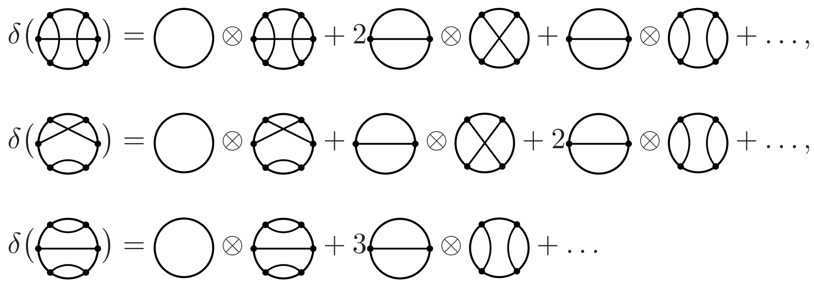
dim = 1. $\mathcal{P}_1 = \mathcal{A}_1^{fr}$ is one-dimensional and spanned by  .

dim = 2. Since

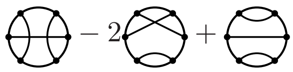


the element  $-$  is primitive. It constitutes a basis of $\mathcal{P}_2$.

dim = 3. The coproducts of the 3 basis elements of $\mathcal{A}_3^{fr}$ are



(Here the dots stand for the terms symmetric to the terms that are shown explicitly.) Looking at these expressions, it is easy to check that the element



is the only, up to multiplication by a scalar, primitive element of $\mathcal{A}_3^{fr}$.

The exact dimensions of $\mathcal{P}_n$ are currently (2009) known up to $n = 12$ (the last three values, corresponding to $n = 10, 11, 12$, were found by J. Kneissler [**Kn0**]):

n	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{P}_n$	1	1	1	2	3	5	8	12	18	27	39	55

We shall discuss the sizes of the spaces $\mathcal{P}_n$, $\mathcal{A}_n$ and $\mathcal{V}_n$ in more detail later (see Sections 5.5 and 14.4).

If the dimensions of $\mathcal{P}_n$ were known for all n, then the dimensions of $\mathcal{A}_n$ would also be known.

Example. Let us find the dimensions of $\mathcal{A}_n^{fr}$, $n \leqslant 5$, assuming that we know the values of $\dim \mathcal{P}_n$ for $n = 1, 2, 3, 4, 5$, which are equal to $1, 1, 1, 2, 3$, respectively. Let p_i be the basis element of $\mathcal{P}_i$, $i = 1, 2, 3$ and denote the bases of $\mathcal{P}_4$ and $\mathcal{P}_5$ as p_{41}, p_{42} and p_{51}, p_{52}, p_{53}, respectively. Nontrivial monomials up to degree 5 that can be made out of these basis elements are:

Degree 2 monomials (1): p_1^2.

Degree 3 monomials (2): p_1^3, $p_1 p_2$.

Degree 4 monomials (4): p_1^4, $p_1^2 p_2$, $p_1 p_3$, p_2^2.

Degree 5 monomials (7): p_1^5, $p_1^3 p_2$, $p_1^2 p_3$, $p_1 p_2^2$, $p_1 p_{41}$, $p_1 p_{42}$, $p_2 p_3$.

A basis of each $\mathcal{A}_n^{fr}$ can be made up of the primitive elements and their products of the corresponding degree. For $n = 0, 1, 2, 3, 4, 5$ we get: $\dim \mathcal{A}_0^{fr} = 1$, $\dim \mathcal{A}_1^{fr} = 1$, $\dim \mathcal{A}_2^{fr} = 1 + 1 = 2$, $\dim \mathcal{A}_3^{fr} = 1 + 2 = 3$, $\dim \mathcal{A}_4^{fr} = 2 + 4 = 6$, $\dim \mathcal{A}_5^{fr} = 3 + 7 = 10$.

The partial sums of this sequence give the dimensions of the spaces of framed Vassiliev invariants: $\dim \mathcal{V}_0^{fr} = 1$, $\dim \mathcal{V}_1^{fr} = 2$, $\dim \mathcal{V}_2^{fr} = 4$, $\dim \mathcal{V}_3^{fr} = 7$, $\dim \mathcal{V}_4^{fr} = 13$, $\dim \mathcal{V}_5^{fr} = 23$.

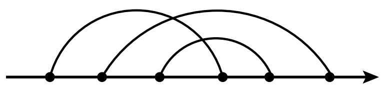
4.6.1. Exercise. Let p_n be the sequence of dimensions of primitive spaces in a Hopf algebra and a_n the sequence of dimensions of the entire algebra. Prove the relation between the generating functions

$$1 + a_1 t + a_2 t^2 + \cdots = \frac{1}{(1-t)^{p_1}(1-t^2)^{p_2}(1-t^3)^{p_3}\ldots}.$$

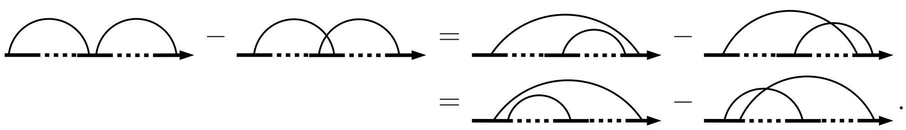
Note that primitive elements of $\mathcal{A}^{fr}$ are represented by rather complicated linear combinations of chord diagrams. A more concise and clear representation can be obtained via connected closed diagrams, to be introduced in the next chapter (Section 5.5).

4.7. Linear chord diagrams

The arguments of this chapter, applied to *long* knots (see 1.8.3), lead us naturally to considering the space of *linear* chord diagrams, that is, diagrams on an oriented line:

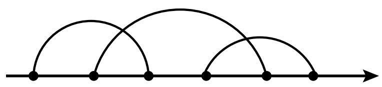


subject to the 4-term relations:



Let us temporarily denote the space of linear chord diagrams with n chords modulo the 4-term relations by $(\mathcal{A}_n^{fr})^{long}$. The space $(\mathcal{A}^{fr})^{long}$ of such chord diagrams of all degrees modulo the 4T relations is a bialgebra; the product in $(\mathcal{A}^{fr})^{long}$ can be defined simply by concatenating the oriented lines.

If the line is closed into a circle, linear 4-term relations become circular (that is, usual) 4-term relations; thus, we have a linear map $(\mathcal{A}_n^{fr})^{long} \to \mathcal{A}_n^{fr}$. This map is evidently onto, as one can find a preimage of any circular chord diagram by cutting the circle at an arbitrary point. This preimage, in general, depends on the place where the circle is cut, so it may appear that this map has a non-trivial kernel. For example, the linear diagram shown above closes up to the same diagram as the one drawn below:



Remarkably, *modulo 4-term relations*, all the preimages of any circular chord diagram are equal in $(\mathcal{A}_3^{fr})^{long}$ (in particular, the two diagrams in the above pictures give the same element of $(\mathcal{A}_3^{fr})^{long}$). This fact is proved by exactly the same argument as the statement that the product of chord diagrams is well-defined (Lemma 4.4.3); we leave it to the reader as an exercise.

Summarizing, we have:

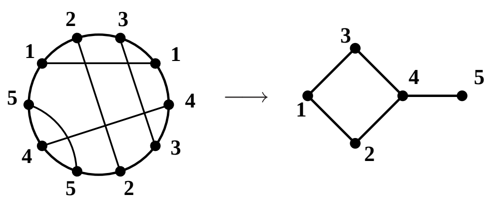
Proposition. *Closing up the line into the circle gives rise to a vector space isomorphism* $(\mathcal{A}^{fr})^{long} \to \mathcal{A}^{fr}$. *This isomorphism is compatible with the multiplication and comultiplication and thus defines an isomorphism of bialgebras.*

A similar statement holds for diagrams modulo 4T and 1T relations. Further, one can consider chord diagrams (and 4T relations) with chords attached to an arbitrary one-dimensional oriented manifold — see Section 5.10.

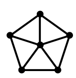
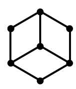
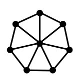
4.8. Intersection graphs

4.8.1. Definition. ([**CD1**]) The *intersection graph* $\Gamma(D)$ of a chord diagram D is the graph whose vertices correspond to the chords of D and whose edges are determined by the following rule: two vertices are connected by an edge if and only if the corresponding chords intersect, and multiple edges are not allowed. (Two chords, a and b, are said to intersect if their endpoints a_1, a_2 and b_1, b_2 appear in the interlacing order a_1, b_1, a_2, b_2 along the circle.)

For example,

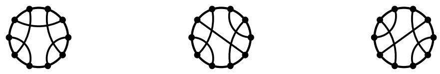


The intersection graphs of chord diagrams are also called *circle graphs* or *interlacement graphs*.

Note that not every graph can be represented as the intersection graph of a chord diagram. For example, the following graphs are not intersection graphs:  ,  ,  .

4.8.2. Exercise. Prove that all graphs with no more than 5 vertices are intersection graphs.

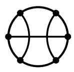
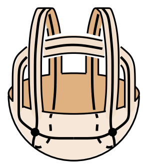
On the other hand, distinct diagrams may have coinciding intersection graphs. For example, there are three different diagrams



with the same intersection graph .

A complete characterization of those graphs that can be realized as intersection graphs was given by A. Bouchet [**Bou2**].

With each chord diagram D we can associate an oriented surface Σ_D by attaching a disc to the circle of D and thickening the chords of D. Then the chords determine a basis in $H_1(\Sigma_D, \mathbb{Z}_2)$ as in the picture below. The intersection matrix for this basis coincides with the adjacency matrix of Γ_D. Using the terminology of singularity theory we may say that the intersection graph Γ_D is the *Dynkin diagram* of the intersection form in $H_1(\Sigma_D, \mathbb{Z}_2)$ constructed for the basis of $H_1(\Sigma_D, \mathbb{Z}_2)$.

$D =$  $\rightsquigarrow$ $\Sigma_D =$  $\rightsquigarrow$ $\Gamma_D =$

Intersection graphs contain a good deal of information about chord diagrams. In [**CDL1**] the following conjecture was stated.

4.8.3. Intersection graph conjecture. *If D_1 and D_2 are two chord diagrams whose intersection graphs are equal, $\Gamma(D_1) = \Gamma(D_2)$, then $D_1 = D_2$ as elements of $\mathcal{A}^{fr}$ (that is, modulo four-term relations).*

Although wrong in general (see Section 9.5.8), this assertion is true in some particular situations:

(1) for all diagrams D_1, D_2 with up to 10 chords (a direct computer check [**CDL1**] up to 8 chords and [**Mu**] for 9 and 10 chords);

(2) when $\Gamma(D_1) = \Gamma(D_2)$ is a tree (see [**CDL2**]) or, more generally, D_1, D_2 belong to the forest subalgebra (see [**CDL3**]);

(3) when $\Gamma(D_1) = \Gamma(D_2)$ is a graph with a single loop (see [**Mel1**]);

(4) for weight systems w coming from standard representations of Lie algebras $\mathfrak{gl}_N$ or $\mathfrak{so}_N$. This means that $\Gamma(D_1) = \Gamma(D_2)$ implies $w(D_1) = w(D_2)$; see Chapter 6, proposition on page 175 and exercise 17 on page 192 of the same chapter;

(5) for the universal $\mathfrak{sl}_2$ weight system and the weight system coming from the standard representation of the Lie superalgebra $\mathfrak{gl}(1|1)$ (see [**ChL**]).

In fact, the intersection graph conjecture can be refined to the following theorem which covers items (4) and (5) above.

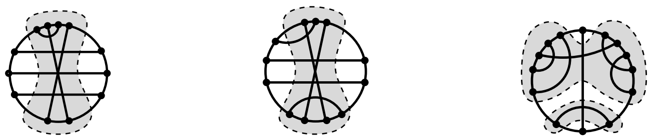
Theorem ([**ChL**]). *The symbol of a Vassiliev invariant that does not distinguish mutant knots depends on the intersection graph only.*

We postpone the discussion of mutant knots, the proof of this theorem and its converse to Section 9.5.

4.8.4. Chord diagrams representing a given graph. To describe all chord diagrams representing a given intersection graph we need the notion of a *share* [**CDL1, ChL**]. Informally, a *share* of a chord diagram is a subset of chords whose endpoints are separated into at most two parts by the endpoints of the complementary chords. More formally,

Definition. A *share* is a part of a chord diagram consisting of two arcs of the outer circle with the following property: each chord one of whose ends belongs to these arcs has both ends on these arcs.

Here are some examples:

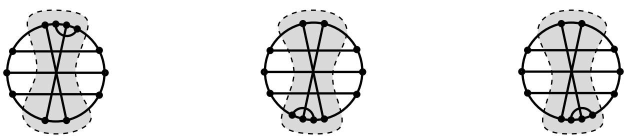


A share   Not a share   Two shares

The complement of a share also is a share. The whole chord diagram is its own share whose complement contains no chords.

Definition. A *mutation of a chord diagram* is another chord diagram obtained by a flip of a share.

For example, three mutations of the share in the first chord diagram above produce the following chord diagrams:



Obviously, mutations preserve the intersection graphs of chord diagrams.

Theorem. *Two chord diagrams have the same intersection graph if and only if they are related by a sequence of mutations.*

This theorem is contained implicitly in papers [**Bou1, GSH**] where chord diagrams are written as *double occurrence words*.

Proof of the theorem. The proof uses Cunningham's theory of graph decompositions [**Cu**].

A *split* of a (simple) graph Γ is a disjoint bipartition $\{V_1, V_2\}$ of its set of vertices $V(\Gamma)$ such that each part contains at least 2 vertices, and with the property that there are subsets $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ such that all the edges of Γ connecting V_1 with V_2 form the complete bipartite graph $K(W_1, W_2)$ with the parts W_1 and W_2. Thus for a split $\{V_1, V_2\}$ the whole graph Γ can be represented as a union of the induced subgraphs $\Gamma(V_1)$ and $\Gamma(V_2)$ linked by a complete bipartite graph.

Another way to think about splits, which is sometimes more convenient and which we shall use in the pictures below, is as follows. Consider two graphs Γ_1 and Γ_2 each with a distinguished vertex $v_1 \in V(\Gamma_1)$ and $v_2 \in V(\Gamma_2)$, respectively, called *markers*. Construct the new graph

$$\Gamma = \Gamma_1 \boxtimes_{(v_1,v_2)} \Gamma_2$$

whose set of vertices is $V(\Gamma) = \{V(\Gamma_1) - v_1\} \cup \{V(\Gamma_2) - v_2\}$, and whose set of edges is

$$\begin{aligned} E(\Gamma) = \{(v_1', v_1'') \in E(\Gamma_1) : v_1' \neq v_1 \neq v_1''\} \cup \{(v_2', v_2'') \in E(\Gamma_2) : v_2' \neq v_2 \neq v_2''\} \\ \cup \{(v_1', v_2') : (v_1', v_1) \in E(\Gamma_1) \text{ and } (v_2, v_2') \in E(\Gamma_2)\} . \end{aligned}$$

Representation of Γ as $\Gamma_1 \boxtimes_{(v_1,v_2)} \Gamma_2$ is called a *decomposition* of Γ, the graphs Γ_1 and Γ_2 are called the *components* of the decomposition. The partition $\{V(\Gamma_1) - v_1, V(\Gamma_2) - v_2\}$ is a split of Γ. Graphs Γ_1 and Γ_2 might be decomposed further giving a finer decomposition of the initial graph Γ. Graphically, we represent a decomposition by pictures of its components where the corresponding markers are connected by a dashed edge.

A *prime* graph is a graph with at least three vertices admitting no splits. A decomposition of a graph is said to be *canonical* if the following conditions are satisfied:

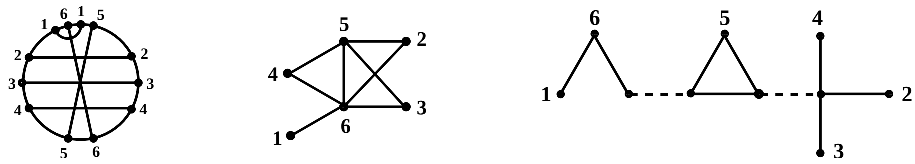
(i) each component is either a prime graph, or a complete graph K_n, or a star S_n, which is the tree with a vertex, the *center*, adjacent to n other vertices;

(ii) no two components that are complete graphs are neighbors, that is, their markers are not connected by a dashed edge;

(iii) the markers of two components that are star graphs connected by a dashed edge are either both centers or both not centers of their components.

W. H. Cunningham proved [**Cu**, Theorem 3] that each graph with at least six vertices possesses a unique canonical decomposition.

Let us illustrate the notions introduced above by an example of canonical decomposition of an intersection graph. We number the chords and the corresponding vertices in our graphs, so that the unnumbered vertices are the markers of the components.

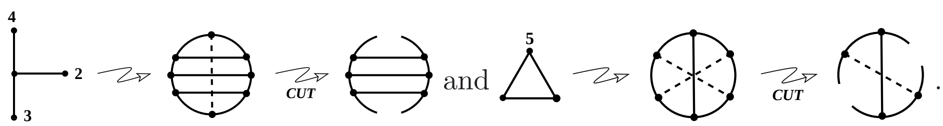


A chord diagram The intersection graph The canonical decomposition

The key observation in the proof of the theorem is that components of the canonical decomposition of any intersection graph admit a unique representation by chord diagrams. For a complete graph and star components, this is obvious. For a prime component, this was proved by A. Bouchet [**Bou1**, Statement 4.4] (see also [**GSH**, Section 6] for an algorithm finding such a representation for a prime graph).

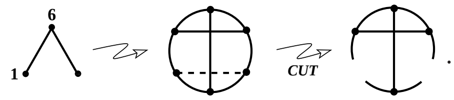
Now, in order to describe all chord diagrams with a given intersection graph, we start with a component of its canonical decomposition. There is only one way to realize the component by a chord diagram. We draw the chord corresponding to the marker as a dashed chord and call it the *marked chord*. This chord indicates the places where we must cut the circle removing the marked chord together with small arcs containing its endpoints. As a result we obtain a chord diagram on two arcs. Repeating the same procedure with the next component of the canonical decomposition, we get another chord diagram on two arcs. We have to glue the arcs of these two diagrams together in the alternating order. There are four possibilities to do this, and they differ by mutations of the share corresponding to one of the two components. This completes the proof of the theorem. □

To illustrate the last stage of the proof consider our standard example and take the star 2-3-4 component first and then the triangle component. We get



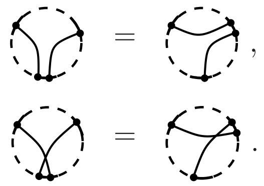
Because of the symmetry, the four ways of glueing these diagrams produce only two distinct chord diagrams with a marked chord:

Repeating the same procedure with the marked chord for the last 1-6 component of the canonical decomposition, we get



Glueing this diagram into the previous two in all possible ways we get the four mutant chord diagrams from page 118.

4.8.5. 2-term relations and the genus of a diagram. A 2-term (or *endpoint sliding*) relation for chord diagrams has the form



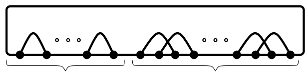
The 4-term relations are evidently a consequence of the 2-term relations; therefore, any function on chord diagrams that satisfies 2-term relations is a weight system. An example of such a weight system is the *genus of a chord diagram* defined as follows.

Replace the outer circle of the chord diagram and all its chords by narrow untwisted bands — this yields an orientable surface with boundary. Attaching a disk to each boundary component gives a closed orientable surface. This is the same as attaching disks to the boundary components of the surface Σ_D from page 117. Its genus is by definition the genus of the chord diagram. The genus can be calculated from the number of boundary components using Euler characteristic. Indeed, the Euler characteristic of the surface with boundary obtained by above described procedure from a chord diagram of degree n is equal to $-n$. If this surface has c boundary components and genus g, then we have $-n = 2 - 2g - c$ while $g = 1 + (n - c)/2$. For example, the two chord diagrams of degree 2 have genera 0 and 1, because the number of connected components of the boundary is 4 and 2, respectively, as one can see in the following picture:

The genus of a chord diagram satisfies 2-term relations, since sliding an endpoint of a chord along another adjacent chord does not change the topological type of the corresponding surface with boundary.

An interesting way to compute the genus from the intersection graph of the chord diagram was found by Moran (see [**Mor**]). Moran's theorem states that the genus of a chord diagram is half the rank of the adjacency matrix over $\mathbb{Z}_2$ of the intersection graph. This theorem can be proved transforming a given chord diagram into the canonical form using the following exercise.

4.8.6. Exercise. A *caravan of* m_1 *"one-humped camels" and* m_2 *"two-humped camels* is the product of m_1 diagrams with 2 crossing chords and m_2 diagrams with one chord:



Show that any chord diagram is equivalent, modulo 2-term relations, to a caravan. Show that the caravans form a basis in the vector space of chord diagrams modulo 2-term relations.

The algebra generated by caravans is thus a quotient algebra of the algebra of chord diagrams.

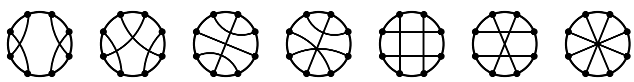
Exercises

(1) A *short* chord is a chord whose endpoints are adjacent, that is, one of the arcs that it bounds contains no endpoints of other chords. In particular, short chords are isolated. Prove that the linear span of all diagrams with a short chord and all four-term relation contains all diagrams with an isolated chord. This means that the restricted one-term relations (only for diagrams with a short chord) imply general one-term relations provided that the four-term relations hold.

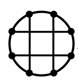
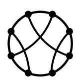
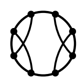
(2) Find the number of different chord diagrams of order n with n isolated chords. Prove that all of them are equal to each other modulo the four-term relations.

(3) Using Table 4.4.2.1 on page 107, find the space of unframed weight systems $\mathcal{W}_4$.

Answer. The basis weight systems are:



1	0	-1	-1	0	-1	-2
0	1	1	2	0	1	3
0	0	0	0	1	1	1

The table shows that the three diagrams ,  and  form a basis in the space $\mathcal{A}_4$.

(4) * Is it true that any chord diagram of order 13 is equivalent to its mirror image modulo 4-term relations?

(5) Prove that the deframing operator $'$ (Section 4.5.6) is a homomorphism of algebras: $(w_1 \cdot w_2)' = w_1' \cdot w_2'$.

(6) Give a proof of Lemma 4.5.8 on page 113.

(7) Find a basis in the primitive space $\mathcal{P}_4$.

Answer. A possible basis consists of the elements $d_6^4 - d_7^4$ and $d_2^4 - 2d_7^4$ from the table on page 107.

(8) Prove that for any primitive element p of degree > 1, $w(p) = w'(p)$ where w' is the deframing of a weight system w.

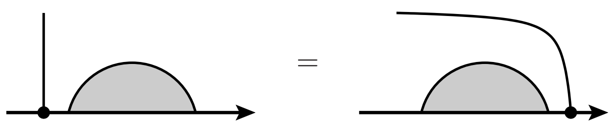
(9) Prove that the symbol of a primitive Vassiliev invariant is a primitive weight system.

(10) Let Θ be the chord diagram with a single chord. By a direct computation, check that $\exp(\alpha\Theta) := \sum_{n=0}^{\infty} \frac{\alpha^n \Theta^n}{n!} \in \widehat{\mathcal{A}}^{fr}$ is a group-like element in the completed Hopf algebra of chord diagrams.

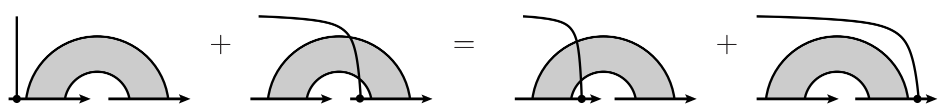
(11) (a) Prove that no chord diagram is equal to 0 modulo 4-term relations.
(b) Let D be a chord diagram without isolated chords. Prove that $D \neq 0$ modulo 1- and 4-term relations.

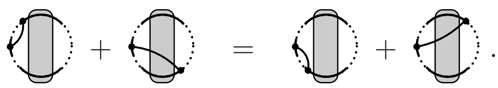
(12) Let $c(D)$ be the number of chord intersections in a chord diagram D. Check that c is a weight system. Find its deframing c'.

(13) **The generalized 4-term relations.**
(a) Prove the following relation:

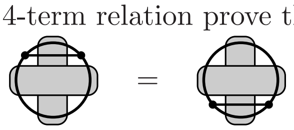


Here the horizontal line is a fragment of the circle of the diagram, while the grey region denotes an arbitrary conglomeration of chords.
(b) Prove the following relation:



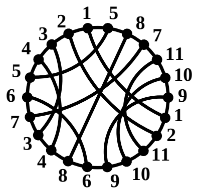
or, in circular form: 

(14) Using the generalized 4-term relation prove the following identity:

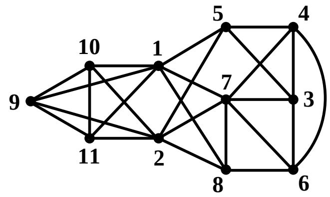


(15) Prove the proposition of the Section 4.7.

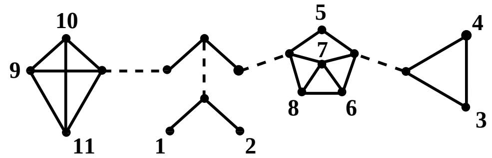
(16) Check that for the chord diagram below, the intersection graph and its canonical decomposition are as shown:



Chord diagram



Intersection graph



Canonical decomposition

(17) ([**LZ**, example 6.4.11]) Prove that $e^{\mathrm{symb}(c_2)}(D)$ is equal to the number of perfect matchings of the intersection graph $\Gamma(D)$. (A *perfect matching* in a graph is a set of disjoint edges covering all the vertices of the graph.)

Part 2

Diagrammatic Algebras

Chapter 5

Jacobi diagrams

In the previous chapter we saw that the study of Vassiliev knot invariants, at least complex-valued, is largely reduced to the study of the algebra of chord diagrams. Here we introduce two different types of diagrams representing elements of this algebra, namely *closed Jacobi diagrams* and *open Jacobi diagrams*. These diagrams provide better understanding of the primitive space $\mathcal{PA}$ and bridge the way to the applications of the Lie algebras in the theory of Vassiliev invariants, see Chapter 6 and Section 11.4.

The name *Jacobi diagrams* is justified by a close resemblance of the basic relations imposed on Jacobi diagrams (STU and IHX) to the Jacobi identity for Lie algebras.

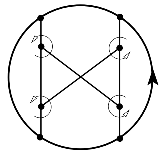
5.1. Closed Jacobi diagrams

5.1.1. Definition. A *closed Jacobi diagram* (or, simply, a *closed diagram*) is a connected trivalent graph with a distinguished embedded oriented cycle, called *Wilson loop*, and a fixed cyclic order of half-edges at each vertex not on the Wilson loop. Half the number of the vertices of a closed diagram is called the *degree*, or *order*, of the diagram. This number is always an integer.

Remark. Some authors (see, for instance, [**HM**]) also include the cyclic order of half-edges at the vertices on the Wilson loop into the structure of a closed Jacobi diagram; this leads to the same theory.

Remark. A Jacobi diagram is allowed to have multiple edges and hanging loops, that is, edges with both ends at the same vertex. It is the possible presence of hanging loops that requires introducing the cyclic order on half-edges rather than edges.

Example. Here is a closed diagram of degree 4:

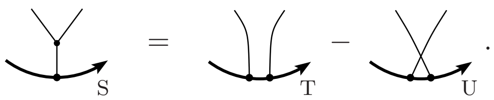


The orientation of the Wilson loop and the cyclic orders of half-edges at the internal vertices are indicated by arrows. In the pictures below, we shall always draw the diagram inside its Wilson loop, which will be assumed to be oriented counterclockwise unless explicitly specified otherwise. Inner vertices will also be assumed to be oriented counterclockwise. (This convention is referred to as the *blackboard orientation*.) Note that the intersection of two edges in the centre of the diagram above is not actually a vertex.

Chord diagrams are closed Jacobi diagrams all of whose vertices lie on the Wilson loop.

Other terms used for closed Jacobi diagrams in the literature include *Chinese character diagrams* [**BN1**], *circle diagrams* [**Kn0**], *round diagrams* [**Wil1**] and *Feynman diagrams* [**KSA**].

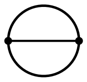
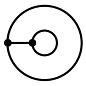
5.1.2. Definition. The vector space of closed diagrams $\mathcal{C}_n$ is the space spanned by all closed diagrams of degree n modulo the *STU relations*:

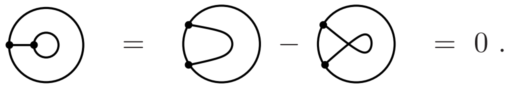


The three diagrams S, T and U must be identical outside the shown fragment. We write $\mathcal{C}$ for the direct sum of the spaces $\mathcal{C}_n$ for all $n \geqslant 0$.

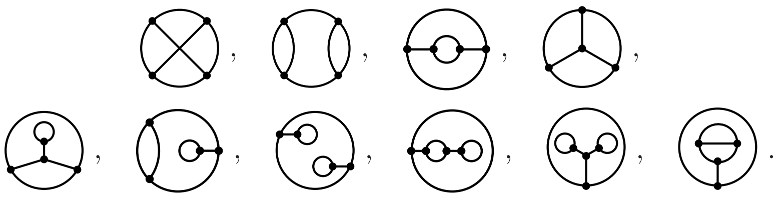
The two diagrams T and U are referred to as the *resolutions* of the diagram S. The choice of the plus and minus signs in front of the two resolutions in the right-hand side of the STU relation, depends on the orientation for the Wilson loop and on the cyclic order of the three edges meeting at the internal vertex of the S-term. Should we reverse one of them, say the orientation of the Wilson loop, the signs of the T- and U-terms change. Indeed,

This remark will be important in Section 5.5.3 where we discuss the problem of detecting knot orientation. One may think of the choice of the direction for the Wilson loop in an STU relation as a choice of the cyclic order "forward-sideways-backwards" at the vertex lying on the Wilson loop. In these terms, the signs in the STU relation depend on the cyclic orders at both vertices of the S-term, the relation above may be thought of as a consequence of the antisymmetry relation AS (see 5.2.2) for the vertex on the Wilson loop, and the STU relation itself can be regarded as a particular case of the IHX relation (see 5.2.3).

5.1.3. Examples. There exist two different closed diagrams of order 1:  ,  , one of which vanishes due to the STU relation:

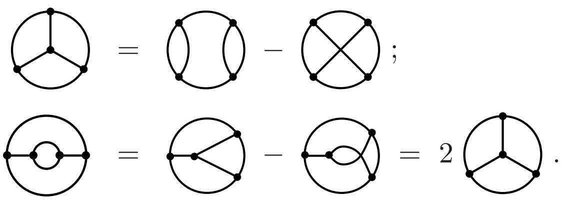


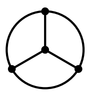
There are ten closed diagrams of degree 2:



The last six diagrams are zero. This is easy to deduce from the STU relations, but the most convenient way of seeing it is by using the AS relations which follow from the STU relations (see Lemma 5.2.5 below).

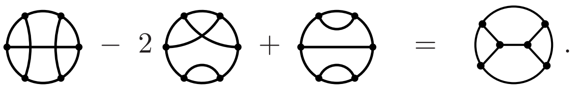
Furthermore, there are at least two relations among the first four diagrams:



It follows that $\dim \mathcal{C}_2 \leqslant 2$. Note that the first of the above equalities gives a concise representation,  , for the basis primitive element of degree 2.

5.1.4. Exercise. Using the STU relations, rewrite the basis primitive element of order 3 in a concise way.

Answer.



We have already mentioned that chord diagrams are a particular case of closed diagrams. Using the STU relations, one can rewrite any closed diagram as a linear combination of chord diagrams. (Examples were given just above.)

A vertex of a closed diagram that lies on the Wilson loop is called *external*; otherwise it is called *internal*. External vertices are also called *legs*. There is an increasing filtration on the space $\mathcal{C}_n$ by subspaces $\mathcal{C}_n^m$ spanned by diagrams with at most m external vertices:

$$\mathcal{C}_n^1 \subset \mathcal{C}_n^2 \subset \ldots \subset \mathcal{C}_n^{2n}.$$

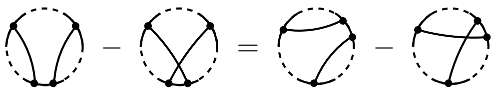
5.1.5. Exercise. Prove that $\mathcal{C}_n^1 = 0$.

Hint. In a diagram with only two legs one of the legs can go all around the circle and change places with the second.

5.2. IHX and AS relations

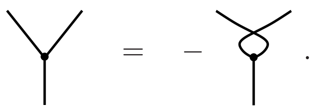
5.2.1. Lemma. *The* STU *relations imply the* 4T *relations for chord diagrams.*

Proof. Indeed, writing the four-term relation in the form



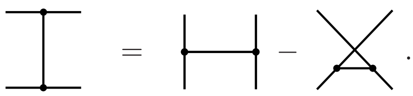
and applying the STU relations to both parts of this equation, we get the same closed diagrams. □

5.2.2. Definition. An AS (=*antisymmetry*) *relation* is:



In other words, a diagram changes sign when the cyclic order of three edges at a trivalent vertex is reversed.

5.2.3. Definition. An IHX *relation* is:



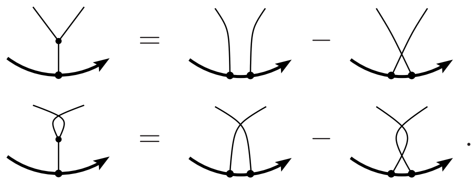
As usual, the unfinished fragments of the pictures denote graphs that are identical (and arbitrary) everywhere but in this explicitly shown fragment.

5.2.4. Exercise. Check that the three terms of the IHX relation "have equal rights". For example, an H turned 90 degrees looks like an I; write an IHX relation starting from that I and check that it is the same as the initial one. Also, a portion of an X looks like an H; write down an IHX relation with that H and check that it is again the same. The IHX relation is in a sense unique; this is discussed in Exercise 15 on page 165.

5.2.5. Lemma. *The* STU *relations imply the* AS *relations for the internal vertices of a closed Jacobi diagram.*

Proof. Induction on the distance (in edges) of the vertex in question from the Wilson loop.

Induction base. If the vertex is adjacent to an external vertex, then the assertion follows by one application of the STU relation:

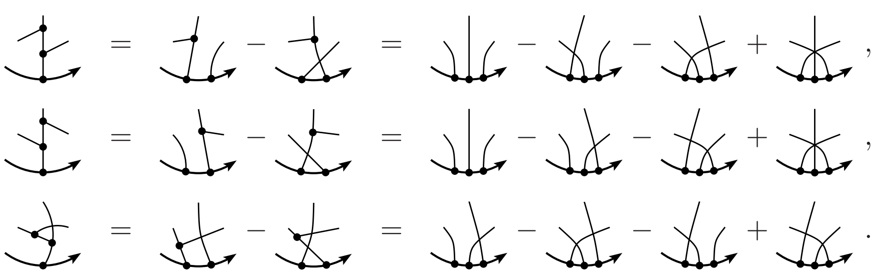


Induction step. Take two closed diagrams f_1 and f_2 that differ only by a cyclic order of half-edges at one internal vertex v. Apply STU relations to both diagrams *in the same way* so that v gets closer to the Wilson loop.

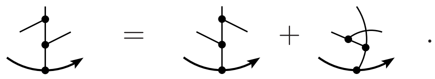
□

5.2.6. Lemma. *The* STU *relations imply the* IHX *relations for the internal edges of a closed diagram.*

Proof. The argument is similar to the one used in the previous proof. We take an IHX relation somewhere inside a closed diagram and, applying the same sequence of STU moves to each of the three diagrams, move the IHX fragment closer to the Wilson loop. The proof of the induction base is shown in these pictures:



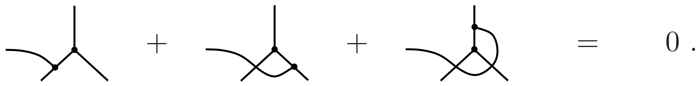
Therefore,



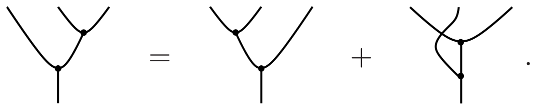
□

5.2.7. Other forms of the IHX relation. The IHX relation can be drawn in several forms, for example:

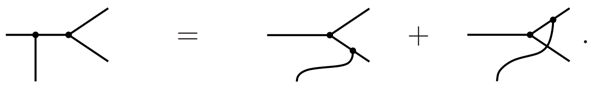
- (rotationally symmetric form)



- (Jacobi form)

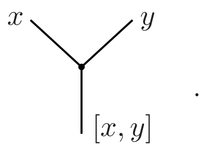


- (Kirchhoff form)



5.2.8. Exercise. By turning your head and pulling the strings of the diagrams, check that all these forms are equivalent.

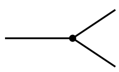
The Jacobi form of the IHX relation can be interpreted as follows. Suppose that to the upper 3 endpoints of each diagram we assign 3 elements of a Lie algebra, x, y and z, while every trivalent vertex, traversed downwards, takes the pair of "incoming" elements into their commutator:

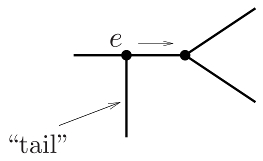


Then the IHX relation means that

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]],$$

which is the classical Jacobi identity. This observation, properly developed, leads to the construction of Lie algebra weight systems — see Chapter 6.

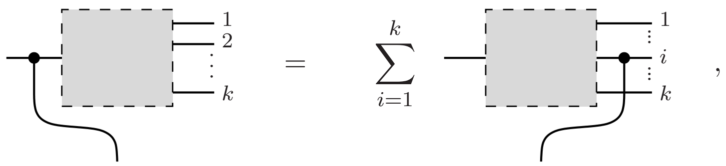
The Kirchhoff presentation is reminiscent of the Kirchhoff's law in electrotechnics. Let us view the portion  of the given graph as a piece of electrical circuit, and the variable vertex as an "electron" e with a "tail" whose endpoint is fixed. Suppose that the electron moves towards a node of the circuit:



Then the IHX relation expresses the well-known Kirchhoff rule: *the sum of currents entering a node is equal to the sum of currents going out of it.* This electrotechnical analogy is very useful, for instance, in the proof of the generalized IHX relation below.

The IHX relation can be generalized as follows:

5.2.9. Lemma. *(Kirchhoff law, or generalized IHX relation). The following identity holds:*

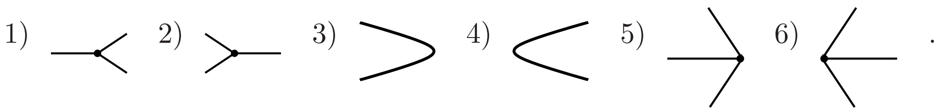


where the grey box is an arbitrary subgraph which has only 3-valent vertices.

Proof. Fix a horizontal line in the plane and consider an immersion of the given graph into the plane with smooth edges, generic with respect to the projection onto this line. More precisely, we assume that (1) the projections of all vertices onto the horizontal line are distinct, (2) when restricted to an arbitrary edge, the projection has only non-degenerate critical points, and (3) the images of all critical points are distinct and different from the images of vertices.

Bifurcation points are the images of vertices and critical points of the projection. Imagine a vertical line that moves from left to right; for every position of this line take the sum of all diagrams obtained by attaching the loose end to one of the intersection points. This sum does not depend on the position of the vertical line, because it does not change when the line crosses one bifurcation point.

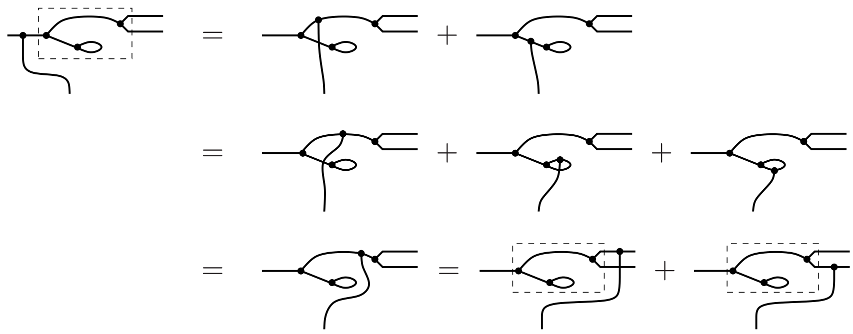
Indeed, bifurcation points fall into six categories:



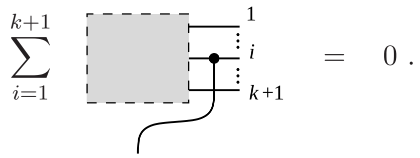
In the first two cases the assertion follows from the IHX relation, in cases 3 and 4 — from the AS relation. Cases 5 and 6 by a deformation of the immersion are reduced to a combination of the previous cases (also, they can be dealt with by one application of the IHX relation in the symmetric form).

□

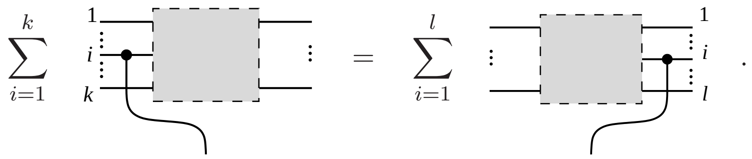
Example.



Remark. The difference between inputs and outputs in the equation of Lemma 5.2.9 is purely notational. We may bend the left-hand leg to the right and move the corresponding term to the right-hand side of the equation, changing its sign because of the antisymmetry relation, and thus obtain:



Or we may prefer to split the legs into two arbitrary subsets, putting one part on the left and another on the right. Then:



5.2.10. A corollary of the AS relation. A simple corollary of the antisymmetry relation in the space $\mathcal{C}$ is that any diagram D containing a hanging loop  is equal to zero. Indeed, there is an automorphism of the diagram that changes the two half-edges of the small circle and thus takes D to $-D$, which implies that $D = -D$ and $D = 0$. This observation also applies to the case when the small circle has other vertices on it and contains a subdiagram, symmetric with respect to the vertical axis. In fact, the assertion is true even if the diagram inside the circle is not symmetric at all. This is a generalization of Exercise 5.1.5, but cannot be proved by the same argument. In Section 5.6 we shall prove a similar statement (Lemma 5.6) about *open* Jacobi diagrams; that proof also applies here.

5.3. Isomorphism $\mathcal{A}^{fr} \simeq \mathcal{C}$

Let $\mathbf{A}_n$ be the set of chord diagrams of order n and $\mathbf{C}_n$ the set of closed diagrams of the same order. We have a natural inclusion $\lambda : \mathbf{A}_n \to \mathbf{C}_n$.

5.3.1. Theorem. *The inclusion λ gives rise to an isomorphism of vector spaces $\lambda : \mathcal{A}_n^{fr} \to \mathcal{C}_n$.*

Proof. We must check:

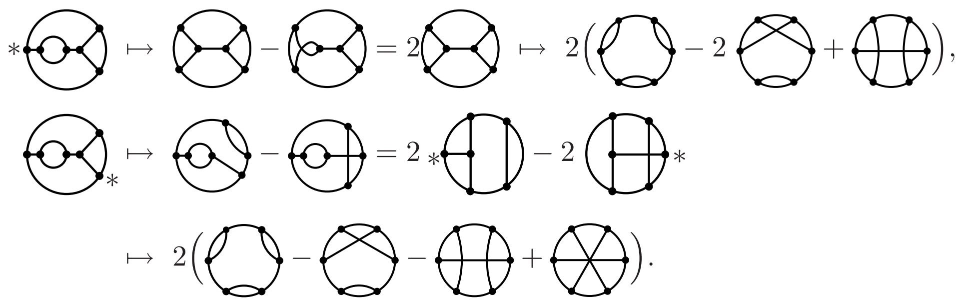
(A) that λ leads to a well-defined linear map from $\mathcal{A}_n^{fr}$ to $\mathcal{C}_n$;

(B) that this map is a linear isomorphism.

Part (A) is easy. Indeed, $\mathcal{A}_n^{fr} = \langle \mathbf{A}_n \rangle / \langle 4\mathrm{T} \rangle$, $\mathcal{C}_n = \langle \mathbf{C}_n \rangle / \langle \mathrm{STU} \rangle$, where angular brackets denote linear span. Lemma 5.2.1 implies that $\lambda(\langle 4\mathrm{T} \rangle) \subseteq \langle \mathrm{STU} \rangle$, therefore the map of the quotient spaces is well-defined.

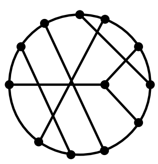
(B) We shall construct a linear map $\rho : \mathcal{C}_n \to \mathcal{A}_n^{fr}$ and prove that it is inverse to λ.

As we mentioned before, any closed diagram by the iterative use of STU relations can be transformed into a combination of chord diagrams. This gives rise to a map $\rho : \mathbf{C}_n \to \langle \mathbf{A}_n \rangle$ which is, however, multivalued, since the result may depend on the specific sequence of relations used. Here is an example of such a situation (the place where the STU relation is applied is marked by an asterisk):



However, the combination $\rho(C)$ is well-defined as an element of $\mathcal{A}_n^{fr}$, that is, modulo the 4T relations. The proof of this fact proceeds by induction on the number k of internal vertices in the diagram C.

If $k = 1$, then the diagram C consists of one tripod and several chords and may look something like this:

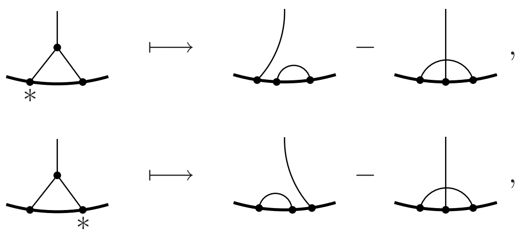


There are 3 ways to resolve the internal triple point by an STU relation, and the fact that the results are the same in $\mathcal{A}_n^{fr}$ is exactly the definition of the 4T relation.

Suppose that ρ is well-defined on closed diagrams with $< k$ internal vertices. Pick a diagram in $\mathbf{C}_n^{2n-k}$. The process of eliminating the triple points starts with a pair of neighboring external vertices. Let us prove, modulo the inductive hypothesis, that if we change the order of these two points, the final result will remain the same.

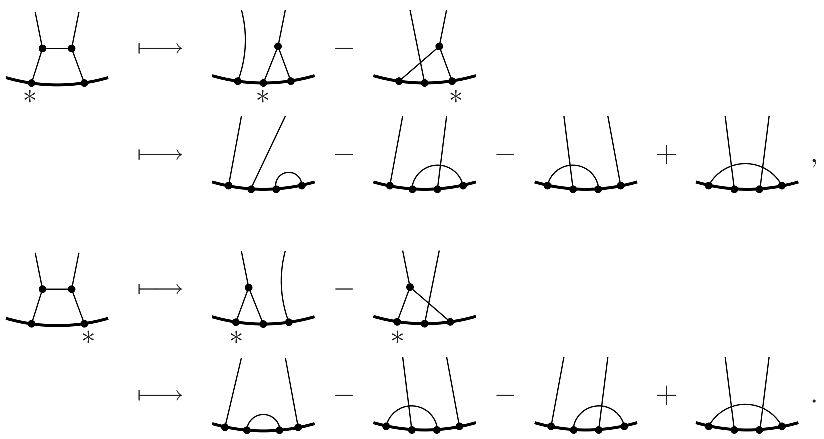
There are 3 cases to consider: the two chosen points on the Wilson loop are (1) adjacent to a common internal vertex, (2) adjacent to neighboring internal vertices, (3) adjacent to non-neighboring internal vertices. The proof for the cases (1) and (2) is shown in the pictures that follow.

(1)



The position of an isolated chord does not matter, because, as we know, the multiplication in $\mathcal{A}^{fr}$ is well-defined.

(2)



After the first resolution, we can choose the sequence of further resolutions arbitrarily, by the inductive hypothesis.

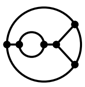
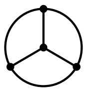
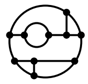
Exercise. Give a similar proof for the case (3).

We thus have a well-defined linear map $\rho : \mathcal{C}_n \to \mathcal{A}_n^{fr}$. The fact that it is two-sided inverse to λ is clear. $\square$

5.4. Product and coproduct in $\mathcal{C}$

Now we shall define a bialgebra structure in the space $\mathcal{C}$.

5.4.1. Definition. The product of two closed diagrams is defined in the same way as for chord diagrams: the two Wilson loops are cut at arbitrary places and then glued together into one loop, in agreement with the orientations:

 $\cdot$  $=$  .

5.4.2. Proposition. *This multiplication is well-defined, that is, it does not depend on the place of cuts.*

Proof. The isomorphism $\mathcal{A}^{fr} \cong \mathcal{C}$ constructed in Theorem 5.3.1 identifies the product in $\mathcal{A}^{fr}$ with the above product in $\mathcal{C}$.

Since the multiplication is well-defined in $\mathcal{A}^{fr}$, it is also well-defined in $\mathcal{C}$. $\square$

To define the *coproduct* in the space $\mathcal{C}$, we need the following definition:

5.4.3. Definition. The *internal graph* of a closed diagram is the graph obtained by stripping off the Wilson loop. A closed diagram is said to be *connected* if its internal graph is connected. The *connected components* of a closed diagram are defined as the connected components of its internal graph.

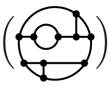
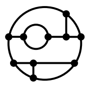
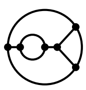
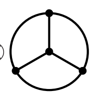
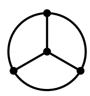
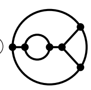
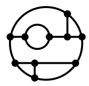
In the sense of this definition, any chord diagram of order n consists of n connected components — the maximal possible number.

Now, the construction of the coproduct proceeds in the same way as for chord diagrams.

5.4.4. Definition. Let D be a closed diagram and $[D]$ the set of its connected components. For any subset $J \subseteq [D]$ denote by D_J the closed diagram with only those components that belong to J and by $D_{\overline{J}}$ the "complementary" diagram ($\overline{J} := [D] \setminus J$). We set

$$\delta(D) := \sum_{J \subseteq [D]} D_J \otimes D_{\overline{J}}.$$

Example.

$\delta\Big($  $\Big) = 1 \otimes$  $+$  $\otimes$  $+$  $\otimes$  $+$  $\otimes 1.$

We know that the algebra $\mathcal{C}$, as a vector space, is spanned by chord diagrams. For chord diagrams, algebraic operations defined in $\mathcal{A}^{fr}$ and $\mathcal{C}$, tautologically coincide. It follows that the coproduct in $\mathcal{C}$ is compatible with its product and that the isomorphisms λ, ρ are, in fact, isomorphisms of bialgebras.

5.5. Primitive subspace of $\mathcal{C}$

By definition, connected closed diagrams are primitive with respect to the coproduct δ. It may sound surprising that the converse is also true:

5.5.1. Theorem. [**BN1**] *The primitive space $\mathcal{P}$ of the bialgebra $\mathcal{C}$ coincides with the linear span of connected closed diagrams.*

Note the contrast of this straightforward characterization of the primitive space in $\mathcal{C}$ with the case of chord diagrams.

Proof. If the primitive space $\mathcal{P}$ were bigger than the span of connected closed diagrams, then, according to Theorem A.2.25, it would contain an element that cannot be represented as a polynomial in connected closed diagrams. Therefore, to prove the theorem it is enough to show that every closed diagram is a polynomial in connected diagrams. This can be done by induction on the number of legs of a closed diagram C. Suppose that the diagram C consists of several connected components (see 5.4.3). The STU relation tells us that we can freely interchange the legs of C modulo closed diagrams with fewer legs. Using such permutations we can separate the connected components of C. This means that modulo closed diagrams with fewer legs C is equal to the product of its connected components. $\square$

5.5.2. Filtration of $\mathcal{P}_n$. The primitive space $\mathcal{P}_n$ cannot be graded by the number of legs k, because the STU relation is not homogeneous with respect to k. However, it can be *filtered*:

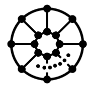
$$0 = \mathcal{P}_n^1 \subseteq \mathcal{P}_n^2 \subseteq \mathcal{P}_n^3 \subseteq \cdots \subseteq \mathcal{P}_n^{n+1} = \mathcal{P}_n \ .$$

where $\mathcal{P}_n^k$ is the subspace of $\mathcal{P}_n$ generated by connected closed diagrams with *at most* k legs.

The connectedness of a closed diagram with $2n$ vertices implies that the number of its legs cannot be bigger than $n+1$. That is why the filtration ends at the term $\mathcal{P}_n^{n+1}$.

The following facts about the filtration are known.

- [**ChV**] The filtration stabilizes even sooner. Namely, $\mathcal{P}_n^n = \mathcal{P}_n$ for even n, and $\mathcal{P}_n^{n-1} = \mathcal{P}_n$ for odd n. Moreover, for even n the quotient space $\mathcal{P}_n^n/\mathcal{P}_n^{n-1}$ has dimension one and is generated by the *wheel* $\overline{w}_n$ with n spokes:

$\overline{w}_n =$ 

n spokes

This fact is related to the Melvin-Morton conjecture (see Section 14.1 and Exercise 13).

- [**Da1**] The quotient space $\mathcal{P}_n^{n-1}/\mathcal{P}_n^{n-2}$ has dimension $[n/6]+1$ for odd n, and 0 for even n.
- [**Da2**] For even n
$$\dim(\mathcal{P}_n^{n-2}/\mathcal{P}_n^{n-3}) = \left[\frac{(n-2)^2+12(n-2)}{48}\right] + 1\ .$$
- For small degrees the dimensions of the quotient spaces $\mathcal{P}_n^k/\mathcal{P}_n^{k-1}$ were calculated by J. Kneissler [**Kn0**] (empty entries in the table are zeroes):

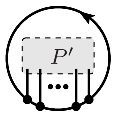
$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	$\dim \mathcal{P}_n$
1		1											1
2		1											1
3		1											1
4		1		1									2
5		2		1									3
6		2		2		1							5
7		3		3		2							8
8		4		4		3		1					12
9		5		6		5		2					18
10		6		8		8		4		1			27
11		8		10		11		8		2			39
12		9		13		15		12		5		1	55

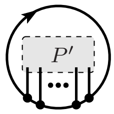
5.5.3. Detecting the knot orientation. One may notice that in the table above all entries with odd k vanish. This means that any connected closed diagram with an odd number of legs is equal to a suitable linear combination of diagrams with fewer legs. This observation is closely related to the problem of distinguishing knot orientation by Vassiliev invariants. The existence of the universal Vassiliev invariant given by the Kontsevich integral reduces the problem of detecting the knot orientation to a purely combinatorial problem. Namely, Vassiliev invariants do not distinguish the orientation of knots if and only if every chord diagram is equal to itself with the reversed orientation of the Wilson loop, modulo the 4T relations (see the Corollary in 9.3). Denote the operation of reversing the orientation by

τ; its action on a chord diagram D is equivalent to a mirror reflection of the diagram as a planar picture, and the question is whether $D = \tau(D)$ in $\mathcal{A}^{fr}$. The followng theorem translates this fact into the language of primitive subspaces.

Theorem. *Vassiliev invariants do not distinguish the orientation of knots if and only if $\mathcal{P}_n^k = \mathcal{P}_n^{k-1}$ for any odd k and arbitrary n.*

To prove the Theorem we need to reformulate the question whether $D = \tau(D)$ in terms of closed diagrams. Reversing the orientation of the Wilson loop on closed diagrams should be done with some caution, see the discussion in 5.1.2 on page 128). The correct way of doing it is carrying the operation τ from chord diagrams to closed diagrams by the isomorphism $\lambda : \mathcal{A}^{fr} \to \mathcal{C}$; then we have the following assertion:

Lemma. *Let* $P =$  *be a closed diagram with k external vertices. Then*

$$\tau(P) \;=\; (-1)^k$$  .

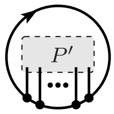
Proof. Represent P as a linear combination of chord diagrams using STU relations, and then reverse the orientation of the Wilson loop of all chord diagrams obtained. After that, convert the resulting linear combination back to a closed diagram. Each application of the STU relation multiplies the result by -1 because of the reversed Wilson loop (see page 128). In total, we have to perform the STU relation $2n - k$ times, where n is the degree of P. Therefore, the result gets multiplied by $(-1)^{2n-k} = (-1)^k$. $\square$

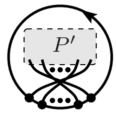
In the particular case $k = 1$ the Lemma asserts that $\mathcal{P}_n^1 = 0$ for all n — this fact appeared earlier as Exercise 5.1.5.

The operation $\tau : \mathcal{C} \to \mathcal{C}$ is, in fact, an algebra automorphism, $\tau(C_1 \cdot C_2) = \tau(C_1) \cdot \tau(C_2)$. Therefore, to check the equality $\tau = \mathrm{id}_{\mathcal{C}}$ it is enough to check it on the primitive subspace, that is, determine whether $P = \tau(P)$ for every connected closed diagram P.

Corollary of the Lemma. *Let $P \in \mathcal{P}^k = \bigoplus_{n=1}^{\infty} \mathcal{P}_n^k$ be a connected closed diagram with k legs. Then* $\tau(P) \equiv (-1)^k P \mod \mathcal{P}^{k-1}$.

Proof of the Corollary. Rotating the Wilson loop in 3-space by 180° about the vertical axis, we get:

$$\tau(P) \;=\; (-1)^k$$ 

$$=\; (-1)^k$$ 

.

The *STU* relations allow us to permute the legs modulo diagrams with fewer number of legs. Applying this procedure to the last diagram we can straighten out all legs and get $(-1)^k P$. □

Proof of the Theorem. Suppose that the Vassiliev invariants do not distinguish the orientation of knots. Then $\tau(P) = P$ for every connected closed diagram P. In particular, for a diagram P with an odd number of legs k we have $P \equiv -P \mod \mathcal{P}^{k-1}$. Hence, $2P \equiv 0 \mod \mathcal{P}^{k-1}$, which means that P is equal to a linear combination of diagrams with fewer legs, and therefore $\dim(\mathcal{P}_n^k/\mathcal{P}_n^{k-1}) = 0$.

Conversely, suppose that Vassiliev invariants do distinguish the orientation. Then there is a connected closed diagram P such that $\tau(P) \neq P$. Choose such P with the smallest possible number of legs k. Let us show that k cannot be even. Consider $X = P - \tau(P) \neq 0$. Since τ is an involution $\tau(X) = -X$. But, in the case of even k, the non-zero element X has fewer legs than k, and $\tau(X) = -X \neq X$, so k cannot be minimal. Therefore, the minimal such k is odd, and $\dim(\mathcal{P}_n^k/\mathcal{P}_n^{k-1}) \neq 0$. □

5.5.4. Exercise. Check that, for invariants of fixed degree, the theorem can be specialized as follows. *Vassiliev invariants of degree $\leqslant n$ do not distinguish the orientation of knots if and only if $\mathcal{P}_m^k = \mathcal{P}_m^{k-1}$ for any odd k and arbitrary $m \leqslant n$.*

5.5.5. Exercise. Similarly to the filtration in the primitive space $\mathcal{P}$, one can introduce the leg filtration in the whole space $\mathcal{C}$. Prove the following version of the above theorem: *Vassiliev invariants of degree n do not distinguish the orientation of knots if and only if $\mathcal{C}_n^k = \mathcal{C}_n^{k-1}$ for any odd k and arbitrary n.*

5.6. Open Jacobi diagrams

The subject of this section is the combinatorial bialgebra $\mathcal{B}$ which is isomorphic to the bialgebras $\mathcal{A}^{fr}$ and $\mathcal{C}$ as a vector space and as a coalgebra, but has a different natural multiplication. This leads to the remarkable fact that in the vector space $\mathcal{A}^{fr} \simeq \mathcal{C} \simeq \mathcal{B}$ there are two multiplications both compatible with one and the same coproduct.

5.6.1. Definition. An *open Jacobi diagram* is a graph with 1- and 3-valent vertices, cyclic order of (half-)edges at every 3-valent vertex and with at least one 1-valent vertex in every connected component.

An open diagram is not required to be connected. It may have loops and multiple edges. We shall see later that, modulo the natural relations any diagram with a loop vanishes. However, it is important to include the diagrams with loops in the definition, because the loops may appear during natural operations on open diagrams, and it is exactly because of this fact that we introduce the cyclic order on half-edges, not on whole edges.

The total number of vertices of an open diagram is even. Half of this number is called the *degree* (or *order*) of an open diagram. We denote the set of all open diagrams of degree n by $\mathbf{B}_n$. The univalent vertices will sometimes be referred to as *legs*.

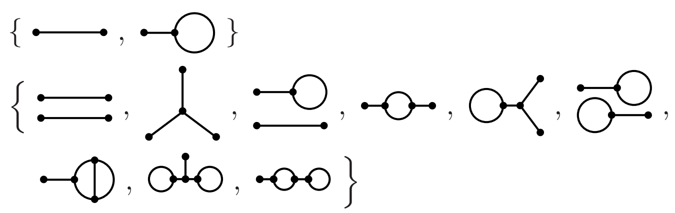
In the literature, open diagrams are also referred to as *1-3-valent diagrams*, *Jacobi diagrams*, *web diagrams* and *Chinese characters*.

Definition. An isomorphism between two open diagrams is a one-to-one correspondence between their respective sets of vertices and half-edges that preserves the vertex-edge adjacency and the cyclic order of half-edges at every vertex.

Example. Below is the complete list of open diagrams of degree 1 and 2, up to isomorphism just introduced.

$\mathbf{B}_1 =$

$\mathbf{B}_2 =$



Most of the elements listed above will be of no importance to us, as they are killed by the following definition.

5.6.2. Definition. The space of open diagrams of degree n is the quotient space

$$\mathcal{B}_n := \langle \mathbf{B}_n \rangle / \langle \text{AS, IHX} \rangle,$$

where $\langle \mathbf{B}_n \rangle$ is the vector space formally generated by all open diagrams of degree n and $\langle \text{AS, IHX} \rangle$ stands for the subspace spanned by all AS and IHX relations (see 5.2.2, 5.2.3). By definition, $\mathcal{B}_0$ is one-dimensional, spanned by the empty diagram, and $\mathcal{B} := \bigoplus_{n=0}^{\infty} \mathcal{B}_n$.

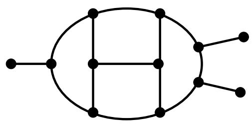
Just as in the case of closed diagrams (Section 5.2.10), the AS relation immediately implies that any open diagram with a loop ( ) vanishes in $\mathcal{B}$. Let us give a most general statement of this observation — valid, in fact, both for open and for closed Jacobi diagrams.

Definition. An *anti-automorphism* of a Jacobi diagram $b \in \mathbf{B}_n$ is a graph automorphism of b such that the cyclic order of half-edges is reversed in an odd number of vertices.

5.6.3. Lemma. *If a diagram $b \in \mathbf{B}_n$ admits an anti-automorphism, then $b = 0$ in the vector space $\mathcal{B}$.*

Proof. Indeed, it follows from the definitions that in this case $b = -b$. $\square$

Example.



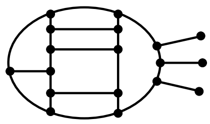
$= \quad 0\,.$

Exercise. Show that $\dim \mathcal{B}_1 = 1$, $\dim \mathcal{B}_2 = 2$.

The relations AS and IHX imply the generalized IHX relation, or Kirchhoff law (Lemma 5.2.9) and many other interesting identities among the elements of the space $\mathcal{B}$. Some of them are proved in the next chapter (Section 7.2.5) in the context of the algebra Γ. Here is one more assertion that makes sense only in $\mathcal{B}$, as its formulation refers to univalent vertices (legs).

Lemma. *If $b \in \mathcal{B}$ is a diagram with an odd number of legs, all of which are attached to one and the same edge, then $b = 0$ modulo AS and IHX relations.*

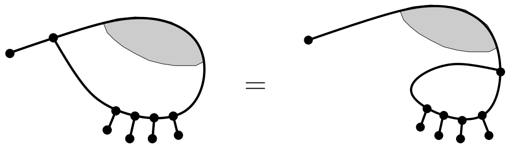
Example.



$= \quad 0\,.$

Note that in this example the diagram does not have an anti-automorphism, so the previous lemma does not apply.

Proof. Any diagram satisfying the premises of the lemma can be put into the form on the left of the next picture. Then by the generalized IHX relation it is equal to the diagram on the right which obviously possesses an anti-automorphism and therefore is equal to zero:



where the grey region is an arbitrary subdiagram. $\square$

In particular, any diagram with exactly one leg vanishes in $\mathcal{B}$. This is an exact counterpart of the corresponding property of closed diagrams (see

Exercise 5.1.5); both facts are, furthermore, equivalent to each other in view of the isomorphism $\mathcal{C} \cong \mathcal{B}$ that we shall speak about later (in Section 5.7).

Conjecture. Any diagram with an odd number of legs is 0 in $\mathcal{B}$.

This important conjecture is equivalent to the conjecture that Vassiliev invariants do not distinguish the orientation of knots (see Section 5.8.3).

Relations AS and IHX, unlike STU, preserve the separation of vertices into 1- and 3-valent. Therefore, the space $\mathcal{B}$ has a much finer grading than $\mathcal{A}^{fr}$. Apart from the main grading by half the number of vertices, indicated by the subscript in $\mathcal{B}$, it also has a grading by the number of univalent vertices

$$\mathcal{B} = \bigoplus_n \bigoplus_k \mathcal{B}_n^k,$$

indicated by the superscript in $\mathcal{B}$, so that $\mathcal{B}_n^k$ is the subspace spanned by all diagrams with k legs and $2n$ vertices in total.

For disconnected diagrams the second grading can, in turn, be refined to a multigrading by the number of legs in each connected component of the diagram:

$$\mathcal{B} = \bigoplus_n \bigoplus_{k_1 \leqslant \ldots \leqslant k_m} \mathcal{B}_n^{k_1,\ldots,k_m}.$$

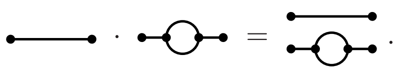
Yet another important grading in the space $\mathcal{B}$ is the grading by the number of *loops* in a diagram, that is, by its first Betti number. In the case of connected diagrams, we have a decomposition:

$$\mathcal{B} = \bigoplus_n \bigoplus_k \bigoplus_l {}^l\mathcal{B}_n^k,$$

The abundance of gradings makes the work with the space $\mathcal{B}$ more convenient than with $\mathcal{C}$, although both are isomorphic, as we shall soon see.

5.6.4. The bialgebra structure on $\mathcal{B}$. Both the product and the coproduct in the vector space $\mathcal{B}$ are defined in a rather straightforward way. We first define the product and coproduct on diagrams, then extend the operations by linearity to the free vector space spanned by the diagrams, and then note that they are compatible with the AS and IHX relations and thus descend to the quotient space $\mathcal{B}$.

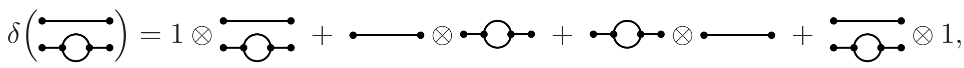
5.6.5. Definition. The product of two open diagrams is their disjoint union.

Example. 

5.6.6. Definition. Let D be an open diagram and $[D]$ — the set of its connected components. For a subset $J \subseteq [D]$, denote by D_J the union of the components that belong to J and by $D_{\overline{J}}$ — the union of the components that do not belong to J. We set

$$\delta(D) := \sum_{J \subseteq [D]} D_J \otimes D_{\overline{J}}.$$

Example.



As the relations in $\mathcal{B}$ do not intermingle different connected components of a diagram, the product of an AS or IHX combination of diagrams by an arbitrary open diagram belongs to the linear span of the relations of the same types. Also, the coproduct of any AS or IHX relation vanishes modulo these relations. Therefore, we have well-defined algebraic operations in the space $\mathcal{B}$, and they are evidently compatible with each other. The space $\mathcal{B}$ thus becomes a graded bialgebra.

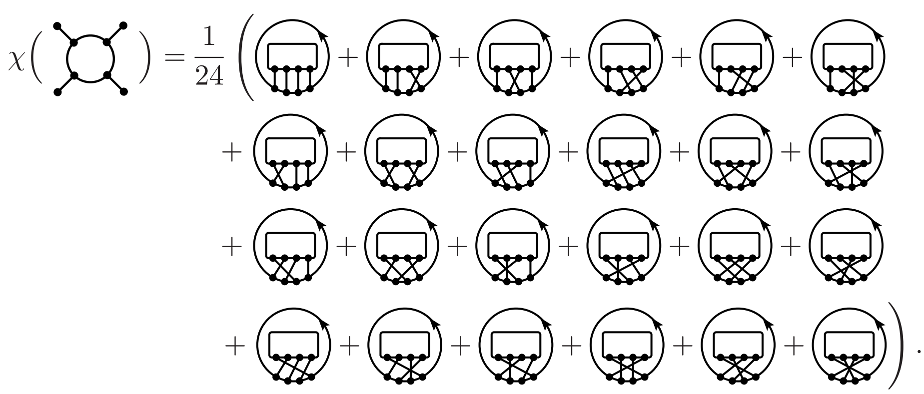
5.7. Linear isomorphism $\mathcal{B} \simeq \mathcal{C}$

In this section we construct a linear isomorphism between vector spaces $\mathcal{B}_n$ and $\mathcal{C}_n$. The question whether it preserves multiplication will be discussed later (Section 5.8). Our exposition follows [**BN1**], with some details omitted, but some examples added.

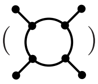
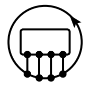
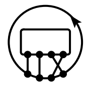
To convert an open diagram into a closed diagram, we join all of its univalent vertices by a Wilson loop. Fix k distinct points on the circle. For an open diagram with k legs $D \in \mathbf{B}_n^k$ there are $k!$ ways of glueing its legs to the Wilson loop at these k points, and we set $\chi(D)$ to be equal to the arithmetic mean of all the resulting closed diagrams. Thus we get the *symmetrization map*

$$\chi : \mathbf{B} \to \mathcal{C}.$$

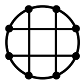
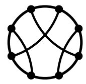
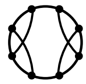
For example,



Scrutinizing these pictures, one can see that 16 out of 24 summands are equivalent to the first diagram, while the remaining 8 are equivalent to the second one. Therefore,

$\chi($  $) = \frac{1}{3}$  $+ \frac{2}{3}$  .

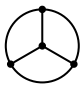
Exercise. Express this element via chord diagrams, using the isomorphism $\mathcal{C} \simeq \mathcal{A}^{fr}$.

Answer:  $- \frac{10}{3}$  $+ \frac{4}{3}$  .

5.7.1. Theorem. *The symmetrization map $\chi : \mathbf{B} \to \mathcal{C}$ descends to a linear map $\chi : \mathcal{B} \to \mathcal{C}$, which is a graded isomorphism between the vector spaces $\mathcal{B}$ and $\mathcal{C}$.*

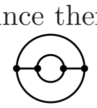
The theorem consists of two parts:

- Easy part: χ is well-defined.
- Difficult part: χ is bijective.

The proof of bijectivity of χ is difficult because not every closed diagram can be obtained by a symmetrization of an open diagram. For example, the diagram  is not a symmetrization of any open diagram, even though it looks very much symmetric. Notice that symmetrizing the internal graph of this diagram we get 0.

Easy part of the theorem. To prove the easy part, we must show that the AS and IHX combinations of open diagrams go to 0 in the space $\mathcal{C}$. This follows from lemmas 5.2.5 and 5.2.6.

Difficult part of the theorem. To prove the difficult part, we construct a linear map τ from $\mathcal{C}$ to $\mathcal{B}$, inverse to χ. This will be done inductively by the number of legs of the diagrams. We shall write τ_k for the restriction of τ to the subspace spanned by diagrams with at most k legs.

There is only one way to attach the only leg of an open diagram to the Wilson loop. Therefore, we can define τ_1 on a closed diagram C with one leg as the internal graph of C. (In fact, both open and closed diagrams with one leg are all zero in $\mathcal{B}$ and $\mathcal{C}$ respectively, see Exercise 5.1.5 and Lemma 5.6). For diagrams with two legs the situation is similar. Every closed diagram with two legs is a symmetrization of an open diagram, since there is only one cyclic order on the set of two elements. For example,  is the symmetrization of the diagram . Therefore, for a closed diagram C with two legs we can define $\tau_2(C)$ to be the internal graph of C.

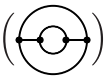
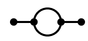
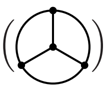
In what follows, we shall often speak of the action of the symmetric group S_k on closed diagrams with k legs. This action preserves the internal graph of a closed diagram and permutes the points where the legs of the internal graph are attached to the Wilson loop. Strictly speaking, to define this action we need the legs of the diagrams to be numbered. We shall always assume that such numbering is chosen; the particular form of this numbering will be irrelevant.

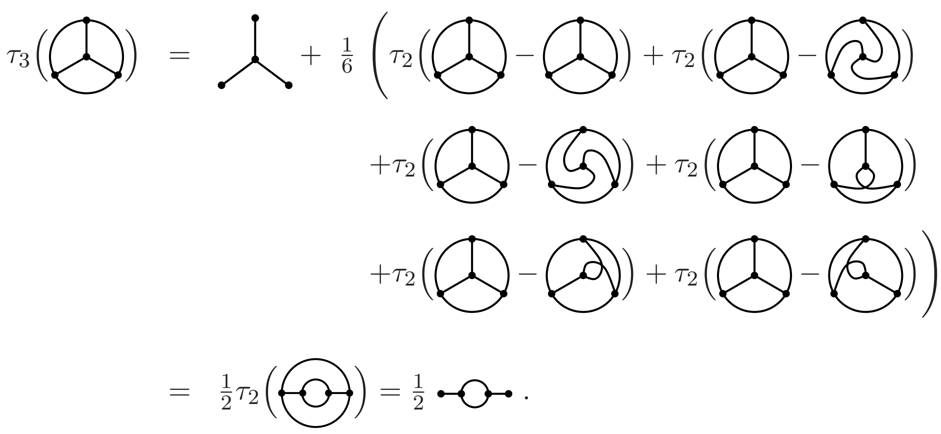
The difference of a closed diagram D and the same diagram whose legs are permuted by some permutation σ, is equivalent, modulo STU relations, to a combination of diagrams with a smaller number of external vertices. For every given D and σ we fix such a linear combination.

Assuming that the map τ is defined for closed diagrams having less than k legs, we define it for a diagram D with exactly k legs by the formula:

$$\tau_k(D) = \widetilde{D} + \frac{1}{k!} \sum_{\sigma \in S_k} \tau_{k-1}(D - \sigma(D)) \ , \tag{5.7.1.1}$$

where $\widetilde{D}$ is the internal graph of D, and $D - \sigma(D)$ is represented as a combination of diagrams with less than k legs according to the choice above.

For example, we know that $\tau\Big($$\Big) =$ , and we want to find $\tau\Big($$\Big)$. By the above formula, we have:



We have to prove the following assertions:

(i) The value $\tau_{k-1}(D - \sigma(D))$ in the formula (5.7.1.1) does not depend on the presentation of $D - \sigma(D)$ as a combination of diagrams with a smaller number of external vertices.

(ii) The map τ respects STU relations.

(iii) $\chi \circ \tau = \mathrm{id}_{\mathcal{C}}$ and τ is surjective.

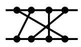
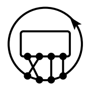
The first two assertions imply that τ is well-defined and the third means that τ is an isomorphism. The rest of the section is dedicated to the proof of these statements.

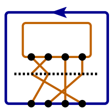
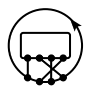
In the vector space spanned by all closed diagrams (with no relations imposed) let $\mathcal{D}^k$ be the subspace spanned by all diagrams with at most k external vertices. We have a chain of inclusions

$$\mathcal{D}^0 \subset \mathcal{D}^1 \subset \mathcal{D}^2 \subset \ldots .$$

We denote by $\mathcal{I}^k$ be the subspace in $\mathcal{D}^k$ spanned by all STU, IHX and antisymmetry relations that do not involve diagrams with more than k external vertices.

5.7.2. Action of permutations on closed diagrams. The action of the symmetric group S_k on closed diagrams with k legs can be represented graphically as the "composition" of a closed diagram with the diagram of the permutation:

$k = 4$; $\sigma = (4132) =$  ; $D =$  ;

$\sigma D =$  $=$  .

5.7.3. Lemma. Let $D \in \mathcal{D}^k$.

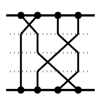
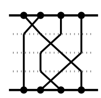
- *Modulo $\mathcal{I}^k$, the difference $D - \sigma D$ belongs to $\mathcal{D}^{k-1}$.*
- *Any choice U_σ of a presentation of σ as a product of transpositions determines in a natural way an element $\Gamma_D(U_\sigma) \in \mathcal{D}^{k-1}$ such that*

$$\Gamma_D(U_\sigma) \equiv D - \sigma D \mod \mathcal{I}^k.$$

- *Furthermore, if U_σ and U'_σ are two such presentations, then $\Gamma_D(U_\sigma)$ is equal to $\Gamma_D(U'_\sigma)$ modulo $\mathcal{I}^{k-1}$.*

This is Lemma 5.5 from [**BN1**]. Rather than giving the details of the proof (which can be found in [**BN1**]) we illustrate it on a concrete example.

Take the permutation $\sigma = (4132)$ and let D be the diagram considered above. Choose two presentations of σ as a product of transpositions:

$U_\sigma = (34)(23)(34)(12) =$  ; $U'_\sigma = (23)(34)(23)(12) =$  .

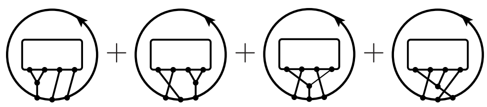
For each of these products we represent $D - \sigma D$ as a sum:

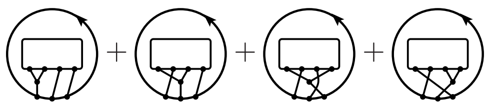
$$\begin{aligned} D - \sigma D &= \big(D - {}^{(12)}D\big) + \big({}^{(12)}D - {}^{(34)(12)}D\big) + \big({}^{(34)(12)}D - {}^{(23)(34)(12)}D\big) \\ &\quad + \big({}^{(23)(34)(12)}D - {}^{(34)(23)(34)(12)}D\big) \end{aligned}$$

and

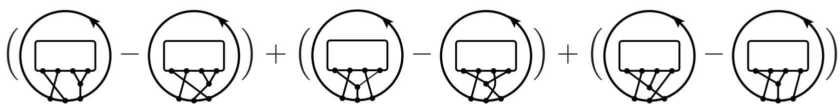
$$\begin{aligned} D - \sigma D &= \big(D - {}^{(12)}D\big) + \big({}^{(12)}D - {}^{(23)(12)}D\big) + \big({}^{(23)(12)}D - {}^{(34)(23)(12)}D\big) \\ &\quad + \big({}^{(34)(23)(12)}D - {}^{(23)(34)(23)(12)}D\big) \ . \end{aligned}$$

Here, the two terms in every pair of parentheses differ only by a transposition of two neighboring legs, so their difference is the right-hand side of an STU relation. Modulo the subspace $\mathcal{I}^4$ each difference can be replaced by the corresponding left-hand side of the STU relation, which is a diagram in $\mathcal{D}^3$. We get

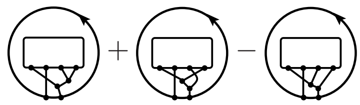
$\Gamma_D(U_\sigma) =$ 

$\Gamma_D(U'_\sigma) =$ 

Now the difference $\Gamma_D(U_\sigma) - \Gamma_D(U'_\sigma)$ equals



Using the STU relation in $\mathcal{I}^3$ we can represent it in the form

$\Gamma_D(U_\sigma) - \Gamma_D(U'_\sigma) =$  $= 0$

which is zero because of the IHX relation.

5.7.4. Proof of assertions (i) and (ii). Let us assume that the map τ, defined by the formula (5.7.1.1), is (i) well-defined on $\mathcal{D}^{k-1}$ and (ii) vanishes on $\mathcal{I}^{k-1}$.

Define $\tau'(D)$ to be equal to $\tau(D)$ if $D \in \mathcal{D}^{k-1}$, and if $D \in \mathcal{D}^k - \mathcal{D}^{k-1}$ set

$$\tau'(D) = \widetilde{D} + \frac{1}{k!} \sum_{\sigma \in S_k} \tau(\Gamma_D(U_\sigma)) \ .$$

Lemma 5.7.3 means that for any given $D \in \mathcal{D}^k$ with exactly k external vertices $\tau(\Gamma_D(U_\sigma)))$ does not depend on a specific presentation U_σ of the permutation σ as a product of transpositions. Therefore, τ' gives a well-defined map $\mathcal{D}^k \to \mathcal{B}$.

Let us now show that τ' vanishes on $\mathcal{I}^k$. It is obvious that τ' vanishes on the IHX and antisymmetry relations since these relations hold in $\mathcal{B}$. So we only need to check the STU relation which relates a diagram D^{k-1} with $k-1$ external vertices and the corresponding two diagrams D^k and U_iD^k with k external vertices, where U_i is a transposition $U_i = (i, i+1)$. Let us apply τ' to the right-hand side of the STU relation:

$$\begin{aligned} \tau'(D^k - U_iD^k) &= \widetilde{D^k} + \tfrac{1}{k!}\sum_{\sigma\in S_k} \tau(\Gamma_{D^k}(U_\sigma)) \\ &\quad -\widetilde{U_iD^k} - \tfrac{1}{k!}\sum_{\sigma'\in S_k} \tau(\Gamma_{U_iD^k}(U_{\sigma'})) \ . \end{aligned}$$

Note that $\widetilde{D^k} = \widetilde{U_iD^k}$. Reparametrizing the first sum, we get

$$\tau'(D^k - U_iD^k) = \frac{1}{k!}\sum_{\sigma\in S_k} \tau(\Gamma_{D^k}(U_\sigma U_i) - \Gamma_{U_iD^k}(U_\sigma)) \ .$$

Using the obvious identity $\Gamma_D(U_\sigma U_i) = \Gamma_D(U_i) + \Gamma_{U_iD^k}(U_\sigma)$ and the fact that $D^{k-1} = \Gamma_D(U_i)$, we now obtain

$$\tau'(D^k - U_iD^k) = \frac{1}{k!}\sum_{\sigma\in S_k} \tau(D^{k-1}) = \tau(D^{k-1}) = \tau'(D^{k-1}) \ ,$$

which means that τ' vanishes on the STU relation, and, hence, on the whole of $\mathcal{I}^k$.

Now, it follows from the second part of Lemma 5.7.3 that $\tau' = \tau$ on $\mathcal{D}^k$. In particular, this means that τ is well-defined on $\mathcal{D}^k$ and vanishes on $\mathcal{I}^k$. By induction, this implies the assertions (i) and (ii).

5.7.5. Proof of assertion (iii). Assume that $\chi \circ \tau$ is the identity for diagrams with at most $k-1$ legs. Take $D \in \mathcal{D}^k$ representing an element of $\mathcal{C}$. Then

$$\begin{aligned} (\chi\circ\tau)(D) &= \chi\Big(\widetilde{D} + \tfrac{1}{k!}\sum_{\sigma\in S_k} \tau(\Gamma_D(U_\sigma))\Big) \\ &= \tfrac{1}{k!}\sum_{\sigma\in S_k} \big(\sigma D + (\chi\circ\tau)(\Gamma_D(U_\sigma))\big) \ . \end{aligned}$$

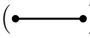
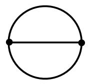
Since $\Gamma_D(U_\sigma)$ is a combination of diagrams with at most $k-1$ legs, by the induction hypothesis $\chi\circ\tau(\Gamma_D(U_\sigma)) = \Gamma_D(U_\sigma)$ and, hence,

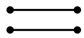
$$(\chi\circ\tau)(D) = \frac{1}{k!}\sum_{\sigma\in S_k}\big(\sigma D + \Gamma_D(U_\sigma)\big) = \frac{1}{k!}\sum_{\sigma\in S_k}\big(\sigma D + D - \sigma D\big) = D \ .$$

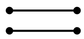
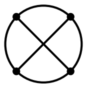
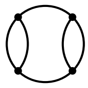
The surjectivity of τ is clear from the definition, so we have established that χ is a linear isomorphism between $\mathcal{B}$ and $\mathcal{C}$. $\square$

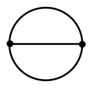
5.8. Relation between $\mathcal{B}$ and $\mathcal{C}$

It is easy to check that the isomorphism χ is compatible with the coproduct in the algebras $\mathcal{B}$ and $\mathcal{C}$. (**Exercise**: pick a decomposable diagram $b \in \mathcal{B}$ and check that $\delta_A(\chi(b))$ and $\chi(\delta_B(b))$ coincide.) However, χ is *not* compatible with the product. For example,

$\chi($  $) =$  .

The square of the element in $\mathcal{B}$ is  . However, the corresponding element of $\mathcal{C}$

$\chi($  $) = \frac{1}{3}$  $+ \frac{2}{3}$ 

is not equal to the square of  .

We can, of course, carry the natural multiplication of the algebra $\mathcal{B}$ to the algebra $\mathcal{C}$ with the help of the isomorphism χ, thus obtaining a bialgebra with two different products, both compatible with one and the same coproduct.

By definition, any connected diagram $p \in \mathcal{B}$ is primitive. Similarly to Theorem 5.5.1 we have:

5.8.1. Theorem. *The primitive space of the bialgebra $\mathcal{B}$ is spanned by connected open diagrams.*

Proof. The same argument as in the proof of Theorem 5.5.1, with a simplification that in the present case we do not have to prove that every element of $\mathcal{B}$ has a polynomial expression in terms of connected diagrams: this holds by definition. $\square$

Although the isomorphism χ does not respect the multiplication, the two algebras $\mathcal{B}$ and $\mathcal{C}$ are isomorphic. This is clear from what we know about their structure: by Milnor–Moore theorem both algebras are commutative polynomial algebras over the corresponding primitive subspaces. But the primitive subspaces coincide, since χ preserves the coproduct! An explicit algebra isomorphism between $\mathcal{B}$ and $\mathcal{C}$ will be the subject of Section 11.3.

Situations of this kind appear in the theory of Lie algebras. Namely, the bialgebra of invariants in the symmetric algebra of a Lie algebra L has a natural map into the centre of the universal enveloping algebra of L. This map, which is very similar in spirit to the symmetrization map χ, is an isomorphism of coalgebras, but does not respect the multiplication. In fact, this analogy is anything but superficial. It turns out that the algebra $\mathcal{C}$ is isomorphic to the centre of the universal enveloping algebra for a certain Casimir Lie algebra in a certain tensor category. For further details see [**HV**].

5.8.2. Unframed version of $\mathcal{B}$. The unframed version of the algebras $\mathcal{A}^{fr}$ and $\mathcal{C}$ are obtained by taking the quotient by the ideal generated by the diagram with 1 chord Θ. Although the product in $\mathcal{B}$ is different, it is easy to see that multiplication in $\mathcal{C}$ by Θ corresponds to multiplication in $\mathcal{B}$ by the *strut* s: the diagram of degree 1 consisting of 2 univalent vertices and one edge. Therefore, the unframed version of the algebra $\mathcal{B}$ is its quotient by the ideal generated by s and we have: $\mathcal{B}' := \mathcal{B}/(s) \cong \mathcal{C}/(\Theta) =: \mathcal{C}'$.

5.8.3. Grading in $\mathcal{B}$ and filtration in $\mathcal{C}$. The space of primitive elements $\mathcal{PB}$ is carried by χ isomorphically onto $\mathcal{PC}$. The space $\mathcal{PC} = \mathcal{P}$ is filtered (see Section 5.5.2), the space $\mathcal{PB}$ is graded (page 144). It turns out that χ intertwines the grading on $\mathcal{B}$ with the filtration on $\mathcal{C}$. Indeed, the definition of χ and the construction of the inverse mapping τ imply two facts:

$$\chi(\mathcal{PB}^i) \subset \mathcal{P}^i \subset \mathcal{P}^k, \text{ if } i < k,$$

$$\tau(\mathcal{P}^k) \subset \bigoplus_{i=1}^{k} \mathcal{PB}^k.$$

Therefore, we have an isomorphism

$$\tau : \mathcal{P}_n^k \longrightarrow \mathcal{PB}_n^1 \oplus \mathcal{PB}_n^2 \oplus \ldots \oplus \mathcal{PB}_n^{k-1} \oplus \mathcal{PB}_n^k \ .$$

and, hence, an isomorphism $\mathcal{P}_n^k/\mathcal{P}_n^{k-1} \cong \mathcal{PB}_n^k$.

Using this fact, we can give an elegant reformulation of the theorem about detecting the orientation of knots (Section 5.5.3, page 140):

Corollary. *Vassiliev invariants do distinguish the orientation of knots if and only if $\mathcal{PB}_n^k \neq 0$ for an odd k and some n.*

Let us clarify that by saying that Vassiliev invariants do distinguish the orientation of knots we mean that there exists a knot K non-equivalent to its inverse K^* and a Vassiliev invariant f such that $f(K) \neq f(K^*)$.

5.8.4. Exercise. Check that in the previous statement the letter $\mathcal{P}$ can be dropped: Vassiliev invariants do distinguish the orientation of knots if and only if $\mathcal{B}_n^k \neq 0$ for an odd k and some n.

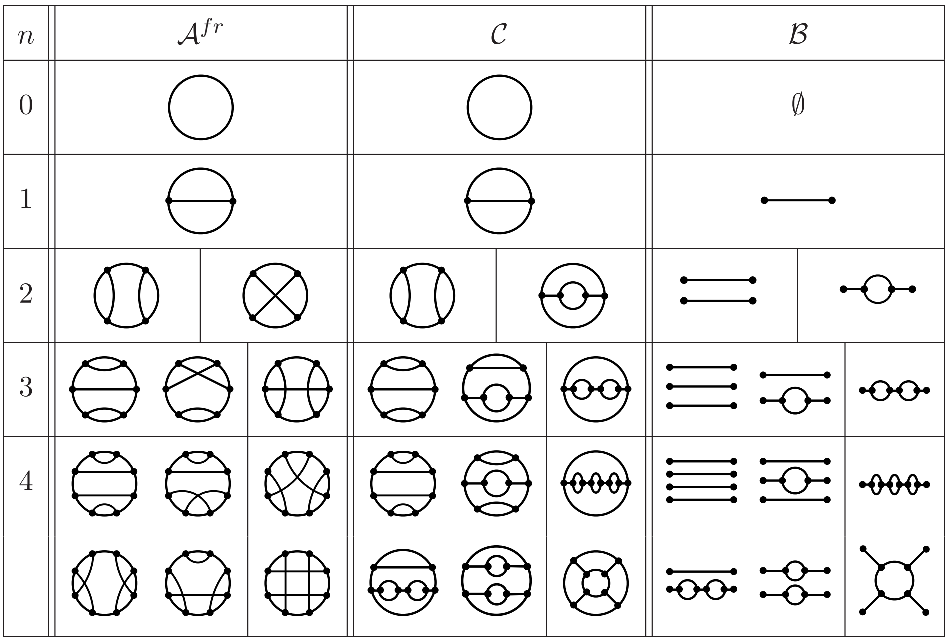
The relation between $\mathcal{C}$ and $\mathcal{B}$ in this respect can also be stated in the form of a commutative diagram:

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\chi"] \arrow[d, "\tau_B"'] & \mathcal{C} \arrow[d, "\tau_C"] \\
\mathcal{B} \arrow[r, "\chi"'] & \mathcal{C}
\end{tikzcd}$$

where χ is the symmetrization isomorphism, τ_C is the orientation reversing map in $\mathcal{C}$ defined by the lemma in Section 5.5.3, while τ_B on an individual diagram from $\mathcal{B}$ acts as multiplication by $(-1)^k$ where k is the number of legs. The commutativity of this diagram is a consequence of the corollary to the above mentioned lemma (see page 140).

5.9. The three algebras in small degrees

Here is a comparative table which displays some linear bases of the algebras $\mathcal{A}^{fr}$, $\mathcal{C}$ and $\mathcal{B}$ in small dimensions.



In every grading up to 4, for each of the three algebras, this table displays a basis of the corresponding homogeneous component. Starting from degree 2, decomposable elements (products of elements of smaller degree) appear on the left, while the new indecomposable elements appear on the right. The bases of $\mathcal{C}$ and $\mathcal{B}$ are chosen to consist of primitive elements and their products. We remind that the difference between the $\mathcal{A}^{fr}$ and $\mathcal{C}$ columns is notational rather than anything else, since chord diagrams are a special case of closed Jacobi diagrams, the latter can be considered as linear combinations of the former, and the two algebras are in any case isomorphic.

5.10. Jacobi diagrams for tangles

In order to define chord diagrams and, more generally, closed Jacobi diagrams, for arbitrary tangles it suffices to make only minor adjustments to the definitions. Namely, one simply replaces the Wilson loop with an arbitrary oriented one-dimensional manifold (the *skeleton* of the Jacobi diagram). In the 4-term relations the points of attachment of chords are allowed to belong to different components of the skeleton, while the STU relations remain the same.

The Vassiliev invariants for tangles with a given skeleton can be described with the help of chord diagrams or closed diagrams with the same skeleton; in fact the Vassiliev-Kontsevich Theorem is valid for tangles and not only for knots.

Open Jacobi diagrams can also be defined for arbitrary tangles. If we consider tangles whose skeleton is not connected, the legs of corresponding open diagrams have to be labeled by the connected components of the skeleton. Moreover, for such tangles there are mixed spaces of diagrams, some of whose legs are attached to the skeleton, while others are "hanging free". Defining spaces of open and mixed diagrams for tangles is a more delicate matter than generalizing chord diagrams: here new relations, called *link relations* may appear in addition to the STU, IHX and AS relations.

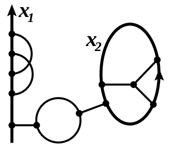
5.10.1. Jacobi diagrams for tangles.

Definition. Let $\boldsymbol{X}$ be a tangle skeleton (see page 28). A *tangle closed Jacobi diagram D with skeleton $\boldsymbol{X}$* is a unitrivalent graph with a distinguished oriented subgraph identified with $\boldsymbol{X}$, a fixed cyclic order of half-edges at each vertex not on $\boldsymbol{X}$, and such that:

- it has no univalent vertices other than the boundary points of $\boldsymbol{X}$;
- each connected component of D contains at least one connected component of $\boldsymbol{X}$.

A tangle Jacobi diagram whose all vertices belong to the skeleton, is called a *tangle chord diagram.* As with usual closed Jacobi diagrams, half the number of the vertices of a closed diagram is called the *degree*, or *order*, of the diagram.

Example. A tangle diagram whose skeleton consists of a line segment and a circle:



The vector space of tangle closed Jacobi diagrams with skeleton $\boldsymbol{X}$ modulo the STU relations is denoted by $\mathcal{C}(\boldsymbol{X})$, or by $\mathcal{C}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ where the $\boldsymbol{x}_i$ are the connected components of $\boldsymbol{X}$. The space $\mathcal{C}_n(\boldsymbol{X})$ is the subspace of $\mathcal{C}(\boldsymbol{X})$ spanned by diagrams of degree n. It is clear that for any $\boldsymbol{X}$ the space $\mathcal{C}_n(\boldsymbol{X})$ is spanned by chord diagrams with n chords.

Two tangle diagrams are considered to be equivalent if there is a graph isomorphism between them which preserves the skeleton and the cyclic order of half-edges at the trivalent vertices outside the skeleton.

Weight systems of degree n for tangles with skeleton $\boldsymbol{X}$ can now be defined as linear functions on $\mathcal{C}_n(\boldsymbol{X})$. The Fundamental Theorem 4.2.1 extends to the present case:

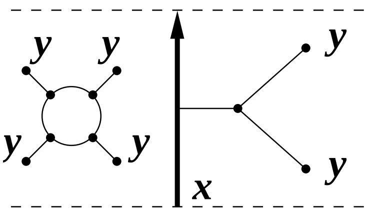
Theorem. *Each tangle weight system of degree* n *is a symbol of some degree* n *Vassiliev invariant of framed tangles.*

In fact, we shall prove this, more general version of the Fundamental Theorem in Chapter 8 and deduce the corresponding statement for knots as a corollary.

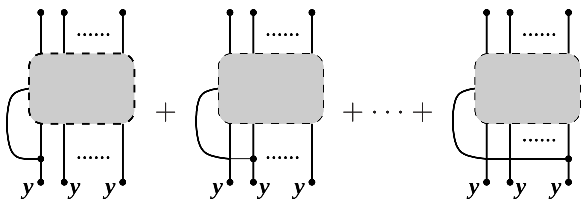
Now, assume that $\boldsymbol{X}$ is a union of connected components $\boldsymbol{x}_i$ and $\boldsymbol{y}_j$ and suppose that the $\boldsymbol{y}_j$ have no boundary.

Definition. A *mixed tangle Jacobi diagram* is a unitrivalent graph with a distinguished oriented subgraph (the *skeleton*) identified with $\cup \boldsymbol{x}_i$, with all univalent vertices, except those on the skeleton, labeled by elements of the set $\{\boldsymbol{y}_j\}$ and a fixed cyclic order of edges at each vertex not on the skeleton, and such that each connected component either contains at least one of the $\boldsymbol{x}_i$, or at least one univalent vertex. A *leg* of a mixed diagram is a univalent vertex that does not belong to the skeleton.

Here is an example of a mixed Jacobi diagram:



Mixed Jacobi diagrams, apart from the usual STU, IHX and antisymmetry relations, are subject to a new kind of relations, called *link relations.* To obtain a link relation, take a mixed diagram, choose one of its legs and one label $\boldsymbol{y}$. For each $\boldsymbol{y}$-labeled vertex, attach the chosen leg to the edge, adjacent to this vertex, sum all the results and set this sum to be equal to 0. The attachment is done according to the cyclic order as illustrated by the following picture:



$= 0.$

Here the shaded parts of all diagrams coincide, the skeleton is omitted from the pictures and the unlabeled legs are assumed to have labels distinct from $\boldsymbol{y}$.

Note that when the skeleton is empty and $\boldsymbol{y}$ is the only label (that is, we are speaking about the usual open Jacobi diagrams), the link relations are an immediate consequence from the Kirchhoff law.

Now, define the vector space $\mathcal{C}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\,|\,\boldsymbol{y}_1,\dots,\boldsymbol{y}_m)$ to be spanned by all mixed diagrams with the skeleton $\cup\boldsymbol{x}_i$ and label in the $\boldsymbol{y}_j$, modulo the STU, IHX, antisymmetry and link relations.

Both closed and open diagrams are particular cases of this construction. In particular, $\mathcal{C}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\,|\,\emptyset)=\mathcal{C}(\boldsymbol{X})$ and $\mathcal{C}(\emptyset\,|\,\boldsymbol{y})=\mathcal{B}$. The latter equality justifies the notation $\mathcal{B}(\boldsymbol{y}_1,\dots,\boldsymbol{y}_m)$ or just $\mathcal{B}(m)$ for the space of m-colored open Jacobi diagrams $\mathcal{C}(\emptyset\,|\,\boldsymbol{y}_1,\dots,\boldsymbol{y}_m)$.

Given a diagram D in $\mathcal{C}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\,|\,\boldsymbol{y}_1,\dots,\boldsymbol{y}_m)$ we can perform "symmetrization of D with respect to the label $\boldsymbol{y}_m$" by taking the average of all possible ways of attaching the $\boldsymbol{y}_m$-legs of D to a circle with the label $\boldsymbol{y}_m$. This way we get the map

$$\chi_{\boldsymbol{y}_m}:\mathcal{C}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\,|\,\boldsymbol{y}_1,\dots,\boldsymbol{y}_m)\to\mathcal{C}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n,\boldsymbol{y}_m\,|\,\boldsymbol{y}_1,\dots,\boldsymbol{y}_{m-1}).$$

Theorem. *The symmetrization map $\chi_{\boldsymbol{y}_m}$ is an isomorphism of vector spaces.*

In particular, iterating $\chi_{\boldsymbol{y}_m}$ we get the isomorphism between the spaces $\mathcal{C}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\,|\,\boldsymbol{y}_1,\dots,\boldsymbol{y}_m)$ and $\mathcal{C}(\boldsymbol{X}\cup\boldsymbol{Y})$, where $\boldsymbol{X}=\cup\boldsymbol{x}_i$ and $\boldsymbol{Y}=\cup\boldsymbol{y}_j$.

Let us indicate the idea of the proof; this will also clarify the origin of the link relations.

Consider the vector space $\mathcal{C}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\,|\,\boldsymbol{y}_1,\dots,\boldsymbol{y}_m^*)$ defined just like $\mathcal{C}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\,|\,\boldsymbol{y}_1,\dots,\boldsymbol{y}_m)$ but without the link relations on the $\boldsymbol{y}_m$-legs. Also, define the space $\mathcal{C}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n,\boldsymbol{y}_m^*\,|\,\boldsymbol{y}_1,\dots,\boldsymbol{y}_{m-1})$ in the same way as

$\mathcal{C}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n, \boldsymbol{y}_m \,|\, \boldsymbol{y}_1, \dots, \boldsymbol{y}_{m-1})$ but with an additional feature that all diagrams have a marked point on the component $\boldsymbol{y}_m$.

Then we have the symmetrization map

$$\chi_{\boldsymbol{y}_m^*} : \mathcal{C}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \,|\, \boldsymbol{y}_1, \dots, \boldsymbol{y}_m^*) \to \mathcal{C}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n, \boldsymbol{y}_m^* \,|\, \boldsymbol{y}_1, \dots, \boldsymbol{y}_{m-1})$$

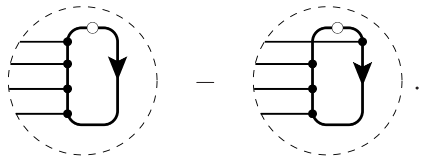
which consists of attaching, in all possible ways, the $\boldsymbol{y}_m$-legs to a pointed circle labeled $\boldsymbol{y}_m$, and taking the average of all the results.

Exercise. Prove that $\chi_{\boldsymbol{y}_m^*}$ is an isomorphism.

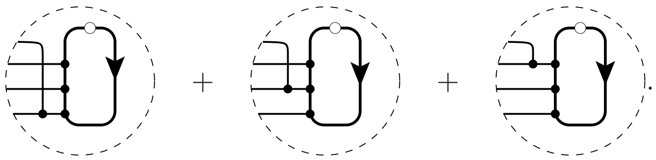
Now, consider the map

$$\mathcal{C}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n, \boldsymbol{y}_m^* \,|\, \boldsymbol{y}_1, \dots, \boldsymbol{y}_{m-1}) \to \mathcal{C}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n, \boldsymbol{y}_m \,|\, \boldsymbol{y}_1, \dots, \boldsymbol{y}_{m-1})$$

that simply forgets the marked point on the circle $\boldsymbol{y}_m$. The kernel of this map is spanned by differences of diagrams of the form



(The diagrams above illustrate the particular case of 4 legs attached to the component $\boldsymbol{y}_m$.) By the STU relations the above is equal to the following "attached link relation":



Exercise. Show that the symmetrization map $\chi_{\boldsymbol{y}_m^*}$ identifies the subspace of link relations in $\mathcal{C}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \,|\, \boldsymbol{y}_1, \dots, \boldsymbol{y}_m^*)$ with the subspace of $\mathcal{C}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n, \boldsymbol{y}_m^* \,|\, \boldsymbol{y}_1, \dots, \boldsymbol{y}_{m-1})$ spanned by all "attached link relations".

5.10.2. Pairings on diagram spaces. There are several kinds of pairings on diagram spaces. The first pairing is induced by the product on tangles; it generalizes the multiplication in the algebra $\mathcal{C}$. This pairing exists between the vector spaces $\mathcal{C}(\boldsymbol{X}_1)$ and $\mathcal{C}(\boldsymbol{X}_2)$ such that the bottom part of $\boldsymbol{X}_1$ coincides with the top part of $\boldsymbol{X}_2$ and these manifold can be concatenated into an oriented 1-manifold $\boldsymbol{X}_1 \circ \boldsymbol{X}_2$. In this case we have the bilinear map

$$\mathcal{C}(\boldsymbol{X}_1) \otimes \mathcal{C}(\boldsymbol{X}_2) \to \mathcal{C}(\boldsymbol{X}_1 \circ \boldsymbol{X}_2),$$

obtained by putting one diagram on top of another.

If $\boldsymbol{X}$ is a collection of n intervals, with one top and one bottom point on each of them, $\boldsymbol{X} \circ \boldsymbol{X}$ is the same thing as $\boldsymbol{X}$ and in this case we have an algebra structure on $\mathcal{C}(\boldsymbol{X})$. This is the algebra of closed Jacobi diagrams for string links on n strands. When $n = 1$, we, of course, come back to the algebra $\mathcal{C}$.

The second multiplication is the *tensor product* of tangle diagrams. It is induced the tensor product of tangles, and consists of placing the diagrams side by side.

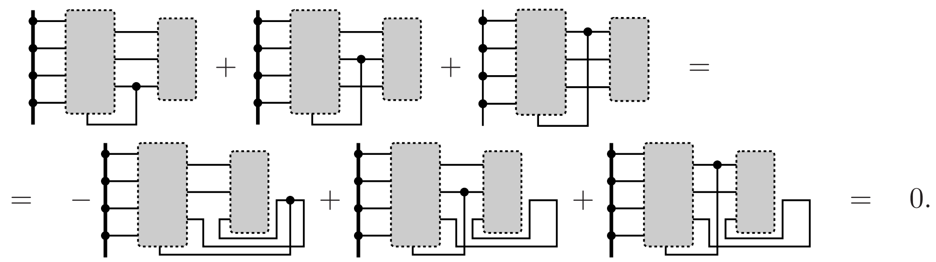
There is yet another pairing on diagram spaces, which is sometimes called "inner product". For diagrams $C \in \mathcal{C}(\boldsymbol{x} \,|\, \boldsymbol{y})$ and $D \in \mathcal{B}(\boldsymbol{y})$ define the diagram $\langle C, D\rangle_{\boldsymbol{y}} \in \mathcal{C}(\boldsymbol{x})$ as the sum of all ways of glueing all the $\boldsymbol{y}$-legs of C to the $\boldsymbol{y}$-legs of D. If the numbers of $\boldsymbol{y}$-legs of C and D are not equal, we set $\langle C, D\rangle_{\boldsymbol{y}}$ to be zero. It may happen that in the process of glueing we get closed circles with no vertices on them (this happens if C and D contain intervals with both ends labeled by $\boldsymbol{y}$). We set such diagrams containing circles to be equal to zero.

5.10.3. Lemma. *The inner product*

$$\langle\ ,\ \rangle_{\boldsymbol{y}} : \mathcal{C}(\boldsymbol{x} \,|\, \boldsymbol{y}) \otimes \mathcal{B}(\boldsymbol{y}) \to \mathcal{C}(\boldsymbol{x})$$

is well-defined.

Proof. We need to show that the class of the resulting diagram in $\mathcal{C}(\boldsymbol{x})$ does not change if we modify the second argument of $\langle\ ,\ \rangle_{\boldsymbol{y}}$ by IHX or antisymmetry relations, and the first argument — by STU or link relations. This is clear for the first three kinds of relations. For link relations it follows from the Kirchhoff rule and the antisymmetry relation. For example, we have



$\square$

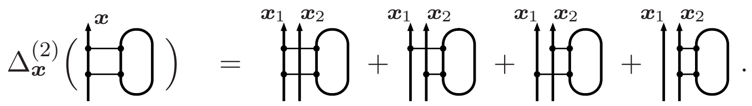
The definition of the inner product can be extended. For example, if two diagrams C, D have the same number of $\boldsymbol{y}_1$-legs and the same number of $\boldsymbol{y}_2$-legs, they can be glued together along the $\boldsymbol{y}_1$-legs and then along the $\boldsymbol{y}_2$-legs. The sum of the results of all such glueings is denoted by $\langle C, D\rangle_{\boldsymbol{y}_1,\boldsymbol{y}_2}$. This construction, clearly, can be generalized further.

5.10.4. Actions of $\mathcal{C}$ and $\mathcal{B}$ on tangle diagrams. While $\mathcal{C}(\boldsymbol{X})$ is not necessarily an algebra, it is more than just a graded vector space. Namely, for each component $\boldsymbol{x}$ of $\boldsymbol{X}$, there is an action of $\mathcal{C}(\boldsymbol{x})$ on $\mathcal{C}(\boldsymbol{X})$, defined as the connected sum along the component $\boldsymbol{x}$. We denote this action by $\#$, as if it were the usual connected sum. More generally, the spaces of mixed tangle diagrams $\mathcal{C}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\,|\,\boldsymbol{y}_1,\ldots,\boldsymbol{y}_m)$ are two-sided modules over $\mathcal{C}(\boldsymbol{x}_i)$ and $\mathcal{B}(\boldsymbol{y}_j)$. The algebra $\mathcal{C}(\boldsymbol{x}_i)$ acts, as before, by the connected sum on the component $\boldsymbol{x}_i$, while the action of $\mathcal{B}(\boldsymbol{y}_j)$ consists of taking the disjoint union with diagrams in $\mathcal{B}(\boldsymbol{y}_j)$. We shall denote the action of $\mathcal{B}(\boldsymbol{y}_j)$ by $\cup$.

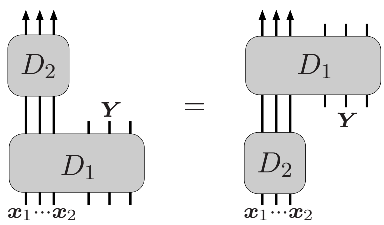
We cannot expect the relation of the module structures on the space of mixed diagrams with the symmetrization map to be straightforward, since the symmetrization map from $\mathcal{B}$ to $\mathcal{C}$ fails to be multiplicative. We shall clarify this remark in 11.3.8.

Exercise. Prove that the above actions are well-defined. In particular, prove that the action of $\mathcal{C}(\boldsymbol{x}_i)$ does not depend on the location where the diagram is inserted into the corresponding component of the tangle diagram, and show that the action of $\mathcal{B}(\boldsymbol{y}_j)$ respects the link relations.

5.10.5. Sliding property. There is one important corollary of the IHX relation (Kirchhoff law), called *sliding property* ([**BLT**]), which holds in the general context of tangle Jacobi diagrams. To formulate it, we need to define the operation $\Delta^{(n)}_{\boldsymbol{x}}:\mathcal{C}(\boldsymbol{x}\cup\boldsymbol{Y})\to\mathcal{C}(\boldsymbol{x}_1\cup\cdots\cup\boldsymbol{x}_n\cup\boldsymbol{Y})$. By definition, $\Delta^{(n)}_{\boldsymbol{x}}(D)$ is the lift of D to the n-th disconnected cover of the line $\boldsymbol{x}$, that is, for each $\boldsymbol{x}$-leg of the diagram D we take the sum over all ways to attach it to $\boldsymbol{x}_i$ for any $i=1,\ldots,n$ (the sum consists of n^k terms, if k is the number of vertices of D belonging to $\boldsymbol{x}$). Example:



Proposition. *(Sliding relation) Suppose that $D\in\mathcal{C}(\boldsymbol{x}\cup\boldsymbol{Y})$; let $D_1=\Delta^{(n)}_{\boldsymbol{x}}(D)$. Then for any diagram $D_2\in\mathcal{C}(\boldsymbol{x}_1\cup\cdots\cup\boldsymbol{x}_n)$ we have $D_1D_2=D_2D_1$. In pictures:*



Proof. Indeed, take the leg in D_1 which is closest to D_2 and consider the sum of all diagrams on $\boldsymbol{x}_1\cup\cdots\cup\boldsymbol{x}_n\cup\boldsymbol{Y}$ where this leg is attached to $\boldsymbol{x}_i$, $i=1,\ldots,n$, while all the other legs are fixed. By Kirchhoff law, this sum is equal to the similar sum where the chosen leg has jumped over D_2. In this way, all the legs jump over D_2 one by one, and the commutativity follows. $\square$

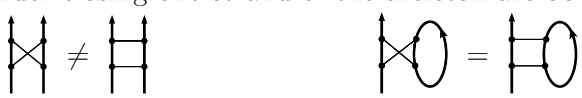
5.10.6. Closing a component of a Jacobi diagram. Recall that long knots can be closed up to produce usual knots. This closure induces a bijection of the corresponding isotopy classes and an isomorphism of the corresponding diagram spaces.

This fact can be generalized to tangles whose skeleton consists of one interval and several circles.

Theorem. *Let $\boldsymbol{X}$ be a tangle skeleton with only one interval component, and $\boldsymbol{X}'$ be a skeleton obtained by closing this component into a circle. The induced map*
$$\mathcal{C}(\boldsymbol{X}) \to \mathcal{C}(\boldsymbol{X}')$$
is an isomorphism of vector spaces.

The proof of this theorem consists of an application of the Kirchhoff's law and we leave it to the reader.

We should point out that closing one component of a skeleton with more that one interval component does not produce an isomorphism of the corresponding diagram spaces. Indeed, let us denote by $\mathcal{A}(2)$ the space of closed diagrams for string links on 2 strands. A direct calculation shows that the two diagrams of order 2 below on the left are different in $\mathcal{A}(2)$, while their images under closing one strand of the skeleton are obviously equal:



The above statements about tangle diagrams, of course, are not arbitrary, but reflect the following topological fact that we state as an exercise:

Exercise. Define the map of closing one component on isotopy classes of tangles with a given skeleton and show that it is bijective if and only if it is applied to tangles whose skeleton has only one interval component.

5.11. Horizontal chord diagrams

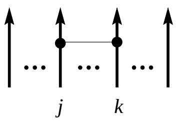
There is yet another diagram algebra which will be of great importance in what follows, namely, the algebra $\mathcal{A}^h(n)$ of *horizontal chord diagrams* on n strands.

A horizontal chord diagram on n strands is a tangle diagram whose skeleton consists of n vertical intervals (all oriented, say, upwards) and all of whose chords are horizontal. Two such diagrams are considered to be equivalent if one can be deformed into the other through horizontal diagrams.

A product of two horizontal diagrams is clearly a horizontal diagram; by definition, the algebra $\mathcal{A}^h(n)$ is generated by the equivalence classes of all such diagrams modulo the horizonal 4T relations 4.1.1.4 (see Section 4.1).

We denote by $\mathbf{1}_n$ the the empty diagram in $\mathcal{A}^h(n)$ which is the multiplicative unit.

Each horizontal chord diagram is equivalent to a diagram whose chords are all situated on different levels, that is, to a product of diagrams of degree 1. Set

$u_{jk} =$ 

$, \qquad 1 \leqslant j < k \leqslant n,$

and for $1 \leqslant k < j \leqslant n$ set $u_{jk} = u_{kj}$. Then $\mathcal{A}^h(n)$ is generated by the u_{jk} subject to the following relations (*infinitesimal pure braid relations*, first appeared in [**Koh2**])

$$[u_{jk}, u_{jl} + u_{kl}] = 0, \quad \text{if } j,k,l \text{ are different,}$$
$$[u_{jk}, u_{lm}] = 0, \quad \text{if } j,k,l,m \text{ are different.}$$

Indeed, the first relation is just the horizontal 4T relation. The second relation is similar to the far commutativity relation in braids. The products of the u_{jk} up to this relation are precisely the equivalence classes of horizontal diagrams.

The algebra $\mathcal{A}^h(2)$ is simply the free commutative algebra on one generator u_{12}.

5.11.1. Proposition. *$\mathcal{A}^h(3)$ is a direct product of the free algebra on two generators u_{12} and u_{23}, and the free commutative algebra on one generator*

$$u = u_{12} + u_{23} + u_{13}.$$

In particular, $\mathcal{A}^h(3)$ is highly non-commutative.

Proof. Choose u_{12}, u_{23} and u as the set of generators for $\mathcal{A}^h(3)$. In terms of these generators all the relations in $\mathcal{A}^h(3)$ can be written as

$$[u_{12}, u] = 0, \quad \text{and} \quad [u_{23}, u] = 0\ .$$

□

For $n > 3$ the multiplicative structure of the algebra $\mathcal{A}^h(n)$ is rather more involved, even though it admits a simple description as a vector space. We shall treat this subject in more detail in Chapter 12, as the algebra $\mathcal{A}^h(n)$ plays the same role in the theory of finite type invariants for pure braids as the algebra $\mathcal{A}$ in the theory of the Vassiliev knot invariants.

We end this section with one property of $\mathcal{A}^h(n)$ which will be useful in Chapter 10.

5.11.2. Lemma. *Let $J, K \subseteq \{1, \ldots, n\}$ be two non-empty subsets with $J \cap K = \emptyset$. Then the element $\sum_{j\in J, k\in K} u_{jk}$ commutes in $\mathcal{A}^h(n)$ with any generator u_{pq} with p and q either both in J or both in K.*

Proof. It is clearly sufficient to prove the lemma for the case when K consists of one element, say k, and both p and q are in J. Now, any u_{jk} commutes with u_{pq} if j is different from both p and q. But $u_{pk} + u_{qk}$ commutes with u_{pq} by the horizontal 4T relation, and this proves the lemma. □

5.11.3. Horizontal diagrams and string link diagrams. Denote by $\mathcal{A}(n)$ be the algebra of closed diagrams for string links. Horizontal diagrams are examples of string link diagrams and the horizontal 4T relations are a particular case of the usual 4T relations, and, hence, there is an algebra homomorphism

$$\mathcal{A}^h(n) \to \mathcal{A}(n).$$

This homomorphism is injective, but this is a surprisingly non-trivial fact; see [**BN8, HM**]. We shall give a proof of this in Chapter 12, see page 358.

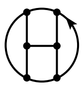
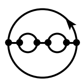
Exercise.

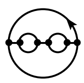
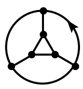
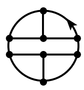
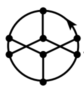
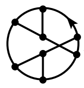
(a) Prove that the chord diagram consisting of one chord connecting the two components of the skeleton belongs to the center of the algebra $\mathcal{A}(2)$.

(b) Prove that any chord diagram consisting of two intersecting chords belongs to the center of the algebra $\mathcal{A}(2)$.

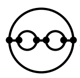
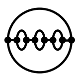
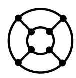
(c) Prove that Lemma 5.11.2 is also valid for $\mathcal{A}(n)$. Namely, show that the element $\sum_{j\in J,k\in K} u_{jk}$ commutes in $\mathcal{A}(n)$ with any chord diagram whose chords have either both ends on the strands in J or on the strands in K.

Exercises

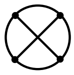
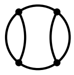
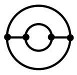
(1) Prove that  = .

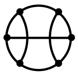
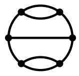
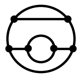
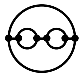
(2) Let $a_1 =$ , $a_2 =$ , $a_3 =$ , $a_4 =$ , $a_5 =$ .

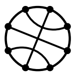
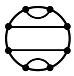
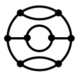
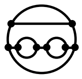
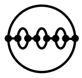
(a) Find a relation between a_1 and a_2.
(b) Represent the sum $a_3 + a_4 - 2a_5$ as a connected closed diagram.
(c) Prove the linear independence of a_3 and a_4 in $\mathcal{C}$.

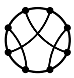
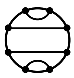
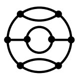
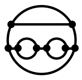
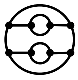
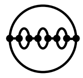
(3) Express the primitive elements ,  and  of degrees 3 and 4 as linear combinations of chord diagrams.

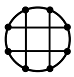
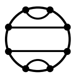
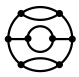
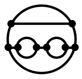
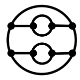
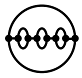
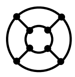
(4) Prove the following identities in the algebra $\mathcal{C}$:

 $=$  $-\frac{1}{2}$  ;

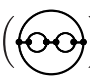
 $=$  $-$  $+\frac{1}{4}$  ;

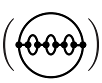
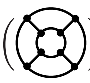
 $=$  $-\frac{3}{2}$  $+\frac{3}{4}$  $-\frac{1}{8}$  ;

 $=$  $-\frac{3}{2}$  $+\frac{1}{2}$  $+\frac{1}{4}$  $-\frac{1}{8}$  ;

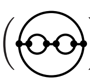
 $=$  $-\,2$  $+$  $+\frac{1}{2}$  $-\frac{1}{2}$  $+$  .

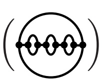
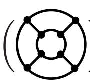
(5) Show that the symbols of the coefficients of the Conway polynomial (Section 2.3) take the following values on the basis primitive diagrams of degree 3 and 4.

$\mathrm{symb}(c_3)\Big($  $\Big) = 0,$

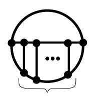
$\mathrm{symb}(c_4)\Big($  $\Big) = 0,$ $\qquad$ $\mathrm{symb}(c_4)\Big($  $\Big) = -2.$

(6) Show that the symbols of the coefficients of the Jones polynomial (Section 3.6.2) take the following values on the basis primitive diagrams of degrees 3 and 4.

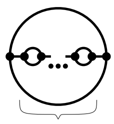
$\mathrm{symb}(j_3)\Big($  $\Big) = -24,$

$\mathrm{symb}(j_4)\Big($  $\Big) = 96,$ $\qquad$ $\mathrm{symb}(j_4)\Big($  $\Big) = 18.$

(7) ([**ChV**]) Let $\overline{t_n} \in \mathcal{P}_{n+1}$ be the closed diagram shown on the right. Prove the following identity

$\overline{t_n} =$ 

n legs

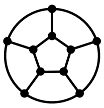
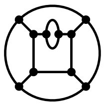
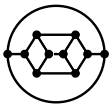
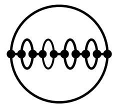
$\overline{t_n} \;=\; \frac{1}{2^n}$ 

n bubbles

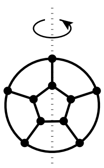
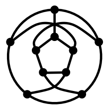
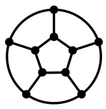
Deduce that $\overline{t_n} \in \mathcal{P}^2_{n+1}$.

(8) Express $\overline{t_n}$ as a linear combination of chord diagrams. In particular, show that the intersection graph of every chord diagram that occurs in this expression is a forest.

(9) ([**ChV**]) Prove the following identity in the space $\mathcal{C}$ of closed diagrams:

 $= \dfrac{3}{4}$  $- \dfrac{1}{12}$  $- \dfrac{1}{48}$  .

Hint. Turn the internal pentagon of the left-hand side of the identity in the 3-space by 180° about the vertical axis. The result will represent the same graph with the cyclic orders at all five vertices of the pentagon changed to the opposite:

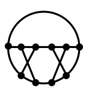
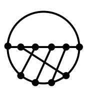
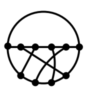
 $= (-1)^5$  $= -$  $+$ (terms with at most 4 legs) .

The last equality follows from the STU relations which allow us to rearrange the legs modulo diagrams with a smaller number of legs. To finish the solution, the reader must figure out the terms in the parentheses.

(10) Prove the linear independence of the three elements in the right-hand side of the last equality, using Lie algebra invariants defined in Chapter 6.

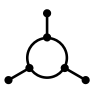
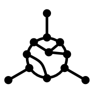
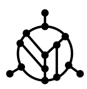
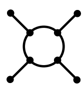
(11) ([**ChV**]) Prove that the primitive space in the algebra $\mathcal{C}$ is generated by the closed diagrams whose internal graph is a tree.

(12) ([**ChV**]) With each permutation σ of n objects associate a closed diagram P_σ acting as in Section 5.7.2 by the permutation on the lower legs of a closed diagram $P_{(12\ldots n)} = \overline{t_n}$ from problem 7. Here are some examples:

$P_{(2143)} =$  ; $P_{(4123)} =$  ; $P_{(4132)} =$  .

Prove that the diagrams P_σ span the vector space $\mathcal{P}_{n+1}$.

(13) ([**ChV**]) Prove that
- $\mathcal{P}_n^n = \mathcal{P}_n$ for even n, and $\mathcal{P}_n^{n-1} = \mathcal{P}_n$ for odd n;
- for even n the quotient space $\mathcal{P}_n^n/\mathcal{P}_n^{n-1}$ has dimension one and generated by the wheel $\overline{w}_n$.

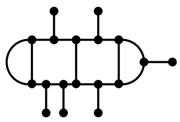
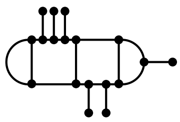
(14) Let $b_1 =$  , $b_2 =$  , $b_3 =$  , $b_4 =$  .

Which of these diagrams are zero in $\mathcal{B}$, that is, vanish modulo AS and IHX relations?

(15) Prove that the algebra generated by all open diagrams modulo the AS and the modified IHX equation $I = aH - bX$, where a and b are arbitrary complex numbers, is isomorphic (equal) to $\mathcal{B}$ if and only if $a = b = 1$, in all other cases it is a free polynomial algebra on one generator.

(16)
- Indicate an explicit form of the isomorphisms $\mathcal{A}^{fr} \cong \mathcal{C} \cong \mathcal{B}$ in the bases given in Section 5.9.
- Compile the multiplication table for $\mathcal{B}_m \times \mathcal{B}_n \to \mathcal{B}_{m+n}$, $m + n \leqslant 4$, for the second product in $\mathcal{B}$ (the one pulled back from $\mathcal{C}$ along the isomorphism $\mathcal{C} \cong \mathcal{B}$).
- Find some bases of the spaces $\mathcal{A}_n^{fr}$, $\mathcal{C}_n$, $\mathcal{B}_n$ for $n = 5$.

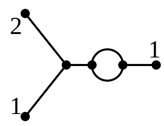
(17) (*J. Kneissler*). Let $\mathcal{B}_n^u$ be the space of open diagrams of degree n with u univalent vertices. Denote by $\omega_{i_1 i_2 \ldots i_k}$ the element of $\mathcal{B}^{i_1+\cdots+i_k}_{i_1+\cdots+i_k+k-1}$ represented by a *caterpillar* diagram consisting of k body segments with $i_1, \ldots, i_k$ "legs", respectively. Using the AS and IHX relations, prove that $\omega_{i_1 i_2 \ldots i_k}$ is well-defined, that is, for inner segments it makes no difference on which side of the body they are drawn. For example,

ω_{0321} =  = 

(18) * (*J. Kneissler*) Is it true that any caterpillar diagram in the algebra $\mathcal{B}$ can be expressed through caterpillar diagrams with even indices $i_1, \ldots, i_k$? Is it true that the primitive space $\mathcal{P}(\mathcal{B})$ (that is, the space spanned by connected open diagrams) is generated by caterpillar diagrams?

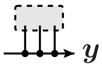
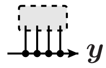
(19) Call a chord diagram $d \in \mathcal{A}^{fr}$ even (odd) if $\bar{d} = d$ (resp. $\bar{d} = -d$), where the bar denotes the reflection of a chord diagram. Prove that under the isomorphism $\chi^{-1} : \mathcal{A}^{fr} \to \mathcal{B}$:
- the image of an even chord diagram is a linear combination of open diagrams with an even number of legs,
- the image of an odd chord diagram in is a linear combination of open diagrams with an odd number of legs.

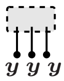
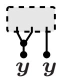
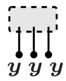
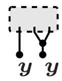
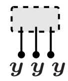
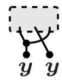
(20) * (The simplest unsolved case of Conjecture 5.6). Is it true that an open diagram with 3 univalent vertices is always equal to 0 as an element of the algebra $\mathcal{B}$?

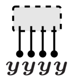
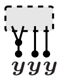
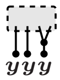
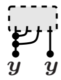
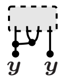
(21) Prove that the diagram 

is equal to 0 in the space $\mathcal{B}(2)$.

(22) Let u_{ij} be the diagram in $\mathcal{A}(3)$ with one chord connecting the ith and the jth component of the skeleton. Prove that for any k the combination $u_{12}^k + u_{23}^k + u_{13}^k$ belongs to the center of $\mathcal{A}(3)$.

(23) Let $D^3 =$  $\in \mathcal{C}(\boldsymbol{X}, \boldsymbol{y})$ and $D^4 =$  $\in \mathcal{C}(\boldsymbol{X}, \boldsymbol{y})$ be tangle diagrams with exactly three and four $\boldsymbol{y}$-legs respectively. Show that

$\mathcal{C}(\boldsymbol{X}|\boldsymbol{y}) \ni \chi_{\boldsymbol{y}}^{-1}(D^3) =$  $+ \frac{1}{2}$  $=$  $+ \frac{1}{2}$  $=$  $- \frac{1}{2}$  ;

$\mathcal{C}(\boldsymbol{X}|\boldsymbol{y}) \ni \chi_{\boldsymbol{y}}^{-1}(D^4) =$  $+ \frac{1}{2}$  $+ \frac{1}{2}$  $+ \frac{1}{8}$  $+ \frac{5}{24}$  .

Hint. Follow the proof of Theorem 5.7.1 on page 147 and then use link relations.

Chapter 6

Lie algebra weight systems

Given a Lie algebra $\mathfrak{g}$ equipped with a non-degenerate invariant bilinear form, one can construct a weight system with values in the center of the universal enveloping algebra $U(\mathfrak{g})$. In a similar fashion one can define a map from the space $\mathcal{B}$ into the ad-invariant part of the symmetric algebra $S(\mathfrak{g})$. These constructions are due to M. Kontsevich [**Kon1**], with basic ideas already appearing in [**Pen**]. If, in addition, we have a finite dimensional representation of the Lie algebra then taking the trace of the corresponding operator we get a numeric weight system. It turns out that these weight systems are the symbols of the quantum group invariants (Section 3.6.6). The construction of weight systems based on representations first appeared in D. Bar-Natan's paper [**BN0**]. The reader is invited to consult the Appendix for basics on Lie algebras and their universal envelopes.

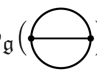
6.1. Lie algebra weight systems for the algebra $\mathcal{A}^{fr}$

6.1.1. Universal Lie algebra weight systems. Kontsevich's construction proceeds as follows. Let $\mathfrak{g}$ be a metrized Lie algebra over $\mathbb{R}$ or $\mathbb{C}$, that is, a Lie algebra with an ad-invariant non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ (see A.1.1). Choose a basis $e_1,\dots,e_m$ of $\mathfrak{g}$ and let $e_1^*,\dots,e_m^*$ be the dual basis with respect to the form $\langle\cdot,\cdot\rangle$.

Given a chord diagram D with n chords, we first choose a base point on its Wilson loop, away from the chords of D. This gives a linear order on the endpoints of the chords, increasing in the positive direction of the Wilson loop. Assign to each chord α an *index*, that is, an integer-valued variable, i_α. The values of i_α will range from 1 to m, the dimension of the Lie algebra. Mark the first endpoint of the chord with the symbol e_{i_α} and the second endpoint with $e^*_{i_\alpha}$.

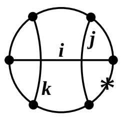
Now, write the product of all the e_{i_α} and all the $e^*_{i_\alpha}$, in the order in which they appear on the Wilson loop of D, and take the sum of the m^n elements of the universal enveloping algebra $U(\mathfrak{g})$ obtained by substituting all possible values of the indexes i_α into this product. Denote by $\varphi_{\mathfrak{g}}(D)$ the resulting element of $U(\mathfrak{g})$.

For example,

$\varphi_{\mathfrak{g}}($  $)$ $= \displaystyle\sum_{i=1}^{m} e_i e_i^* =: c$

is the *quadratic Casimir element* associated to the chosen invariant form.

Another example: if

$D =$  ,

then

$$\varphi_{\mathfrak{g}}(D) = \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} e_i e_j e_k e_i^* e_k^* e_j^*.$$

6.1.2. Theorem. *The above construction has the following properties:*

(1) *the element $\varphi_{\mathfrak{g}}(D)$ does not depend on the choice of the base point on the diagram,*
(2) *it does not depend on the choice of the basis $\{e_i\}$ of the Lie algebra,*
(3) *it belongs to the ad-invariant subspace $U(\mathfrak{g})^{\mathfrak{g}}$ of the universal enveloping algebra $U(\mathfrak{g})$ (which coincides with the center $ZU(\mathfrak{g})$).*
(4) *the function $D \mapsto \varphi_{\mathfrak{g}}(D)$ satisfies 4-term relations,*
(5) *the resulting mapping $\varphi_{\mathfrak{g}} : \mathcal{A}^{fr} \to ZU(\mathfrak{g})$ is a homomorphism of algebras.*

Proof. (1) Introducing a base point means that a circular chord diagram is replaced by a linear chord diagram (see Section 4.7). Modulo 4-term relations, this map is an isomorphism, and, hence, the assertion follows from (4).

(2) An exercise in linear algebra: take two different bases $\{e_i\}$ and $\{f_j\}$ of $\mathfrak{g}$ and reduce the expression for $\varphi_{\mathfrak{g}}(D)$ in one basis to the expression in another using the transition matrix between the two bases. Technically, it is enough to do this exercise only for $m = \dim\mathfrak{g} = 2$, since the group of transition matrices $GL(m)$ is generated by linear transformations in 2-dimensional planes.

(3) It is enough to prove that $\varphi_{\mathfrak{g}}(D)$ commutes with any basis element e_r. By property (2), we can choose the basis to be orthonormal with respect to the ad-invariant form $\langle\cdot,\cdot\rangle$, so that $e_i^* = e_i$ for all i. Now, the commutator of e_r and $\varphi_{\mathfrak{g}}(D)$ can be expanded into a sum of $2n$ expressions, similar to $\varphi_{\mathfrak{g}}(D)$, only with one of the e_i replaced by its commutator with e_r. Due to the antisymmetry of the structure constants c_{ijk} (Lemma A.1.6), these expressions cancel in pairs that correspond to the ends of each chord.

To take a concrete example,

$$\begin{aligned}
&[e_r, \sum_{ij} e_ie_je_ie_j] \\
&= \sum_{ij}[e_r,e_i]e_je_ie_j + \sum_{ij} e_i[e_r,e_j]e_ie_j + \sum_{ij} e_ie_j[e_r,e_i]e_j + \sum_{ij} e_ie_je_i[e_r,e_j] \\
&= \sum_{ijk} c_{rik}e_ke_je_ie_j + \sum_{ijk} c_{rjk}e_ie_ke_ie_j + \sum_{ijk} c_{rik}e_ie_je_ke_j + \sum_{ijk} c_{rjk}e_ie_je_ie_k \\
&= \sum_{ijk} c_{rik}e_ke_je_ie_j + \sum_{ijk} c_{rjk}e_ie_ke_ie_j + \sum_{ijk} c_{rki}e_ke_je_ie_j + \sum_{ijk} c_{rkj}e_ie_ke_ie_j.
\end{aligned}$$

Here the first and the second sums cancel with the third and the fourth sums, respectively.

(4) We still assume that the basis $\{e_i\}$ is $\langle\cdot,\cdot\rangle$–orthonormal. Then one of the pairwise differences of the chord diagrams that constitute the 4 term relation in equation (4.1.1.3) (page 98) is sent by $\varphi_{\mathfrak{g}}$ to

$$\sum c_{ijk} \ldots e_i \ldots e_j \ldots e_k \ldots ,$$

while the other goes to

$$\sum c_{ijk} \ldots e_j \ldots e_k \ldots e_i \ldots$$

Due to the cyclic symmetry of the structure constants c_{ijk} in an orthonormal basis (see Lemma A.1.6 on page 419), these two expressions are equal.

(5) Using property (1), we can place the base point in the product diagram $D_1 \cdot D_2$ between D_1 and D_2. Then the identity $\varphi_{\mathfrak{g}}(D_1 \cdot D_2) = \varphi_{\mathfrak{g}}(D_1)\varphi_{\mathfrak{g}}(D_2)$ becomes evident. □

Remark. If D is a chord diagram with n chords, then

$$\varphi_{\mathfrak{g}}(D) = c^n + \{\text{terms of degree less than } 2n \text{ in } U(\mathfrak{g})\},$$

where c is the quadratic Casimir element as on page 168. Indeed, we can permute the endpoints of chords on the circle without changing the highest term of $\varphi_{\mathfrak{g}}(D)$ since all the additional summands arising as commutators have degrees smaller than $2n$. Therefore, the highest degree term of $\varphi_{\mathfrak{g}}(D)$ does not depend on D. Finally, if D is a diagram with n isolated chords, that is, the n-th power of the diagram with one chord, then $\varphi_{\mathfrak{g}}(D) = c^n$.

By definition, the center $ZU(\mathfrak{g})$ of the universal enveloping algebra is precisely the $\mathfrak{g}$-invariant subspace $U(\mathfrak{g})^{\mathfrak{g}} \subset U(\mathfrak{g})$. According to the Harish-Chandra theorem (see [**Hum**]) for a semi-simple Lie algebra $\mathfrak{g}$, the center $ZU(\mathfrak{g})$ is isomorphic to the algebra of polynomials in certain variables $c_1 = c, c_2, \dots, c_r$, where $r = rank(\mathfrak{g})$.

6.1.3. $\mathfrak{sl}_2$-weight system. Consider the Lie algebra $\mathfrak{sl}_2$ of 2×2 matrices with zero trace. It is a three-dimensional Lie algebra spanned by the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with the commutators

$$[H,E] = 2E, \qquad [H,F] = -2F, \qquad [E,F] = H\ .$$

We will use the symmetric bilinear form $\langle x, y\rangle = \mathrm{Tr}(xy)$:

$$\langle H,H\rangle = 2,\ \langle H,E\rangle = 0,\ \langle H,F\rangle = 0,\ \langle E,E\rangle = 0,\ \langle E,F\rangle = 1,\ \langle F,F\rangle = 0.$$

One can easily check that it is ad-invariant and non-degenerate. The corresponding dual basis is

$$H^* = \frac{1}{2}H, \qquad E^* = F, \qquad F^* = E,$$

and, hence, the Casimir element is $c = \frac{1}{2}HH + EF + FE$.

The center $ZU(\mathfrak{sl}_2)$ is isomorphic to the algebra of polynomials in a single variable c. The value $\varphi_{\mathfrak{sl}_2}(D)$ is thus a polynomial in c. In this section, following [**ChV**], we explain a combinatorial procedure to compute this polynomial for a given chord diagram D.

The algebra $\mathfrak{sl}_2$ is simple, hence, any invariant form is equal to $\lambda\langle\cdot,\cdot\rangle$ for some constant λ. The corresponding Casimir element c_λ, as an element of the universal enveloping algebra, is related to $c = c_1$ by the formula $c_\lambda = \frac{c}{\lambda}$. Therefore, the weight system

$$\varphi_{\mathfrak{sl}_2}(D) = c^n + a_{n-1}c^{n-1} + a_{n-2}c^{n-2} + \dots + a_2c^2 + a_1c$$

and the weight system corresponding to $\lambda\langle\cdot,\cdot\rangle$

$$\varphi_{\mathfrak{sl}_2,\lambda}(D) = c_\lambda^n + a_{n-1,\lambda}c_\lambda^{n-1} + a_{n-2,\lambda}c_\lambda^{n-2} + \dots + a_{2,\lambda}c_\lambda^2 + a_{1,\lambda}c_\lambda$$

are related by the formula $\varphi_{\mathfrak{sl}_2,\lambda}(D) = \frac{1}{\lambda^n}\cdot\varphi_{\mathfrak{sl}_2}(D)\big|_{c=\lambda\cdot c_\lambda}$, or

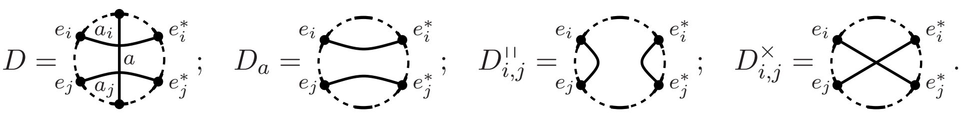
$$a_{n-1} = \lambda a_{n-1,\lambda}, \quad a_{n-2} = \lambda^2 a_{n-2,\lambda}, \ \dots\ a_2 = \lambda^{n-2}a_{2,\lambda}, \quad a_1 = \lambda^{n-1}a_{1,\lambda}.$$

Theorem. *Let $\varphi_{\mathfrak{sl}_2}(\cdot)$ be the weight system associated to $\mathfrak{sl}_2$ and the invariant form $\langle\cdot,\cdot\rangle$. Take a chord diagram D and choose a chord a of D. Then*

$$\varphi_{\mathfrak{sl}_2}(D) = (c-2k)\varphi_{\mathfrak{sl}_2}(D_a) + 2\sum_{1\leqslant i<j\leqslant k}\Big(\varphi_{\mathfrak{sl}_2}(D_{i,j}^{||}) - \varphi_{\mathfrak{sl}_2}(D_{i,j}^{\times})\Big),$$

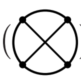
where:
- *k is the number of chords that intersect the chord a;*
- *D_a is the chord diagram obtained from D by deleting the chord a;*
- *$D^{||}_{i,j}$ and $D^{\times}_{i,j}$ are the chord diagrams obtained from D_a in the following way:*

Consider an arbitrary pair of chords a_i and a_j different from the chord a and such that each of them intersects a. The chord a divides the circle into two arcs. Denote by e_i and e_j the endpoints of a_i and a_j that belong to the left arc and by e^*_i, e^*_j the endpoints of a_i and a_j that belong to the right arc. There are three ways to connect four points e_i, e^*_i, e_j, e^*_j by two chords. In D_a, we have the case $(e_i, e^*_i), (e_j, e^*_j)$. Let $D^{||}_{i,j}$ be the diagram with the connection $(e_i, e_j), (e^*_i, e^*_j)$. Let $D^{\times}_{i,j}$ be the diagram with the connection $(e_i, e^*_j), (e^*_i, e_j)$. All other chords are the same in all the diagrams:

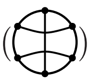
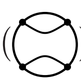
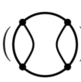
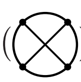


The theorem allows one to compute $\varphi_{\mathfrak{sl}_2}(D)$ recursively, because each of the three diagrams $D_a, D^{||}_{i,j}$ and $D^{\times}_{i,j}$ has one chord less than D.

Examples.

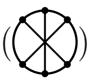
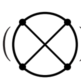
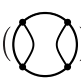
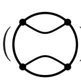
(1) $\varphi_{\mathfrak{sl}_2}($  $) = (c-2)c$. In this case, $k = 1$ and the sum in the theorem is zero, since there are no pairs (i, j).

(2)

$\varphi_{\mathfrak{sl}_2}($  $) = (c-4)\varphi_{\mathfrak{sl}_2}($  $) + 2\varphi_{\mathfrak{sl}_2}($  $) - 2\varphi_{\mathfrak{sl}_2}($  $)$

$$= (c-4)c^2 + 2c^2 - 2(c-2)c \quad = \quad (c-2)^2 c.$$

(3)

$\varphi_{\mathfrak{sl}_2}($  $) = (c-4)\varphi_{\mathfrak{sl}_2}($  $) + 2\varphi_{\mathfrak{sl}_2}($  $) - 2\varphi_{\mathfrak{sl}_2}($  $)$

$$= (c-4)(c-2)c + 2c^2 - 2c^2 \quad = \quad (c-4)(c-2)c.$$

Remark. Choosing the invariant form $\lambda\langle\cdot,\cdot\rangle$, we obtain a modified relation

$$\varphi_{\mathfrak{sl}_2,\lambda}(D) = \Big(c_\lambda - \frac{2k}{\lambda}\Big)\varphi_{\mathfrak{sl}_2,\lambda}(D_a) + \frac{2}{\lambda}\sum_{1\leqslant i<j\leqslant k}\Big(\varphi_{\mathfrak{sl}_2,\lambda}(D^{||}_{i,j}) - \varphi_{\mathfrak{sl}_2,\lambda}(D^{\times}_{i,j})\Big).$$

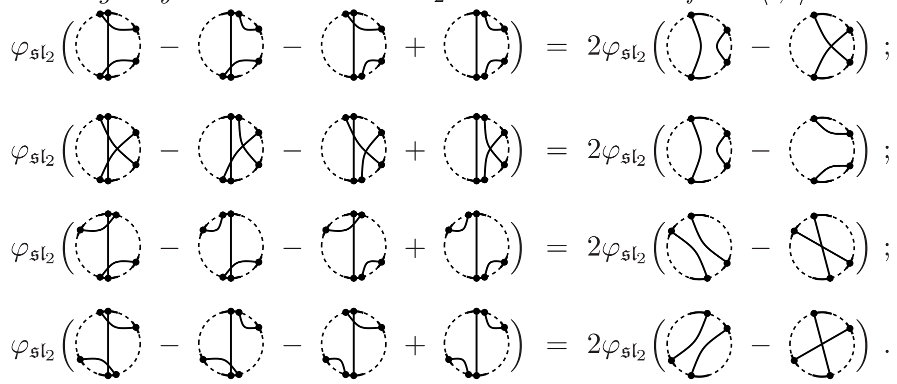
If $k = 1$, the second summand vanishes. In particular, for the Killing form ($\lambda = 4$) and $k = 1$ we have

$$\varphi_{\mathfrak{g}}(D) = (c - 1/2)\varphi_{\mathfrak{g}}(D_a).$$

It is interesting that the last formula is valid for any simple Lie algebra $\mathfrak{g}$ with the Killing form and any chord a having only one intersection with other chords. See Exercise 8 for a generalization of this fact.

Exercise. Deduce the theorem from the following lemma by induction (in case of difficulty see the proof in [**ChV**]).

Lemma (6-term relations for the universal $\mathfrak{sl}_2$ weight system)**.** *Let $\varphi_{\mathfrak{sl}_2}(\cdot)$ be the weight system associated to $\mathfrak{sl}_2$ and the invariant form $\langle\cdot,\cdot\rangle$. Then*



These relations also providee a recursive way to compute $\varphi_{\mathfrak{sl}_2}(D)$ as the two chord diagrams on the right-hand side have one chord less than the diagrams on the left-hand side, and the last three diagrams on the left-hand side are simpler than the first one since they have less intersections between their chords. See Section 6.2.3 for a proof of this Lemma.

6.1.4. Weight systems associated with representations. The construction of Bar-Natan, in comparison with that of Kontsevich, uses one additional ingredient: a representation of a Lie algebra.

A *linear representation* is a homomorphism of Lie algebras $T : \mathfrak{g} \to \mathrm{End}(V)$. It extends to a homomorphism of associative algebras $U(T) : U(\mathfrak{g}) \to \mathrm{End}(V)$. The composition of following three maps (with the last map being the trace)

$$\mathcal{A} \stackrel{\varphi_{\mathfrak{g}}}{\to} U(\mathfrak{g}) \stackrel{U(T)}{\to} \mathrm{End}(V) \stackrel{\mathrm{Tr}}{\to} \mathbb{C}$$

by definition gives the *weight system associated with the representation*

$$\varphi_{\mathfrak{g}}^T = \mathrm{Tr} \circ U(T) \circ \varphi_{\mathfrak{g}}$$

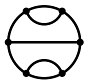
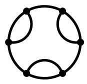
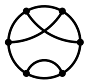
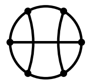
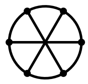
(by abuse of notation, we shall sometimes write $\varphi_{\mathfrak{g}}^V$ instead of $\varphi_{\mathfrak{g}}^T$.

If the representation T is irreducible, then, according to the Schur Lemma [**Hum**], every element of the center $ZU(\mathfrak{g})$ is represented (via $U(T)$) by a scalar operator $\mu \cdot \mathrm{id}_V$. Therefore its trace equals $\varphi_{\mathfrak{g}}^T(D) = \mu \dim V$. The number $\mu = \frac{\varphi_{\mathfrak{g}}^T(D)}{\dim V}$, as a function of the chord diagram D, is a weight system which is clearly multiplicative.

6.1.5. Example: algebra $\mathfrak{sl}_2$ with the standard representation. Consider the standard 2-dimensional representation St of $\mathfrak{sl}_2$. Then the Casimir element is represented by the matrix

$$c = \frac{1}{2}HH + EF + FE = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix} = \frac{3}{2} \cdot \mathrm{id}_2.$$

In degree 3 we have the following weight systems

D					
$\varphi_{\mathfrak{sl}_2}(D)$	c^3	c^3	$c^2(c-2)$	$c(c-2)^2$	$c(c-2)(c-4)$
$\varphi_{\mathfrak{sl}_2}^{St}(D)$	$27/4$	$27/4$	$-9/4$	$3/4$	$15/4$
$\varphi_{\mathfrak{sl}_2}'^{St}(D)$	0	0	0	12	24

Here the last row represents the unframed weight system obtained from $\varphi_{\mathfrak{sl}_2}^{St}(\cdot)$ by the deframing procedure from Section 4.5.6. A comparison of this computation with the one from Section 3.6.2 shows that $\mathrm{symb}(j_3) = -\frac{1}{2}\varphi_{\mathfrak{sl}_2}'^{St}$. See Exercises 14 and 15 at the end of the chapter for more information about these weight systems.

6.1.6. $\mathfrak{gl}_N$ with standard representation. Consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}_N$ of all $N \times N$ matrices and its standard representation St. Fix the trace of the product of matrices as the preferred ad-invariant form: $\langle x, y \rangle = \mathrm{Tr}(xy)$.

The algebra $\mathfrak{gl}_N$ is generated by matrices e_{ij} with 1 on the intersection of i-th row with j-th column and zero elsewhere. We have $\langle e_{ij}, e_{kl} \rangle = \delta_i^l \delta_j^k$, where δ is the Kronecker delta. Therefore, the duality between $\mathfrak{gl}_N$ and $(\mathfrak{gl}_N)^*$ defined by $\langle \cdot, \cdot \rangle$ is given by the formula $e_{ij}^* = e_{ji}$.

One can check that $[e_{ij}, e_{kl}] \neq 0$ only in the following cases:

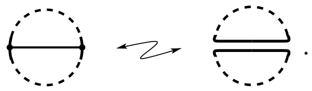
- $[e_{ij}, e_{jk}] = e_{ik}$, if $i \neq k$,
- $[e_{ij}, e_{ki}] = -e_{kj}$, if $j \neq k$,
- $[e_{ij}, e_{ji}] = e_{ii} - e_{jj}$, if $i \neq j$,

This gives the following formula for the Lie bracket as a tensor in $\mathfrak{gl}_N^* \otimes \mathfrak{gl}_N^* \otimes \mathfrak{gl}_N$:

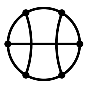
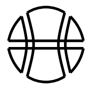
$$[\cdot,\cdot] = \sum_{i,j,k=1}^{N} (e_{ij}^* \otimes e_{jk}^* \otimes e_{ik} - e_{ij}^* \otimes e_{ki}^* \otimes e_{kj}).$$

D. Bar-Natan found the following elegant way of computing the weight system $\varphi_{\mathfrak{gl}_N}^{St}(\cdot)$.

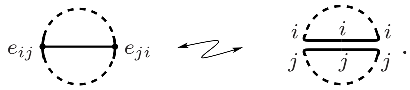
Theorem ([**BN0**])**.** *Denote by $s(D)$ the number of connected components of the curve obtained by doubling all chords of a chord diagram D.*



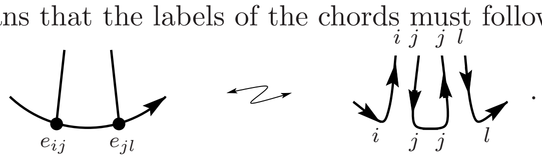
Then $\varphi_{\mathfrak{gl}_N}^{St}(D) = N^{s(D)}$.

Example. For $D =$  we obtain the picture . Here $s(D) = 2$, hence $\varphi_{\mathfrak{gl}_N}^{St}(D) = N^2$.

Proof. For each chord we attach a matrix e_{ij} to one endpoint and the matrix e_{ji} to another endpoint. The pair (ij) is an index associated with the given chord; different chords have different indices that vary in an independent way. We can label the two copies of the chord as well as the four pieces of the Wilson loop adjacent to its endpoints, by indices i and j as follows:



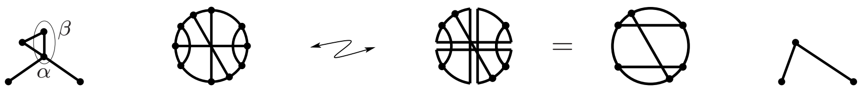
To compute the value of the weight system $\varphi_{\mathfrak{gl}_N}^{St}(D)$, we must sum up the products $\ldots e_{ij}e_{kl}\ldots$. Since we deal with the standard representation of $\mathfrak{gl}_N$, the product should be understood as the genuine matrix multiplication, unlike the case of weight systems with values in the universal enveloping algebra. Since $e_{ij} \cdot e_{kl} = \delta_{jk} \cdot e_{il}$, we obtain a non-zero summand only if $j = k$. This means that the labels of the chords must follow the pattern:



Therefore, all the labels on one and the same connected component of the curve obtained are equal. Now, if we take the whole product of matrices along the circle, we will get the operator e_{ii} whose trace is 1. Then we sum up all these ones over all possible labelings. The number of labelings is equal to the number of possibilities to assign an index i, j, l,… to each connected component of the curve obtained by doubling all the chords. Each component giving exactly N possibilities, the total number becomes $N^{s(D)}$. $\square$

Proposition. *The weight system* $\varphi^{St}_{\mathfrak{gl}_N}(D)$ *depends only on the intersection graph of* D.

Proof. Induction on the number of chords in D. If all chords are isolated, then the assertion is trivial. Otherwise, pick up two intersecting chords α and β in D and double them according to the theorem. Straightening the pinced circle with two double chords into a new Wilson loop, we get a chord diagram D'' having two chords less than D. One can see that the intersection graph $\Gamma(D'')$ depends only on the intersection graph $\Gamma(D)$.



Indeed, for any two vertices i and j different from α, β their connectivity (by an edge) in $\Gamma(D'')$ either coincides with their connectivity in $\Gamma(D)$ or changes to the opposite depending on their location with respect to α and β according to the rules:

location of i and j in $\Gamma(D)$	*connectivity of i and j in $\Gamma(D'')$*
i and j are connected with both α and β	the same
one of i and j is connected with both α and β, and another one is connected with only one of them	opposite
one of i and j is connected with both α and β, and another one is not connected with them	the same
both i and j are connected with one and the same of α and β	the same
one of i and j is connected with one of α and β, and another one is connected with another one	opposite
one of i and j is connected with one of α and β, and another one is not connected with them	the same
none of i and j is connected with α and β	the same

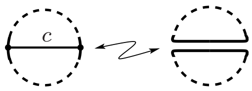
$\square$

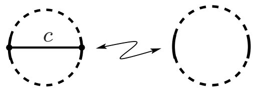
See exercise 9 at the end of this chapter for another proof of this proposition.

6.1.7. $\mathfrak{sl}_N$ with standard representation. Here we describe the weight system $\varphi^{St}_{\mathfrak{sl}_N}(D)$ associated with the Lie algebra $\mathfrak{sl}_N$, the invariant form $\langle x, y\rangle = \mathrm{Tr}(xy)$, and its standard representation by $N \times N$ matrices with zero trace.

Following Section 3.6.2, introduce *a state* σ for a chord diagram D as an arbitrary function on the set $[D]$ of chords of D with values in the set

$\{1, -\frac{1}{N}\}$. With each state σ we associate an immersed plane curve obtained from D by resolutions of all its chords according to s:



, if $\sigma(c) = 1$;


, if $\sigma(c) = -\dfrac{1}{N}$.

Let $|\sigma|$ denote the number of components of the curve obtained in this way.

Theorem. $\varphi^{St}_{\mathfrak{sl}_N}(D) = \sum\limits_{\sigma} \Big(\prod\limits_{c} \sigma(c)\Big)\, N^{|\sigma|}$ *, where the product is taken over all n chords of D, and the sum is taken over all 2^n states for D.*

One can prove this theorem in the same way as we did for $\mathfrak{gl}_N$ picking an appropriate basis for the vector space $\mathfrak{sl}_N$ and then working with the product of matrices (see exercise 13). However, we prefer to prove it in a different way, via several reformulations using the algebra structure of weight systems which is dual to the coalgebra structure of chord diagrams (Section 4.5).

Reformulation 1. *For a subset $J \subseteq [D]$ (the empty set and the whole $[D]$ are allowed) of chords of D, let D_J be the chord diagram formed by chords from J, let $s(D_J)$ denote the number of connected components of the curve obtained by doubling of all chords of D_J, let $|J|$ be the number of chords in J, and let $n - |J| = |\overline{J}|$ stand for the number of chords in $\overline{J} = [D] \setminus J$. Then*

$$\varphi^{St}_{\mathfrak{sl}_N}(D) = \sum_{J \subseteq [D]} (-1)^{n-|J|} N^{s(D_J)-n+|J|}.$$

This assertion is obviously equivalent to the theorem where, for every state s, the subset J consists of all chords c with value $s(c) = 1$.

Consider the weight system $e^{-\frac{\mathbf{I}_1}{N}}$ from Section 4.5.6, which is equal to the constant $\frac{1}{(-N)^n}$ on any chord diagram with n chords.

Reformulation 2.

$$\varphi^{St}_{\mathfrak{sl}_N} = e^{-\frac{\mathbf{I}_1}{N}} \cdot \varphi^{St}_{\mathfrak{gl}_N} \ .$$

Indeed, by the definition of the product of weight systems (Section 4.5),

$$\left(e^{-\frac{\mathbf{I}_1}{N}} \cdot \varphi^{St}_{\mathfrak{gl}_N}\right)(D) := \left(e^{-\frac{\mathbf{I}_1}{N}} \otimes \varphi^{St}_{\mathfrak{gl}_N}\right)(\delta(D)) \ ,$$

where $\delta(D)$ is the coproduct (Section 4.4) of the chord diagram D. It splits D into two complementary parts $D_{\overline{J}}$ and D_J: $\delta(D) = \sum\limits_{J \subseteq [D]} D_{\overline{J}} \otimes D_J$. The weight system $\varphi^{St}_{\mathfrak{gl}_N}(D_J)$ gives $N^{s(D_J)}$. The remaining part is given by $e^{-\frac{\mathbf{I}_1}{N}}(D_{\overline{J}})$.

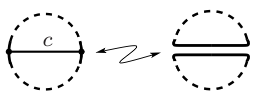
Reformulation 3.

$$\varphi^{St}_{\mathfrak{gl}_N} = e^{\frac{\mathbf{I}_1}{N}} \cdot \varphi^{St}_{\mathfrak{sl}_N} \ .$$

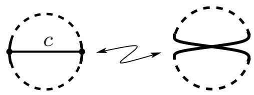
The equivalence of this formula to the previous one follows from the fact that the weight systems $e^{-\frac{\mathbf{I}_1}{N}}$ and $e^{\frac{\mathbf{I}_1}{N}}$ are inverse to each other as elements of the completed algebra of weight systems.

Proof. We will prove the theorem in reformulation 3. The Lie algebra $\mathfrak{gl}_N$ is a direct sum of $\mathfrak{sl}_N$ and the trivial one-dimensional Lie algebra generated by the identity matrix id_N. Its dual is $\mathrm{id}_N^* = \frac{1}{N}\mathrm{id}_N$. We can choose a basis for the vector space $\mathfrak{gl}_N$ consisting of the basis for $\mathfrak{sl}_N$ and the unit matrix id_N. To every chord we must assign either a pair of dual basic elements of $\mathfrak{sl}_N$, or the pair $(\mathrm{id}_N, \frac{1}{N}\mathrm{id}_N)$, which is equivalent to forgetting the chord and multiplying the obtained diagram by $\frac{1}{N}$. This means precisely that we are applying the weight system $e^{\frac{\mathbf{I}_1}{N}}$ to the chord subdiagram $D_{\overline{J}}$ formed by forgotten chords, and the weight system $\varphi^{St}_{\mathfrak{sl}_N}$ to the chord subdiagram D_J formed by the remaining chords. $\square$

6.1.8. $\mathfrak{so}_N$ with the standard representation. In this case *a state* σ for D is a function on the set $[D]$ of chords of D with values in the set $\{1/2, -1/2\}$. The rule of resolution of a chord according to its state is



$, \text{ if } \sigma(c) = \dfrac{1}{2};$



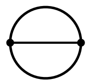
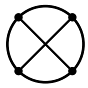
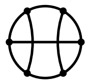
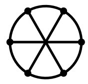
$, \text{ if } \sigma(c) = -\dfrac{1}{2}.$

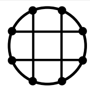
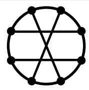
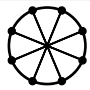
As before, $|\sigma|$ denotes the number of components of the obtained curve.

Theorem ([**BN0, BN1**]). $\varphi^{St}_{\mathfrak{so}_N}(D) = \sum\limits_{\sigma} \Big(\prod\limits_{c} \sigma(c)\Big) N^{|\sigma|}$, *where the product is taken over all n chords of D, and the sum is taken over all 2^n states for D.*

We leave the proof of this theorem to the reader as an exercise (number 16 at the end of the chapter).

Here is the table of values of $\varphi^{St}_{\mathfrak{so}_N}(D)$ for basic diagrams of small degree:

D				
$\varphi_{\mathfrak{so}_N}(D)$	$\frac{1}{2}(N^2-N)$	$\frac{1}{4}(N^2-N)$	$\frac{1}{8}(N^2-N)$	$\frac{1}{8}N(-N^2+4N-3)$

D			
$\varphi_{\mathfrak{so}_N}(D)$	$\frac{1}{16}N(3N^2-8N+5)$	$\frac{1}{8}N(N-1)^2$	$\frac{1}{16}N(N^3-4N^2+6N-3)$

Exercises 17—21 contain additional information about this weight system.

6.1.9. $\mathfrak{sp}_{2N}$ with the standard representation. It turns out that $\varphi^{St}_{\mathfrak{sp}_{2N}}(D) = (-1)^{n+1}\varphi_{\mathfrak{so}_{-2N}}(D)$, where the last notation means the formal substitution of $-2N$ instead of the variable N in the polynomial $\varphi_{\mathfrak{so}_N}(D)$, and n, as usual, means the number of chords of D. This implies that the weight system $\varphi^{St}_{\mathfrak{sp}_{2N}}$ does not provide any new knot invariant. Some details about it can be found in [**BN0, BN1**].

It would be interesting to find a combinatorial description of weight systems for the exceptional simple Lie algebras E_6, E_7, E_8, F_4, G_2.

6.2. Lie algebra weight systems for the algebra $\mathcal{C}$

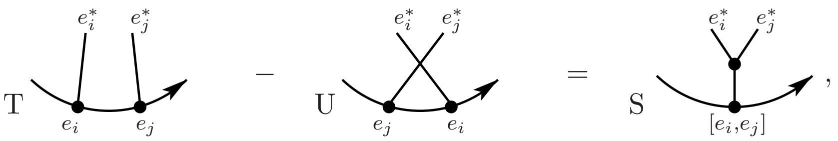
We would like to extend the weight system $\varphi_{\mathfrak{g}}(\cdot)$ to the weight system $\rho_{\mathfrak{g}}(\cdot)$ for the algebra $\mathcal{C}$ in such a way that the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi_{\mathfrak{g}}"] \arrow[d, "\hat{\lambda}"'] & U(\mathfrak{g}) \\ \mathcal{C} \arrow[ur, "\rho_{\mathfrak{g}}"'] & \end{tikzcd}$$

becomes commutative. Here $\hat{\lambda}$ is the isomorphism of $\mathcal{A}$ and $\mathcal{C}$ from section 5.3.1. (Since $\mathcal{A}$ and $\mathcal{C}$ are isomorphic, the word 'extension' is not quite appropriate; we use it because the set of closed diagrams generating $\mathcal{C}$ is strictly wider than the set of chord diagrams generating $\mathcal{A}$ and we are going to explain the rule of computing the value of ρ on any closed diagram.)

The STU relation (Section 5.1.2), defining the algebra $\mathcal{C}$, gives us a hint. Namely, if we assign elements e_i, e_j to the endpoints of chords of the T- and

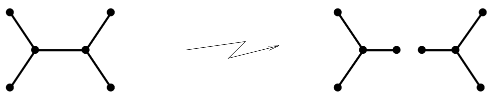
U- diagrams from the STU relations,



then it is natural to assign the commutator $[e_i, e_j]$ to the trivalent vertex in the S-diagram. The formal construction goes as follows.

For any closed Jacobi diagram $C \in \mathbf{C}_n$ let $V = \{v_1, \ldots, v_m\}$ be the set of its external vertices (lying on the Wilson loop) ordered according to the orientation of the loop. We construct a tensor $T_{\mathfrak{g}}(C) \in \mathfrak{g}^{\otimes m}$ whose i-th tensor factor $\mathfrak{g}$ corresponds to the element v_i of the set V. Then we put $\rho_{\mathfrak{g}}(C)$ to be equal to the image of the tensor $T_{\mathfrak{g}}(C)$ in $U(\mathfrak{g})$ under the natural projection. This is the weight system for the algebra $\mathcal{C}$.

To construct the tensor $T_{\mathfrak{g}}(C)$, we cut all the edges between the trivalent vertices of C. This splits C into a union of elementary pieces (tripods), each consisting of one trivalent vertex and three univalent vertices. Here is an example:

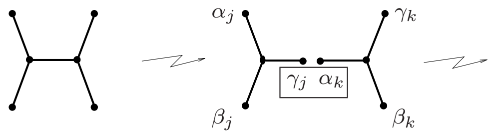


With each elementary piece we associate a copy of the tensor $-J \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ whose factors correspond to the univalent vertices in agreement with the cyclic ordering of the edges. This tensor is defined as follows. Consider the Lie bracket $[\cdot, \cdot]$ as an element of $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$. Identification of $\mathfrak{g}^*$ and $\mathfrak{g}$ via $\langle \cdot, \cdot \rangle$ provides the tensor $J \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$.

6.2.1. Exercise. Use the properties of $[\cdot, \cdot]$ and $\langle \cdot, \cdot \rangle$ to prove that the tensor J thus obtained is skew-symmetric under the permutations of the three tensor factors (see Lemma A.1.6 on page 419).

This result ensures that the previous construction makes sense. Although it does not really matter, please note that we associate the tensor $-J$, not J, to each vertex. This convention proves useful in the case $\mathfrak{g} = \mathfrak{gl}_N$: it makes the graphical algorithm for the calculation of the weight system look more natural.

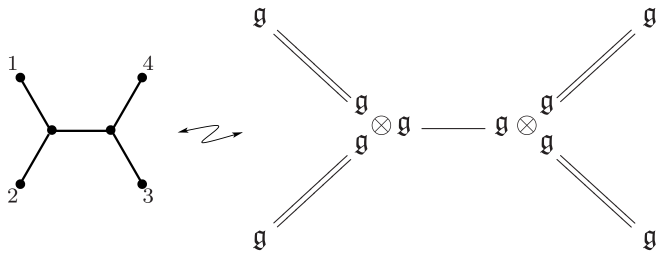
The tensor $T_{\mathfrak{g}}(C)$ corresponding to C is combined from these elementary ones in the following way. Let us restore C from the tripod pieces. Each time two univalent vertices are put together, we contract the two tensor factors corresponding to these vertices by taking the form $\langle \cdot, \cdot \rangle$ on them. For example, if $-J = \sum \alpha_j \otimes \beta_j \otimes \gamma_j$, then the tensor we relate to the union of two pieces will be:



$$\sum_{i,j,k} \langle \gamma_j, \alpha_k \rangle \alpha_j \otimes \beta_j \otimes \beta_k \otimes \gamma_k.$$

Gluing and contracting in such a way over all the edges previously cut we obtain an element of the tensor product of several copies of $\mathfrak{g}$ whose factors correspond to the univalent vertices of C. This is what we denote by $T_{\mathfrak{g}}(C)$.

Speaking pictorially, we suspend a copy of the tensor cube of $\mathfrak{g}$ with the distinguished element $-J$ living inside it at every internal vertex of C, then take the tensor product over all internal vertices and compute the complete contraction of the resulting tensor, applying the bilinear form $\langle \cdot, \cdot \rangle$ to every pair of factors that correspond to the endpoints of one internal edge, for example:



$$\longrightarrow \quad \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}\ ,$$

where the numbers at univalent vertices on the graph correspond to the order of tensor factors from left to right in the resulting space $\mathfrak{g}^{\otimes 4}$. Doing this with all internal vertices and edges we get the tensor $T_{\mathfrak{g}}(C) \in \mathfrak{g}^{\otimes m}$

Note that the correspondence $C \to T_{\mathfrak{g}}(C)$ satisfies the AS and IHX relations:

(1) the AS relation follows from the fact that the tensor J changes sign under odd permutations of the three factors in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$.

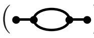
(2) the IHX relation is a corollary of the Jacobi identity in $\mathfrak{g}$.

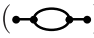
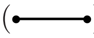
Moreover, the element $\rho_{\mathfrak{g}}(C)$, which is the image of the tensor $T_{\mathfrak{g}}(C)$ under the natural projection $\mathfrak{g}^{\otimes m} \to U(\mathfrak{g})$, satisfies also the STU relation:

(3) the STU relation follows from the definition of commutator in the universal enveloping algebra.

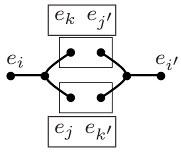
To exemplify this construction, we prove the following lemma relating the tensor corresponding to a 'bubble' with the quadratic Casimir tensor.

6.2.2. Lemma. *For the Killing form $\langle\cdot,\cdot\rangle^K$ as the preferred invariant form we have*

$T_{\mathfrak{g}}($$) \;=\; T_{\mathfrak{g}}($$)\,.$

For the bilinear form $\mu\langle\cdot,\cdot\rangle^K$ the rule changes as follows: $T_{\mathfrak{g}}($$) = \frac{1}{\mu}T_{\mathfrak{g}}($$)$.

Proof. The cut and glue procedure above gives the following tensor written in an orthonormal basis $\{e_i\}$.



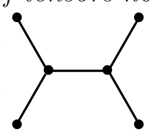
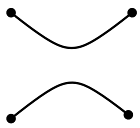
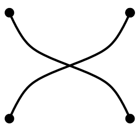
$$\rightsquigarrow \sum_{i,i'} \sum_{k,j,k',j'} c_{ijk}c_{i'j'k'}\langle e_k, e_{j'}\rangle^K \langle e_j, e_{k'}\rangle^K e_i \otimes e_{i'}$$
$$= \sum_{i,i'}\Big(\sum_{j,k} c_{ijk}c_{i'kj}\Big) e_i \otimes e_{i'}\ ,$$

where c_{ijk} are the structure constants: $J = \sum\limits_{i,j,k=1}^{d} c_{ijk} e_i \otimes e_j \otimes e_k$.

To compute the coefficient $\Big(\sum\limits_{j,k} c_{ijk}c_{i'kj}\Big)$ let us find the value of the Killing form $\langle e_{i'}, e_i\rangle^K = \mathrm{Tr}(\mathrm{ad}_{e_{i'}}\mathrm{ad}_{e_i})$. Since $\mathrm{ad}_{e_i}(e_j) = \sum\limits_k c_{ijk}e_k$ and $\mathrm{ad}_{e_{i'}}(e_k) = \sum\limits_l c_{i'kl}e_l$, the (j,l)-entry of the matrix of the product $\mathrm{ad}_{e_{i'}}\mathrm{ad}_{e_i}$ will be $\sum\limits_k c_{ijk}c_{i'kl}$. Therefore $\langle e_{i'}, e_i\rangle^K = \sum\limits_{j,k} c_{ijk}c_{i'kj}$. Orthonormality of the basis $\{e_i\}$ implies that $\sum\limits_{j,k} c_{ijk}c_{i'kj} = \delta_{i,i'}$. This means that the tensor in the right-hand side equals $\sum\limits_i e_i \otimes e_i$, which is the quadratic Casimir tensor from the left-hand side. $\square$

6.2.3. The $\mathfrak{sl}_2$-weight system.

Theorem ([**ChV**]). *For the invariant form $\langle x,y\rangle = Tr(xy)$ the following equality of tensors holds:*

$T_{\mathfrak{sl}_2}\Big($$\Big) \;=\; 2T_{\mathfrak{sl}_2}\Big($$\Big) - 2T_{\mathfrak{sl}_2}\Big($$\Big)\,.$

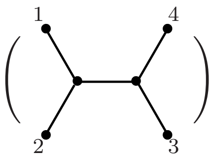
If the chosen invariant form is $\lambda\langle\cdot,\cdot\rangle$, then the coefficient 2 in this equation is replaced by $\frac{2}{\lambda}$.

Proof. For the algebra $\mathfrak{sl}_2$ the Casimir tensor and the Lie bracket tensor are

$$C = \frac{1}{2}H \otimes H + E \otimes F + F \otimes E \ ;$$

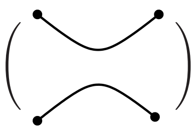
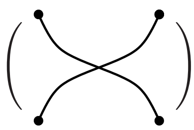
$$-J = -H\otimes F\otimes E+F\otimes H\otimes E+H\otimes E\otimes F-E\otimes H\otimes F-F\otimes E\otimes H+E\otimes F\otimes H.$$

Then the tensor corresponding to the right-hand side is (we numerate the vertices according to the tensor factors)

$T_{\mathfrak{sl}_2}$ 

$$= -H\otimes F\otimes H\otimes E+H\otimes F\otimes E\otimes H+F\otimes H\otimes H\otimes E$$
$$-F\otimes H\otimes E\otimes H-H\otimes E\otimes H\otimes F+H\otimes E\otimes F\otimes H+E\otimes H\otimes H\otimes F-E\otimes H\otimes F\otimes H$$
$$+2F\otimes E\otimes F\otimes E-2F\otimes E\otimes E\otimes F-2E\otimes F\otimes F\otimes E+2E\otimes F\otimes E\otimes F$$

$$= 2\Big(\tfrac{1}{4}H\otimes H\otimes H\otimes H+\tfrac{1}{2}H\otimes E\otimes F\otimes H+\tfrac{1}{2}H\otimes F\otimes E\otimes H+\tfrac{1}{2}E\otimes H\otimes H\otimes F$$
$$+E\otimes E\otimes F\otimes F+E\otimes F\otimes E\otimes F+\tfrac{1}{2}F\otimes H\otimes H\otimes E+F\otimes E\otimes F\otimes E+F\otimes F\otimes E\otimes E\Big)$$
$$-2\Big(\tfrac{1}{4}H\otimes H\otimes H\otimes H+\tfrac{1}{2}H\otimes E\otimes H\otimes F+\tfrac{1}{2}H\otimes F\otimes H\otimes E+\tfrac{1}{2}E\otimes H\otimes F\otimes H$$
$$+E\otimes E\otimes F\otimes F+E\otimes F\otimes F\otimes E+\tfrac{1}{2}F\otimes H\otimes E\otimes H+F\otimes E\otimes E\otimes F+F\otimes F\otimes E\otimes E\Big)$$

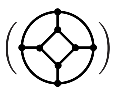
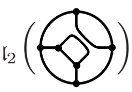
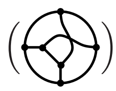
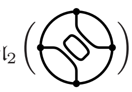
$= 2T_{\mathfrak{sl}_2}$  $- 2T_{\mathfrak{sl}_2}$  .

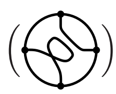
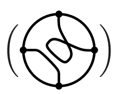
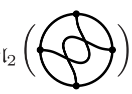
$\square$

Remark. During the reduction of a closed diagram according to this theorem a closed circle different from the Wilson loop may occur (see the example below). In this situation we replace the circle by the numeric multiplier $3 = \dim \mathfrak{sl}_2$ which is the trace of the identity operator in the adjoint representation of $\mathfrak{sl}_2$.

Remark. In the context of weight systems this relation was first noted in [**ChV**]. Then it was rediscovered several times. But in more general context of graphical notation for tensors it was known to R. Penrose [**Pen**] circa 1955. In a certain sense, this relation goes back to Euler and Lagrange because it is an exact counterpart of the classical "$bac - cab$" rule, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$, for the ordinary cross product of vectors in 3-space.

Example.

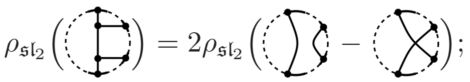
$\rho_{\mathfrak{sl}_2}$  $= 2\rho_{\mathfrak{sl}_2}$  $- 2\rho_{\mathfrak{sl}_2}$  $= 4\rho_{\mathfrak{sl}_2}$ 

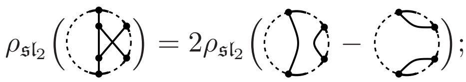
$-4\rho_{\mathfrak{sl}_2}$  $- 4\rho_{\mathfrak{sl}_2}$  $+ 4\rho_{\mathfrak{sl}_2}$ 

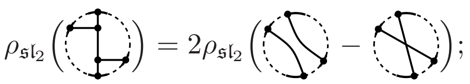
$$= 12c^2 - 4c^2 - 4c^2 + 4c^2 = 8c^2 \ .$$

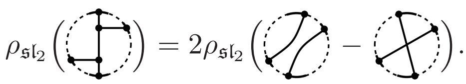
The next corollary implies the 6-term relation from Section 6.1.3.

Corollary.





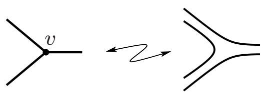
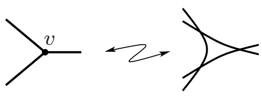




6.2.4. $\mathfrak{gl}_N$ with the standard representation. For a closed diagram $C \in \mathcal{C}_n$ with the set IV of t internal trivalent vertices we double each *internal* edge (not a piece of the Wilson loop) and count the number of components of the obtained curve as before. The only problem here is how to connect the lines near an internal vertex. This can be decided by means of a state function $s : IV \to \{-1, 1\}$.

Theorem ([**BN1**]). *Let $\rho^{St}_{\mathfrak{gl}_N}(\cdot)$ be the weight system associated to the Lie algebra $\mathfrak{gl}_N$, its standard representation, and the invariant form $\langle x, y\rangle = Tr(xy)$.*

For a closed diagram C and a state $s : IV \to \{-1, 1\}$ double every internal edge and connect the lines together in a neighborhood of a vertex $v \in IV$ according to the state s:

 , *if $s(v) = 1$;*  , *if $s(v) = -1$.*

Let $|s|$ denote the number of components of the curve obtained in this way. Then

$$\rho^{St}_{\mathfrak{gl}_N}(C) = \sum_s \Big(\prod_v s(v)\Big)\ N^{|s|}\ ,$$

where the product is taken over all t internal vertices of C, and the sum is taken over all the 2^t states for C.

Proof. A straightforward way to prove this theorem is to use the STU relation and the theorem of Section 6.1.6. However we prefer a different way based on the graphical notation for tensors developed below.

Using the identification of $(\mathfrak{gl}_N)^*$ with $\mathfrak{gl}_N$ and generators e_{ij} indicated at the beginning of Section 6.1.6 we can rewrite the Lie bracket as

$$J = \sum_{i,j,k=1}^{N} (e_{ji} \otimes e_{kj} \otimes e_{ik} - e_{ji} \otimes e_{ik} \otimes e_{kj}).$$

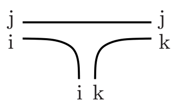
To each internal trivalent vertex we associate the tensor

$$-J = \sum_{i,j,k=1}^{N} (e_{ji} \otimes e_{ik} \otimes e_{kj} - e_{ji} \otimes e_{kj} \otimes e_{ik}) \ .$$

6.2.5. Graphical notation for tensors. Here we introduce a system of graphical notation for a special class of the elements of $(\mathfrak{gl}_N)^{\otimes n}$ similar to that invented by R. Penrose in [**Pen**] (see also [**BN1**]). Consider a plane diagram (T-*diagram*) that consists of n pairs of mutually close points connected pairwise by n lines. Formally speaking, a T-diagram is a set X of cardinality $2n$ endowed with two involutions without fixed points (or, in other words, two partitions of X into n two-point subsets). In the pictures below, X is the set of all protruding endpoints of the lines which indicate the second partition, while the pairs of the first partition are the endpoints that are drawn close to each other.

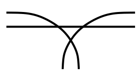
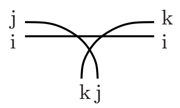
Given a T-diagram, we can write out the corresponding element of the n-th tensor power of $\mathfrak{gl}_N$ in the following way. We put the same index at either end of each line, and then, traveling around the encompassing circle, write a factor e_{ij} each time we encounter a pair of neighboring points, the first of which is assigned the index i and the second, the index j. Then we consider the sum of all such tensors when all the indices range from 1 to N. In a sense, this procedure gives a graphical substitute for the formal Einstein summation rule in multi-index expressions. We must admit that in general the element of $\otimes^n \mathfrak{gl}_N$ thus obtained may depend on the choice of the starting point of the circle, — but the corresponding element of the symmetric power, obtained by symmetrization, will not depend on this choice.

For example, to find the tensor corresponding to the diagram  we take three indices i, j and k and write:



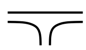
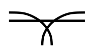
$$\leftrightsquigarrow \quad \sum_{i,j,k=1}^{N} e_{ji} \otimes e_{ik} \otimes e_{kj}.$$

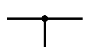
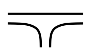
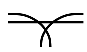
Here is another example:

 $\leftrightsquigarrow$ 

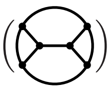
$$\leftrightsquigarrow \quad \sum_{i,j,k=1}^{N} e_{ji} \otimes e_{kj} \otimes e_{ik}.$$

These two examples are sufficient to graphically express the tensor J:

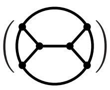
$J \quad = \quad$  $-$ 

With every trivalent vertex of an open diagram we associate a copy of the tensor $-J$:  $\leftrightsquigarrow$  $-$ 

depending on the value of state on the vertex. After all such resolutions we get a linear combination of 2^t tensor monomials representing the tensor $T_{\mathfrak{gl}_N}(C)$. Then taking the corresponding element in the universal enveloping algebra $U(\mathfrak{gl}_N)$ and the trace of its standard representation we get the polynomial weight system $\rho^{St}_{\mathfrak{gl}_N}(C)$. Now repeating the argument at the end of the proof on p.174 we get the Theorem. $\square$

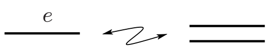
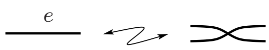
Example. Let us compute the value $\rho^{St}_{\mathfrak{gl}_N}\Big($  $\Big)$. There are four resolutions of the triple points:

$\prod s(v)=1$	$\prod s(v)=-1$	$\prod s(v)=-1$	$\prod s(v)=1$
$\lvert s\rvert=4$	$\lvert s\rvert=2$	$\lvert s\rvert=2$	$\lvert s\rvert=2$

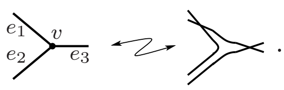
Therefore, $\rho^{St}_{\mathfrak{gl}_N}\Big($  $\Big) = N^4 - N^2$.

Other properties of the weight system $\rho^{St}_{\mathfrak{gl}_N}(\cdot)$ are formulated in exercises 29 - 33.

6.2.6. $\mathfrak{so}_N$ with standard representation. Now a *state* for $C \in \mathcal{C}_n$ will be a function $s : IE \to \{-1, 1\}$ on the set IE of internal edges (those which are not on the Wilson loop). The value of a state indicates the way of doubling the corresponding edge:

 , if $s(e) = 1$;  , if $s(e) = -1$.

In a neighborhood of a trivalent vertex we connect the lines in a standard fashion. For example, if the values of the state on three edges e_1, e_2, e_3 meeting at a vertex v are $s(e_1) = -1$, $s(e_2) = 1$, and $s(e_3) = -1$, then we have



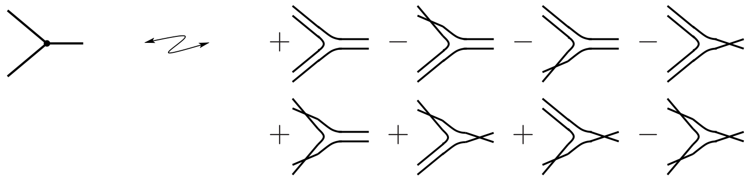
As usual, $|s|$ denotes the number of components of the curve obtained in this way.

Theorem ([**BN1**]). *Let $\rho^{St}_{\mathfrak{so}_N}(\cdot)$ be the weight system associated to the Lie algebra $\mathfrak{so}_N$, its standard representation, and the invariant form $\langle x, y\rangle = Tr(xy)$. Then*

$$\rho^{St}_{\mathfrak{so}_N}(C) = 2^{\#(IV)-\#(IE)} \sum_s \Big(\prod_e s(e)\Big)\ N^{|s|}\ ,$$

where the product is taken over all internal edges of C, the sum is taken over all states for C, and $\#(IV)$, $\#(IE)$ denote the numbers of internal vertices and edges respectively.

The Theorem can be reformulated as follows. For an internal vertex v consider one of the following 8 resolutions (two for each edge incident to v) together with the corresponding sign:

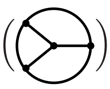


Now let s be a choice of one of these resolution for every internal trivalent vertex, and let $sign(s)$ be the product of the corresponding signs over all internal trivalent vertices. Then

$$\rho^{St}_{\mathfrak{so}_N}(C) = \frac{1}{4^{\#(IV)}} \sum_s sign(s) N^{|s|} \ ,$$

In this form the Theorem can be easily proved by using the STU relation and the theorem of Section 6.1.8.

Example.

$\rho^{St}_{\mathfrak{so}_N}\Big($  $\Big)$

$$\begin{aligned} &= \tfrac{1}{4}(N^3 - N^2 - N^2 - N^2 + N + N + N - N) \\ &= \tfrac{1}{4}(N^3 - 3N^2 + 2N) \ = \ \tfrac{1}{4}N(N-1)(N-2) \ . \end{aligned}$$

6.3. Lie algebra weight systems for the algebra $\mathcal{B}$

In this section for a metrized Lie algebra $\mathfrak{g}$ we construct a weight system $\psi : \mathcal{B} \to S(\mathfrak{g})$, defined on the space of open diagrams $\mathcal{B}$ and taking values in the complete symmetric algebra of the vector space $\mathfrak{g}$ (in fact, even in its $\mathfrak{g}$-invariant subspace $S(\mathfrak{g})^{\mathfrak{g}}$).

Let $O \in \mathbf{B}$ be an open diagram. Denote by V the set of its legs (univalent vertices). Just as in Section 6.2 we construct a tensor

$$T_{\mathfrak{g}}(O) \in \bigotimes_{v \in V} \mathfrak{g},$$

where the symbol $\bigotimes\limits_{v \in V} \mathfrak{g}$ means the tensor product whose factors correspond to the elements of the set V: it is defined like the usual tensor product, only instead of linearly ordered arrays $(g_1, ..., g_m)$, $g_i \in \mathfrak{g}$ we use families of elements of $\mathfrak{g}$ indexed by the set V.

A numbering ν of the set of univalent vertices V yields an isomorphism between $\bigotimes_{v\in V} \mathfrak{g}$ and the conventional tensor power $\bigotimes_{i=1}^{m} \mathfrak{g} = \mathfrak{g}^{\otimes m}$ (with a linear order of the factors). This isomorphism takes $T_\mathfrak{g}(O)$ to $T^\nu_\mathfrak{g}(O)$. The average over all numberings ν is the element of the symmetric power which we wanted to construct:

$$\psi_\mathfrak{g}(O) = \frac{1}{m!}\sum_{\nu\in S_m} T^\nu_\mathfrak{g}(O) \in S^m(\mathfrak{g}).$$

Algorithmically, this step means that we forget the symbols of the tensor product and view the result as an ordinary commutative polynomial in the variables that correspond to a basis of $\mathfrak{g}$.

6.3.1. The formal PBW theorem. The relation between the Lie algebra weight system for chord diagrams and for open diagrams is expressed by the following theorem.

Theorem. *For any metrized Lie algebra $\mathfrak{g}$ the diagram*

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\psi_\mathfrak{g}} & S(\mathfrak{g}) \\ {\scriptstyle\chi}\big\downarrow & & \big\downarrow{\scriptstyle\beta_\mathfrak{g}} \\ \mathcal{A} & \xrightarrow[\varphi_\mathfrak{g}]{} & U(\mathfrak{g}) \end{array}$$

is commutative.

Proof. The assertion becomes evident as soon as one recalls the definitions of all the constituents of the diagram: the isomorphism χ between the algebras $\mathcal{A}$ and $\mathcal{B}$ described in section 5.7, the weight systems $\varphi_\mathfrak{g}$ and $\psi_\mathfrak{g}$, defined in sections 6.1 and 6.3, and $\beta_\mathfrak{g}$ — the Poincaré–Birkhoff–Witt isomorphism taking an element $x_1x_2...x_n$ into the arithmetic mean of $x_{i_1}x_{i_2}...x_{i_n}$ over all permutations $(i_1, i_2, ..., i_n)$ of the set $\{1, 2, \dots, n\}$. Its restriction to the invariant subspace $S(\mathfrak{g})^\mathfrak{g}$ is an isomorphism with the center of $U(\mathfrak{g})$. $\square$

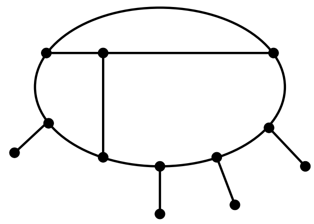
6.3.2. Example. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{so}_3$. It has a basis $\{a, b, c\}$ which is orthonormal with respect to the Killing form $\langle\cdot,\cdot\rangle^K$ and with the commutators $[a, b] = c$, $[b, c] = a$, $[c, a] = b$. So as a metrized Lie algebra $\mathfrak{so}_3$ is isomorphic to Euclidean 3-space with the cross product as a Lie bracket. The tensor that we put in every 3-valent vertex in this case is

$$\begin{aligned} J &= a\wedge b\wedge c \\ &= a\otimes b\otimes c + b\otimes c\otimes a + c\otimes a\otimes b - b\otimes a\otimes c - c\otimes b\otimes a - a\otimes c\otimes b. \end{aligned}$$

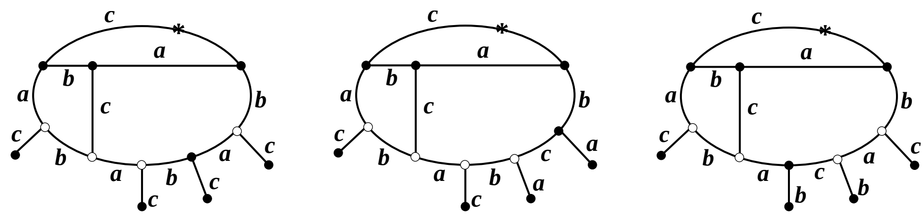
Since the basis is orthonormal, the only way to get a non-zero element in the process of contraction along the edges is to choose the same basic element on either end of each edge. On the other hand, the formula for

J shows that in every vertex we must choose a summand with different basic elements corresponding to the 3 edges. This leads to the following algorithm to compute the tensor $T_{\mathfrak{so}_3}(O)$ for a given diagram O: one must list all 3-colorings of the edges of the graph by 3 colors a, b, c such that the 3 colors at every vertex are always different, then sum up the tensor products of the elements written on the legs, each taken with the sign $(-1)^s$, where s is the number of *negative* vertices (i. e. vertices where the colors, read counterclockwise, come in the negative order a, c, b).

For example, consider the diagram (the *Pont-Neuf diagram with parameters* $(1,3)$ in the terminology of O. Dasbach [**Da3**], see also p.412 below):

$O =$ 

It has 18 edge 3-colorings, which can be obtained from the following three by permutations of (a,b,c):



In these pictures, negative vertices are marked by small empty circles. Writing the tensors in the counterclockwise order starting from the marked point, we get:

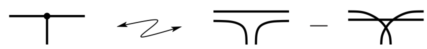
$$\begin{aligned}
&2(a\otimes a\otimes a\otimes a+b\otimes b\otimes b\otimes b+c\otimes c\otimes c\otimes c)\\
&+a\otimes b\otimes b\otimes a+a\otimes c\otimes c\otimes a+b\otimes a\otimes a\otimes b\\
&+b\otimes c\otimes c\otimes b+c\otimes a\otimes a\otimes c+c\otimes b\otimes b\otimes c\\
&+a\otimes a\otimes b\otimes b+a\otimes a\otimes c\otimes c+b\otimes b\otimes a\otimes a\\
&+b\otimes b\otimes c\otimes c+c\otimes c\otimes a\otimes a+c\otimes c\otimes b\otimes b.
\end{aligned}$$

This tensor is not symmetric with respect to permutations of the 4 tensor factors (although — we note this in passing — it is symmetric with respect to the permutations of the 3 letters a, b, c). Symmetrizing, we get:

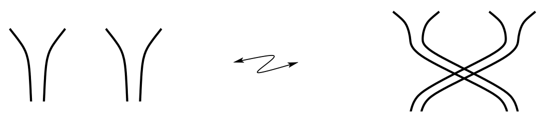
$$\psi_{\mathfrak{so}_3}(O) = 2(a^2+b^2+c^2)^2.$$

This example shows that the weight system defined by the Lie algebra $\mathfrak{so}_3$, is closely related to the 4-color theorem, see [**BN3**] for details.

6.3.3. The $\mathfrak{gl}_N$ weight system for the algebra $\mathcal{B}$. We are going to adapt the graphical notation for tensors from Section 6.2.4 to this situation. Namely, with every trivalent vertex of an open diagram O we associate two resolutions (T-diagrams):



After all such resolutions we get a linear combination of 2^t T-diagrams, where t is the number of trivalent vertices of O. Each diagram consists of m pairs of points and $2m$ lines connecting them, and represents a tensor from ${\mathfrak{gl}_N}^{\otimes m}$. Permuting the tensor factors in ${\mathfrak{gl}_N}^{\otimes m}$ corresponds to interchanging the pairs of points in the diagram:



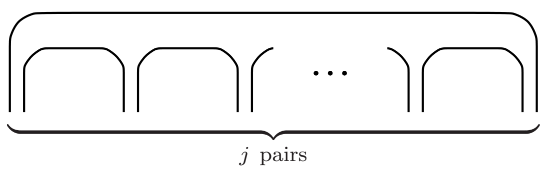
Therefore, if we consider a T-diagram up to an arbitrary permutation of the pairs, it correctly defines an element of $U_n(\mathfrak{gl}_N)/U_{n-1}(\mathfrak{gl}_N) \simeq S^n(\mathfrak{gl}_N)$. On a purely combinatorial level, a T-diagram considered up to an arbitrary permutation of pairs of legs is a set of cardinality $2n$ endowed with a splitting into n ordered pairs and a splitting into n unordered pairs. Thus, the linear combination of T-diagrams which is considered up to such permutations represents the element $\psi_{\mathfrak{gl}_N}(C)$.

6.3.4. The center of $U(\mathfrak{gl}_N)$. It is known [**Zh**] that $ZU(\mathfrak{gl}_N)$ is generated by N variables c_j, $1 \leqslant j \leqslant N$ (*generalized Casimir elements*):

$$c_j = \sum_{i_1,\dots,i_j=1}^{N} e_{i_1 i_2} e_{i_2 i_3} e_{i_3 i_4} \dots e_{i_{j-1} i_j} e_{i_j i_1}$$

as a free commutative polynomial algebra.

In the graphical notation

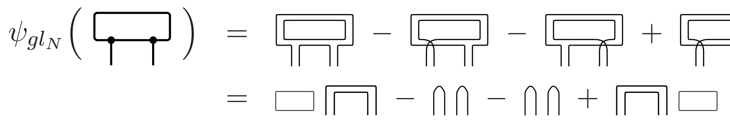
$c_j =$ 

.

In particular, $c_1 = \bigcap = \sum\limits_{i=1}^{N} e_{ii}$ is the unit matrix (note that it is *not* the unit of the algebra $S(\mathfrak{g})$), $c_2 = \sqcap = \sum\limits_{i_1,i_2=1}^{N} e_{i_1i_2}e_{i_2i_1}$ is the quadratic Casimir element.

It is convenient to extend the list $c_1, \ldots, c_N$ of our variables by setting $c_0 = N$. The graphical notation for c_0 will be a circle. (Indeed, a circle has no legs, so it must correspond to an element of the 0-th symmetric power of $\mathfrak{g}$, which is the ground field. When the index written on the circle runs from 1 to N, we must take the sum of ones in the quantity N, thus getting N.)

Example.



$= 2(c_0c_2 - c_1^2)$.

Exercises

(1) Let $(\mathfrak{g}_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathfrak{g}_2, \langle\cdot,\cdot\rangle_2)$ be two metrized Lie algebras. Then their direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is also a metrized Lie algebra with respect to the form $\langle\cdot,\cdot\rangle_1 \oplus \langle\cdot,\cdot\rangle_2$. Prove that $\varphi_{\mathfrak{g}_1\oplus\mathfrak{g}_2} = \varphi_{\mathfrak{g}_1} \cdot \varphi_{\mathfrak{g}_2}$.

The general aim of exercises (2)-(8) is to compare the behavior of $\varphi_{\mathfrak{sl}_2}(D)$ with that of the chromatic polynomial of a graph. In these exercises we use the form $2\langle\cdot,\cdot\rangle$ as the invariant form.

(2) * Prove that $\varphi_{\mathfrak{sl}_2}(D)$ depends only on the intersection graph $\Gamma(D)$ of the chord diagram D (see Section 4.8).

(3) Prove that the polynomial $\varphi_{\mathfrak{sl}_2}(D)$ has alternating coefficients.

(4) Show that for any chord diagram D the polynomial $\varphi_{\mathfrak{sl}_2}(D)$ is divisible by c.

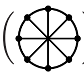
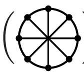
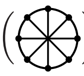
(5) * Prove that the sequence of coefficients of the polynomial $\varphi_{\mathfrak{sl}_2}(D)$ is unimodal (i.e. its absolute values form a sequence with only one maximum).

(6) Let D be a chord diagram with n chords for which $\Gamma(D)$ is a tree. Prove that $\varphi_{\mathfrak{sl}_2}(D) = c(c-2)^{n-1}$.

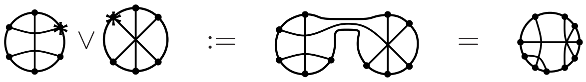
(7) Prove that the highest three coefficients of the polynomial $\varphi_{\mathfrak{sl}_2}(D)$ are

$$\varphi_{\mathfrak{sl}_2}(D) = c^n - e \cdot c^{n-1} + (e(e-1)/2 - t + 2q) \cdot c^{n-2} - \ldots\ ,$$

where e is the number of intersections of chords of D; t and q are the numbers of triangles and quadrangles of D respectively. A triangle is a subset of three chords of D with all pairwise intersections. A quadrangle of D is an unordered subset of four chords a_1, a_2, a_3, a_4 which form a circle of length four. This means that, after a suitable relabeling, a_1 intersects a_2 and a_4, a_2 intersects a_3 and a_1, a_3 intersects a_4 and a_2, a_4 intersects a_1 and a_3 and any other intersections are allowed. For example,

$e\Big($  $\Big) = 6,$ $t\Big($  $\Big) = 4,$ $q\Big($  $\Big) = 1$.

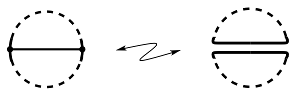
(8) (*A. Vaintrob* [**Vai2**]). Define *vertex multiplication* of chord diagrams as follows:



Of course, the result depends of the choice of vertices where multiplication is performed. Prove that for any choice

$$\varphi_{\mathfrak{sl}_2}(D_1 \vee D_2) = \frac{\varphi_{\mathfrak{sl}_2}(D_1) \cdot \varphi_{\mathfrak{sl}_2}(D_2)}{c} .$$

(9) (*S. Lando* [**LZ**], *B. Mellor* [**Mel2**]). Let $s(D)$ be the number of connected components of the curve obtained by doubling all chords of a chord diagram D, and N be a formal variable. Consider the adjacency matrix M of the intersection graph of D as a matrix over the field of two elements $\mathbb{F}_2 = \{0, 1\}$. Prove that $s(D) - 1$ is equal to the corank of M (over $\mathbb{F}_2$), and deduce from this that $\varphi^{St}_{\mathfrak{gl}_N}(D)$ depends only on the intersection graph $\Gamma(D)$.

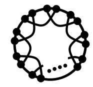


Essentially the same weight system was independently rediscovered by B. Bollobás and O. Riordan [**BR2**] who used it to produce a polynomial invariant of ribbon graphs generalizing the Tutte polynomial [**BR3**].

(10) (*D. Bar-Natan, S. Garoufalidis* [**BNG**]) Prove that the weight system $\varphi^{St}_{\mathfrak{gl}_N}$ satisfies the 2-term relations (see 4.8.5). (A similar fact for $\varphi^{St}_{\mathfrak{so}_N}$ can be found in [**Mel2**].)

(11) (*D. Bar-Natan, S. Garoufalidis* [**BNG**]) Let c_n be the coefficient of t^n in the Conway polynomial and D, a chord diagram of degree n. Prove that $\mathrm{symb}(c_n)(D)$ equals, modulo 2, the determinant of the adjacency matrix of the intersection graph $\Gamma(D)$.

(12) Let D_n be a chord diagram with n chords whose intersection graph is a circle. Prove that $\varphi^{St}_{\mathfrak{gl}_N}(D_n) = \varphi^{St}_{\mathfrak{gl}_N}(D_{n-2})$. Deduce from this that $\varphi^{St}_{\mathfrak{gl}_N}(D_n) = N^2$ for odd n, and $\varphi^{St}_{\mathfrak{gl}_N}(D_n) = N^3$ for even n.

$D_n =$ 

n chords

(13) Work out a proof of the theorem from Section 6.1.7 about $\mathfrak{sl}_N$ weight system with standard representation, similar to the one given in Section 6.1.6. Use the basis of the vector space $\mathfrak{sl}_N$ consisting of the matrices e_{ij} for $i \neq j$ and the matrices $e_{ii} - e_{i+1,i+1}$.

(14) Prove that $\varphi'^{St}_{\mathfrak{sl}_N} \equiv \varphi'^{St}_{\mathfrak{gl}_N}$.

Hint. $$\varphi'^{St}_{\mathfrak{sl}_N} = e^{-\frac{N^2-1}{N}\mathbf{I}_1} \cdot \varphi^{St}_{\mathfrak{sl}_N} = e^{-N\mathbf{I}_1} \cdot \varphi^{St}_{\mathfrak{gl}_N} = \varphi'^{St}_{\mathfrak{gl}_N}.$$

(15) Compare the symbol of the coefficient j_n of the Jones polynomial (section 3.6.2) with the weight system coming form $\mathfrak{sl}_2$, and prove that
$$\mathrm{symb}(j_n) = \frac{(-1)^n}{2}\varphi'^{St}_{\mathfrak{sl}_2} \ .$$

Hint. Compare the formula for $\varphi'^{St}_{\mathfrak{sl}_2}$ from the previous problem and the formula for symb(j_n) from Section 3.6.2, and prove that
$$(|s|-1) \equiv \#\{\text{chords } c \text{ such that } s(c) = 1\} \mod 2\ .$$

(16) Work out a proof of the theorem from Section 6.1.8 about $\mathfrak{so}_N$ weight system in standard representation. Use the basis of $\mathfrak{so}_N$ formed by matrices $e_{ij} - e_{ji}$ for $i < j$. (In case of difficulty consult [**BN0, BN1**].)

(17) Work out a proof, similar to the proof of the Proposition from Section 6.1.6, that $\varphi^{St}_{\mathfrak{so}_N}(D)$ depends only on the intersection graph of D.

(18) (*B. Mellor* [**Mel2**]). For any subset $J \subseteq [D]$, let M_J denotes *marked adjacency matrix* of the intersection graph of D over the filed $\mathbb{F}_2$, that is the adjacency matrix M only the diagonal elements corresponding to elements of J are switched to 1. Prove that
$$\varphi^{St}_{\mathfrak{so}_N}(D) = \frac{N^{n+1}}{2^n} \sum_{J\subseteq[D]} (-1)^{|J|} N^{-\mathrm{rank}(M_J)}\ ,$$
where the rank is computed as the rank of a matrix over $\mathbb{F}_2$. This gives another proof of the fact that $\varphi^{St}_{\mathfrak{so}_N}(D)$ depends only on the intersection graph $\Gamma(D)$.

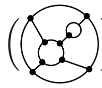
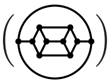
(19) Show that $N = 0$ and $N = 1$ are roots of polynomial $\varphi^{St}_{\mathfrak{so}_N}(D)$ for any chord diagram D.

(20) Let D be a chord diagram with n chords, such that the intersection graph $\Gamma(D)$ is a tree. Show that $\varphi^{St}_{\mathfrak{so}_N}(D) = \frac{1}{2^n}N(N-1)$.

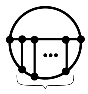
(21) Let D_n be a chord diagram from problem 12. Prove that

a). $\varphi^{St}_{\mathfrak{so}_N}(D_n) = \frac{1}{2}\left(\varphi^{St}_{\mathfrak{so}_N}(D_{n-2}) - \varphi^{St}_{\mathfrak{so}_N}(D_{n-1})\right)$;

b). $\varphi^{St}_{\mathfrak{so}_N}(D_n) = \frac{1}{(-2)^n} N(N-1)(a_{n-1}N - a_n)$, where the recurrent sequence a_n is defined by $\quad a_0 = 0, \quad a_1 = 1, \quad a_n = a_{n-1} + 2a_{n-2}$.

(22) Compute $\rho_{\mathfrak{sl}_2}\Bigg($  $\Bigg)$, $\rho_{\mathfrak{sl}_2}\Bigg($  $\Bigg)$, and show that these two closed diagrams are linearly independent.

(23) Let $\overline{t_n} \in \mathcal{C}_{n+1}$ be a closed diagram with n legs as shown in the figure.
Show that $\rho_{\mathfrak{sl}_2}(\overline{t_n}) = 2^n c$.

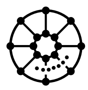
$\overline{t_n} =$ 

n legs

(24) Let $\overline{w_n} \in \mathcal{C}_n$ be a wheel with n spokes.
Show that

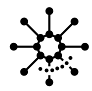
$\rho_{\mathfrak{sl}_2}(\overline{w_2}) = 4c, \quad \rho_{\mathfrak{sl}_2}(\overline{w_3}) = 4c$, and

$\rho_{\mathfrak{sl}_2}(\overline{w_n}) = 2c \cdot \rho_{\mathfrak{sl}_2}(\overline{w_{n-2}}) + 2\rho_{\mathfrak{sl}_2}(\overline{w_{n-1}}) - 2^{n-1}c.$

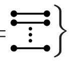
$\overline{w_n} =$ 

n spokes

(25) Let $w_{2n} \in \mathcal{B}_{2n}$ be a wheel with $2n$ spokes and $(\bullet\!\!-\!\!\bullet)^n \in \mathcal{B}_n$ be the n-th power of the element $\bullet\!\!-\!\!\bullet$ in the algebra $\mathcal{B}$.

$w_{2n} =$ 

$2n$ spokes

$(\bullet\!\!-\!\!\bullet)^n =$  n segments

Show that for the tensor $T_{\mathfrak{sl}_2}$ from 6.2.3 the following equality holds $T_{\mathfrak{sl}_2}(w_{2n}) = 2^{n+1}\ T_{\mathfrak{sl}_2}((\bullet\!\!-\!\!\bullet)^n)$.

Therefore, $\psi_{\mathfrak{sl}_2}(w_{2n}) = 2^{n+1}\ \psi_{\mathfrak{sl}_2}((\bullet\!\!-\!\!\bullet)^n)$.

(26) Let $p \in \mathcal{P}_n^k \subset \mathcal{C}_n$ be a primitive element of degree $n > 1$ with at most k external vertices. Show that $\rho_{\mathfrak{sl}_2}(p)$ is a polynomial in c of degree $\leqslant k/2$.

Hint. Use the theorem from 6.2.3 and the calculation of $\rho_{\mathfrak{sl}_2}(\overline{t_3})$ from exercise (23).

(27) Let $\varphi'_{\mathfrak{sl}_2}$ be the deframing of the weight system $\varphi_{\mathfrak{sl}_2}$ according to the procedure of Section 4.5.6. Show that for any element $D \in \mathcal{A}_n$, $\varphi'_{\mathfrak{sl}_2}(D)$ is a polynomial in c of degree $\leqslant [n/2]$

Hint. Use the previous exercise, exercise (8) of Chapter 4, and Section 5.5.2.

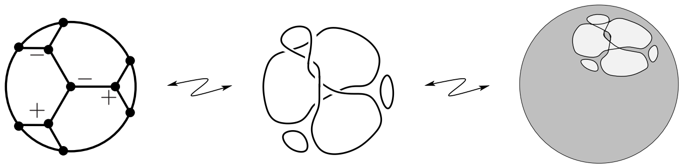
(28) Denote by V_k the k-dimensional irreducible representation of $\mathfrak{sl}_2$ (see Appendix A.1.1). Let $\varphi'^{V_k}_{\mathfrak{sl}_2}$ be the corresponding weight system. Show that for any element $D \in \mathcal{A}_n$ of degree n, $\varphi'^{V_k}_{\mathfrak{sl}_2}(D)/k$ is a polynomial in k of degree at most n.

Hint. The quadratic Casimir number in this case is $\frac{k^2-1}{2}$.

(29) Let $C \in \mathcal{C}_n$ $(n > 1)$ be a closed diagram (*primitive element*) which remains connected after deleting the Wilson loop. Prove that $\rho^{St}_{\mathfrak{gl}_N}(C) = \rho^{St}_{\mathfrak{sl}_N}(C)$.

Hint. For the Lie algebra $\mathfrak{gl}_N$ the tensor $J \in \mathfrak{gl}_N^{\otimes 3}$ lies in the subspace $\mathfrak{sl}_N^{\otimes 3}$.

(30) Consider a closed diagram $C \in \mathcal{C}_n$ and a $\mathfrak{gl}_N$-state s for it (see p. 183). Construct a surface $\Sigma_s(C)$ by attaching a disk to the Wilson loop, replacing each edge by a narrow band and gluing the bands together at trivalent vertices — flatly, if $s = 1$, and with a twist, if $s = -1$. Here is an example:

$C =$  $=: \Sigma_s(C)$.

a). Show that the surface $\Sigma_s(C)$ is orientable.

b). Compute the Euler characteristic of $\Sigma_s(C)$ in terms of C, and show that it depends only on the degree n of C.

c). Prove that $\rho^{St}_{\mathfrak{gl}_N}(C)$ is an odd polynomial for even n, and it is an even polynomial for odd n.

(31) Show that $N = 0$, $N = -1$, and $N = 1$ are roots of the polynomial $\rho^{St}_{\mathfrak{gl}_N}(C)$ for any closed diagram $C \in \mathcal{C}_n$ $(n > 1)$.

(32) Compute $\rho^{St}_{\mathfrak{gl}_N}(\overline{t_n})$, where $\overline{t_n}$ is the closed diagram from problem 23.

Answer. For $n \geqslant 1$, $\rho^{St}_{\mathfrak{gl}_N}(\overline{t_n}) = N^n(N^2 - 1)$.

(33) For the closed diagram $\overline{w_n}$ as in the problem 24, prove that $\rho^{St}_{\mathfrak{gl}_N}(\overline{w_n}) = N^2(N^{n-1} - 1)$ for odd n, and $\rho^{St}_{\mathfrak{gl}_N}(\overline{w_n}) = N(N^n + N^2 - 2)$ for even n.

Hint. Prove the recurrent formula $\rho^{St}_{\mathfrak{gl}_N}(\overline{w_n}) = N^{n-1}(N^2 - 1) + \rho^{St}_{\mathfrak{gl}_N}(\overline{w_{n-2}})$ for $n \geqslant 3$.

(34) Extend the definition of the weight system $\mathrm{symb}(c_n)(\cdot)$ of the coefficient c_n of the Conway polynomial to $\mathcal{C}_n$, and prove that

$$\mathrm{symb}(c_n)(C) = \sum_s \Big(\prod_v s(v)\Big)\delta_{1,|s|} \ ,$$

where the states s are precisely the same as in the theorem of Section 6.2.4 for the weight system $\rho^{St}_{\mathfrak{gl}_N}(\cdot)$. In other words, prove that $\mathrm{symb}(c_n)(C)$ equals to the coefficient at N in the polynomial $\rho^{St}_{\mathfrak{gl}_N}(C)$. In particular, show that $\mathrm{symb}(c_n)(\overline{w_n}) = -2$ for even n, and $\mathrm{symb}(c_n)(\overline{w_n}) = 0$ for odd n.

(35) Show that $N = 0$, and $N = 1$ are roots of the polynomial $\rho^{St}_{\mathfrak{so}_N}(C)$ for any closed diagram $C \in \mathcal{C}_n$ $(n > 0)$.

(36) a). Let $C \in \mathcal{C}$ be a closed diagram with at least one internal trivalent vertex. Prove that $N = 2$ is a root of the polynomial $\rho^{St}_{\mathfrak{so}_N}(C)$.

b). Deduce that $\rho^{St}_{\mathfrak{so}_2}(C) = 0$ for any primitive closed diagram C.

Hint. Consider the eight states that differ only on three edges meeting at an internal vertex (see p.186). Show that the sum over these eight states, $\sum \mathrm{sign}(s)2^{|s|}$, equals zero.

(37) Prove that $\rho^{St}_{\mathfrak{so}_N}(\overline{t_n}) = \frac{N-2}{2}\rho^{St}_{\mathfrak{so}_N}(\overline{t_{n-1}})$ for $n > 1$.
In particular, $\rho^{St}_{\mathfrak{so}_N}(\overline{t_n}) = \frac{(N-2)^n}{2^{n+1}}N(N-1)$.

(38) Using some bases in $\mathcal{A}_2$ and $\mathcal{B}_2$, find the matrix of the isomorphism χ, then calculate (i.e. express as polynomials in the standard generators) the values on the basic elements of the weight systems $\varphi_{\mathfrak{g}}$ and $\psi_{\mathfrak{g}}$ for the Lie algebras $\mathfrak{g} = \mathfrak{so}_3$ and $\mathfrak{g} = \mathfrak{gl}_N$ and check the validity of the relation $\beta \circ \psi = \varphi \circ \chi$ in this particular case.

(39) Prove that the mapping $\psi : \mathcal{B} \to S(\mathfrak{g})$ is well-defined, i.e. that the IHX relation follows from the Jacobi identity.

Chapter 7

Algebra of 3-graphs

The algebra of 3-graphs, introduced in [**DKC**], is obtained by a construction which is very natural from an abstract point of view. The elements of this algebra differ from closed diagrams in that they do not have any distinguished cycle, like Wilson loop; they differ from open diagrams in that these graphs are regular, without univalent vertices. So it looks as a simplification of both algebras $\mathcal{C}$ and $\mathcal{B}$. Strictly speaking, there are two different algebra structures on the same vector space Γ of 3-graphs given by the edge (section 7.2) and the vertex (section 7.3) multiplications. These algebras are closely related to the Vassiliev invariants in several ways. Namely,

- the vector space Γ is isomorphic to the subspace $\mathcal{P}^2$ of the primitive space $\mathcal{P}$ of Jacobi diagrams spanned by connected diagrams with 2 legs (section 7.4.2);
- the algebra Γ acts on the primitive space $\mathcal{P}$ in two ways, via edge, and via vertex multiplications (see sections 7.4.1 and 7.4.3);
- these actions behave nicely with respect to Lie algebra weight systems (see chapter 6); as a consequence, the algebra Γ is as good a tool for the proof of existence of non-Lie-algebraic weight systems as the algebra Λ in Vogel's original approach (section 7.7);
- and finally, the space Γ describes the combinatorics of finite type invariants of integral homology 3-spheres in much the same way as the space of chord diagrams describes the combinatorics of Vassiliev knot invariants. This topic, however, lies outside of the scope of our book and we refer an interested reader to [**Oht1**].

Unlike $\mathcal{C}$ and $\mathcal{B}$, the algebra Γ does not have any natural coproduct.

7.1. The space of 3-graphs

7.1.1. Definition. A *3-graph* is a connected 3-valent graph with a fixed *rotation*. The *rotation* is the choice of a cyclic order of edges at every vertex, i. e. one of the two cyclic permutations in the set of three edges adjacent to this vertex. Note that the number of vertices of a 3-valent graph is always even, and the number of edges a multiple of 3. Half the number of the vertices will be referred to as the *degree* (or *order*) of a 3-graph.

In this definition, *graphs* are allowed to have multiple edges and loops. In the case of graphs with loops, the notion of rotation requires the following refinement: the cyclic order is introduced not in the set of *edges* incident to a given vertex, but in the corresponding set of *half-edges*.

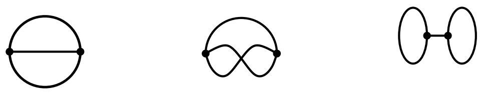
A topology free combinatorial definition of the set of half-edges can be given as follows. Let E, V be the sets of edges and vertices, respectively, of the graph under study. Then the set of half-edges is a 'double fibering' $E \stackrel{\alpha}{\leftarrow} H \stackrel{\beta}{\rightarrow} V$, that is a set H supplied with two projections $\alpha : H \to E$ and $\beta : H \to V$ such that (a) the inverse image of every edge $\alpha^{-1}(e)$ consists of two elements, (b) the cardinality of each set $\beta^{-1}(v)$ equals the valency of the vertex v, (c) for any point $h \in H$ the vertex $\beta(h)$ is one of the two endpoints of the edge $\alpha(h)$. Then the rotation is a cyclic permutation in the inverse image of each vertex $\beta^{-1}(v)$.

One can forget about edges and vertices altogether and adopt the following clear-cut, although less pictorial, definition of a 3-graph: this is *a set of cardinality* $6n$ *equipped with two permutations, one with the cyclic structure* $(3)(3)\dots(3)$*, another with the cyclic structure* $(2)(2)\dots(2)$*, with the requirement that there is no proper subset invariant under both permutations.* The relation to the previous definition consists in that H is the set of half-edges, the first permutation corresponds to the rotation in the vertices and the second, to the transition from one endpoint of an edge to another. A vertex (resp. edge) of the graph is an orbit of the first (resp. second) permutation.

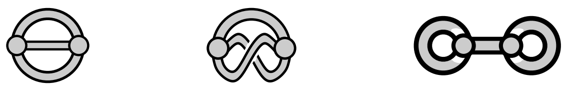
7.1.2. Definition. Two 3-graphs are said to be *isomorphic* if there is a one-to-one correspondence between their sets of half-edges which preserves the rotation and induces a usual graph isomorphism — or, in other words, which preserves both structural permutations.

Remark. It is convenient to include the circle into the set of 3-graphs viewing it as a graph with a zero number of vertices.

Example. Up to an isomorphism, there are three different 3-graphs of degree 1:



Remark. Graphs with rotation are often called *ribbon graphs* (see [**LZ**]), because such a graph can be represented as an orientable surface with boundary having the form of a narrow strip along the skeleton that has the shape of the given graph:

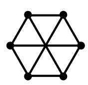


The construction of the surfaces starting from a graph proceeds as follows. Replace each vertex and each edge by an oriented disk (imagine that disks for vertices are 'round' while the disks for edges are 'oblong'). Then glue them together along pieces of boundary in agreement with the orientation and so that the attachment of edge-disks to vertex-disks follows the cyclic order prescribed at the vertices in the direction of the boundary induced by the chosen orientation of the disks.

7.1.3. Definition. The space Γ_n is the quotient space of the vector space over $\mathbb{Q}$ spanned by connected 3-graphs of degree n (i. e., having $2n$ vertices) modulo the AS and IHX relations (see p. 130). By definition, Γ_0 is one-dimensional and spanned by the circle.

That is, the space of 3-graphs Γ differs from the space of open Jacobi diagrams $\mathcal{B}$ (p. 141) in that here we consider only connected diagrams without univalent vertices, while the space $\mathcal{B}$ is spanned by diagrams with necessarily at least one univalent vertex in each connected component.

7.1.4. Exercise.
Check that the 3-graph on the right is equal to zero as an element of the space Γ_3.



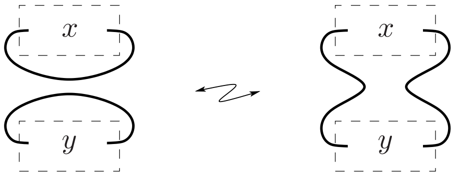
7.2. Edge multiplication

In the graded space

$$\Gamma = \Gamma_0 \oplus \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \dots$$

there is a natural structure of a commutative algebra.

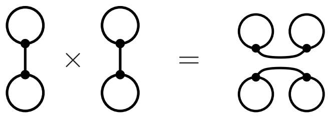
7.2.1. Definition. The *edge product* of two non-zero 3-graphs x and y is the 3-graph obtained in the following way. Choose arbitrarily an edge in x and an edge in y. Cut each of these two edges in the middle and attach the endpoints that appear on one edge, to the endpoints appearing on the other.



$= x \cdot y$.

The edge product of 3-graphs can be understood as the connected sum of x and y along the chosen edges, and also as the result of insertion of one graph, say x, into an edge of y.

Remark. The product of two connected graphs may yield a disconnected graph, for example:



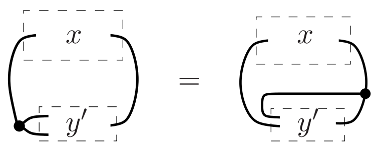
This happens, however, only in the case when each of the two graphs becomes disconnected after cutting the chosen edge, and in this case both graphs are 0 modulo AS and IHX relations (see Lemma 7.2.7(b) below).

7.2.2. Theorem. *The edge product of 3-graphs, viewed as an element of the space* Γ*, is well-defined.*

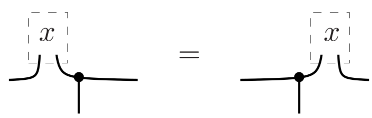
Note that, as soon as this assertion is proved, one immediately sees that the multiplication is commutative.

The claim that the product is well-defined consists of two parts. First, that modulo the AS and IHX relations the product does not depend on the *choice* of the two edges x and y which are cut and pasted. Second, that it does not depend on the *way* they are put together (clearly, the two loose ends of one graph can be glued to the two loose ends of another graph in two different ways). These two facts are established in the following two lemmas.

7.2.3. Lemma.
(**a**) *A vertex with an attached edge can be dragged through a subgraph that has two legs.*



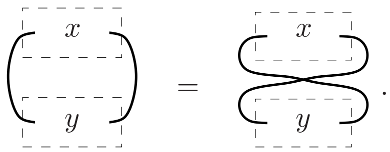
(**b**) *A subgraph with two legs can be carried through a vertex.*



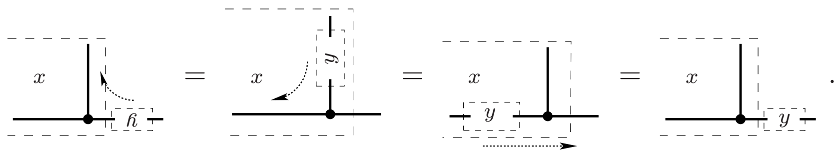
Proof. Taking a closer look at these pictures, the reader will understand that assertions (a) and (b) have exactly the same meaning and in fact are nothing but a particular case ($k = 1$) of the Kirchhoff law (see page 133). $\square$

Another wording for Lemma 7.2.3 is: *the results of insertion of a 3-graph x into two adjacent edges of 3-graph y, are equal.* Since y is connected, this implies that the product does not depend on the choice of the edge in y.

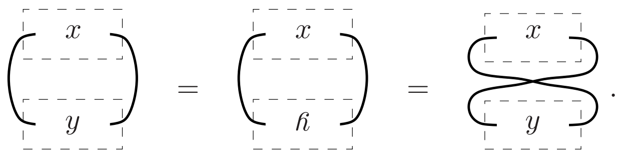
7.2.4. Lemma. *Two different ways to glue together two graphs with edges cut give the same result in the space* Γ*:*



Proof. Choosing the vertex of x which is nearest to the right exit of y in the product, one can, by Lemma 7.2.3, effectuate the following manoeuvres:



Therefore,



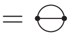
The lemma is proved, and edge multiplication of 3-graphs is thus well defined. $\square$

The edge product of 3-graphs is extended to the product of arbitrary elements of the space Γ in the usual way, by linearity.

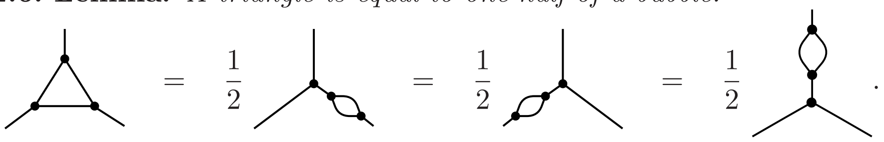
Corollary. *The edge product in* Γ *is well-defined, associative, and distributive.*

This follows from the fact that the product of 3-graphs is well-defined and a linear combination of either AS or IHX type, when multiplied by an arbitrary graph, is a combination of the same type. Distributivity and associativity are obvious.

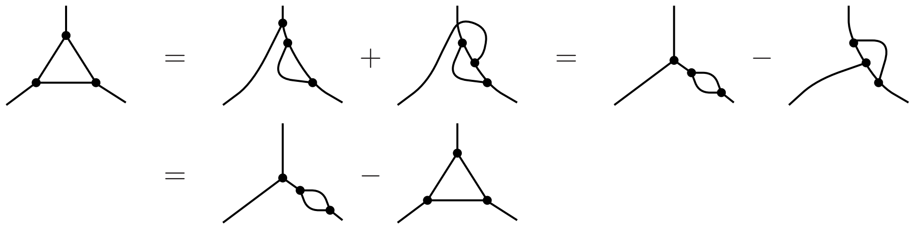
7.2.5. Some identities. We will prove several identities elucidating the structure of the 3-graph algebra. They are easy consequences of the defining relations. Note that these identities make sense and hold also in the algebras of closed and open diagrams, $\mathcal{C}$ and $\mathcal{B}$.

The first identity (Lemma 7.2.6) links the two operations defined in the set of 3-graphs: *the insertion of a triangle into a vertex* (which is a particular case of the vertex multiplication we shall study later) and *the insertion of a bubble into an edge.* To insert a bubble into an edge of a graph is the same thing as to multiply this graph by the element $\beta =$  $\in \Gamma_1$. The correctness of multiplication in Γ implies that the result of the bubble insertion does not depend on the chosen edge; then it follows from Lemma 7.2.6 that the result of the triangle insertion does not depend on the choice of the vertex.

7.2.6. Lemma. *A triangle is equal to one half of a bubble:*



Proof.



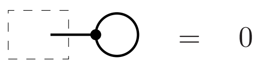
□

Corollary. *If a bubble (or a triangle) travels along a connected component of a diagram, the latter does not change.*

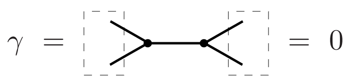
It follows that there is a well-defined operator of bubble insertion $\Gamma \to \Gamma : x \mapsto \beta x$ that raises the grading by 1. For a long time it was conjectured that this operator is injective. Recently, Pierre Vogel [**Vo2**] proved that it has non-trivial kernel.

The second lemma describes two classes of 3-graphs which are equal to 0 in the algebra Γ, i. e. modulo the AS and IHX relations.

7.2.7. Lemma.
(**a**) *A graph with a loop is* 0 *in* Γ.

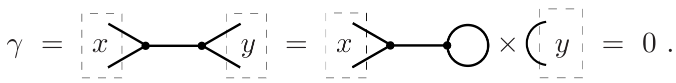


(**b**) *More generally, if the edge connectivity of the graph* γ *is* 1, *i. e., it becomes disconnected after removal of an edge, then* $\gamma = 0$.



Proof. (a) A graph with a loop is zero because of antisymmetry. Indeed, changing the rotation in the vertex of the loop yields a graph which is, on one hand, isomorphic to the initial one, and on the other hand, differs from it by a sign.

(b) Such a graph can be represented as a product of two graphs, one of which is a graph with a loop, i.e., zero according to (a):



□

7.2.8. The Zoo. The next table shows the dimensions d_n and displays the bases of the vector spaces Γ_n for $n \leqslant 11$, obtained by computer calculations.

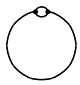
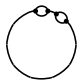
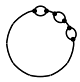
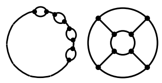
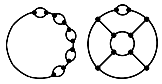
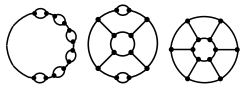
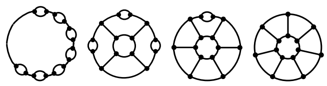
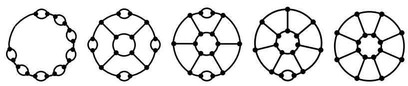
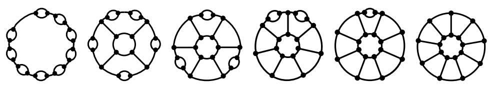
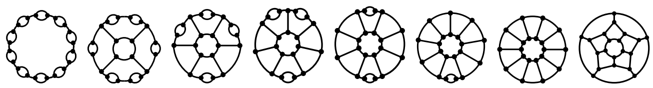
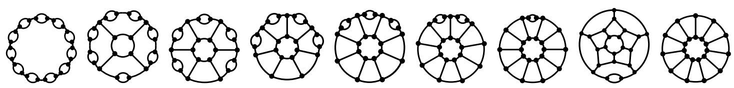
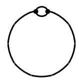
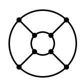
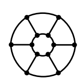
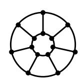
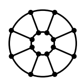
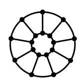
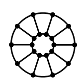
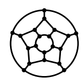
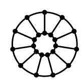
n	d_n	additive generators
1	1	
2	1	
3	1	
4	2	
5	2	
6	3	
7	4	
8	5	
9	6	
10	8	
11	9	

Table 7.2.8.1. Additive generators of the algebra of 3-graphs Γ

Note that the column for d_n coincides with the column for $k = 2$ in the table of primitive spaces on page 139. This will be proved in Proposition 7.4.2.

From the table one can see that the following elements can be chosen as multiplicative generators of the algebra Γ up to degree 11 (notations in the lower line mean: β — 'bubble', ω_i — 'wheels', δ — 'dodecahedron'):

1	4	6	7	8	9	10		11
								
β	ω_4	ω_6	ω_7	ω_8	ω_9	ω_{10}	δ	ω_{11}

7.2.9. Conjecture. *The algebra* Γ *is generated by planar graphs.*

The reader might have noted that the table of additive generators does not contain the elements ω_4^2 of degree 8 and $\omega_4\omega_6$ of degree 10. This is not accidental. It turns out that the following relations (found by A. Kaishev [**Kai**]) hold in the algebra Γ:

$$\begin{aligned}\omega_4^2 &= \frac{5}{384}\beta^8 - \frac{5}{12}\beta^4\omega_4 + \frac{5}{2}\beta^2\omega_6 - \frac{3}{2}\beta\omega_7,\\ \omega_4\omega_6 &= \frac{305}{27648}\beta^{10} - \frac{293}{864}\beta^6\omega_4 + \frac{145}{72}\beta^4\omega_6 - \frac{31}{12}\beta^3\omega_7 + 2\beta^2\omega_8 - \frac{3}{4}\beta\omega_9.\end{aligned}$$

In fact, as we shall see in section 7.3, it is true in general that *the product of an arbitrary pair of homogeneous elements of* Γ *of positive degree belongs to the ideal generated by* β.

Since there are non-trivial relations between the generators, the algebra of 3-graphs, in contrast to the algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, is not free and hence does not possess the structure of a Hopf algebra.

7.3. Vertex multiplication

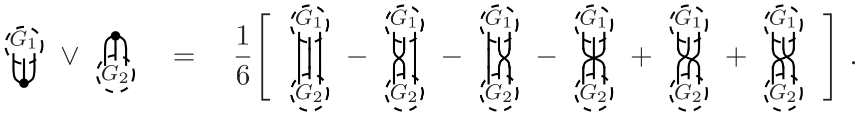
In this section we define the vertex multiplication on the space

$$\Gamma_{\geqslant 1} = \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4 \oplus \dots\ .$$

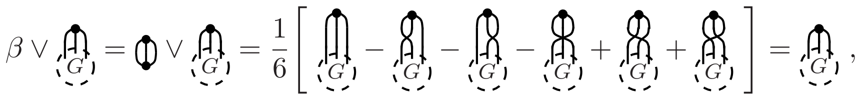
It is different from Γ in that it does not contain the one-dimensional subspace Γ_0 spanned by the circle, the only graph without vertices.

7.3.1. Definition. The *vertex product* of two 3-graphs G_1 and G_2 is the 3-graph obtained in the following way. Choose arbitrarily a vertex in G_1 and a vertex in G_2. Cut each of these two vertices together with the half-edges incident to them. Then attach the endpoints that appear on one graph, to the endpoints appearing on the other. There are six possibilities for this. Take the alternating average of all of them. We take with the sign minus those three attachments where the cyclic orders at two removed vertices are coherent and with the plus sign those three which switch the cyclic orders.

Pictorially, if we draw the graphs G_1, G_2 like this $G_1 =$ , $G_2 =$ , then to draw their vertex product we have to merge them and insert a permutation of the three strands between them. Then we take the result with the sign of the permutation and average it over all six permutations:



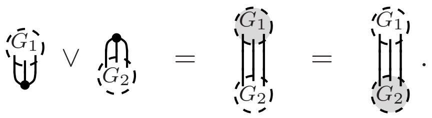
As an example, let us compute the vertex multiplication by the theta graph:



because all summands in the brackets are equal to each other due to the AS relation. So β is going to be the unit for the algebra $\Gamma_{\geqslant 1}$ with vertex multiplication.

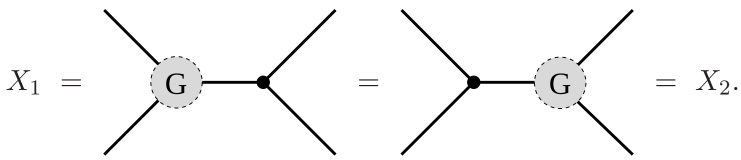
The vertex product of 3-graphs can be understood as the average result of insertions of one graph, say G_1, into a vertex of G_2 over all possible ways.

To shorten the notation, we will draw diagrams with shaded disks, understanding them as alternated linear combinations of six graphs like above. For example:



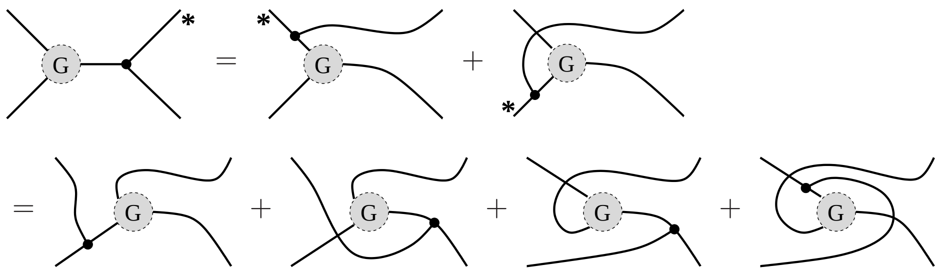
7.3.2. Theorem. *The vertex multiplication in* $\Gamma_{\geqslant 1}$ *is well-defined, commutative and associative.*

Proof. We must only prove that the following equality holds due to the AS and IHX relations:



where G denotes an arbitrary subgraph with three legs (and each picture is the alternating sum of six diagrams).

By the Kirchhoff law we have:



(the stars indicate the place where the tail of the 'moving electron' is fixed in Kirchhoff's relation). Now, in the last line the first and the fourth diagrams are equal to X_2, while the sum of the second and the third diagrams is equal to $-X_1$ (again, by an application of Kirchhoff's rule). We thus have $2X_1 = 2X_2$ and therefore $X_1 = X_2$.

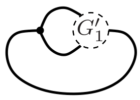
Commutativity and associativity are obvious. □

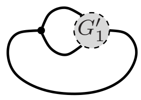
7.3.3. Remark. The vertex multiplication shifts the degrees by one. So it is an operation of degree -1: $\Gamma_n \vee \Gamma_m \subset \Gamma_{n+m-1}$.

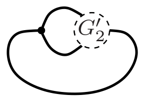
7.3.4. Proposition. (**Relation between two multiplications in Γ.**) *The edge multiplication "$\cdot$" in the algebra of 3-graphs Γ is related to the vertex multiplication "$\vee$" as follows:*

$$G_1 \cdot G_2 = \beta \cdot (G_1 \vee G_2)$$

Proof. Choose a vertex in each of the given graphs G_1 and G_2 and call what remains G_1' and G_2', respectively:

$G_1 =$ 

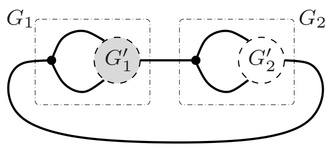
$=$ 

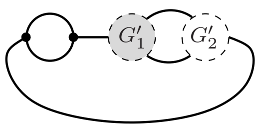
, $G_2 =$ 

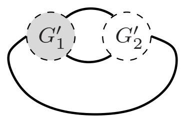
,

where, as explained above, the shaded region means an alternating average over the six permutations of the three legs.

Then, by theorem 7.3.2 we have:

$G_1 \cdot G_2 =$ 

$=$ 

$= \beta \cdot$ 

$= \beta \cdot (G_1 \vee G_2)$.

□

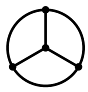
7.4. Action of Γ on the primitive space $\mathcal{P}$

7.4.1. Edge action of Γ on $\mathcal{P}$. As we know (section 5.5) the primitive space $\mathcal{P}$ of the algebra $\mathcal{C}$ is spanned by connected diagrams, i. e. closed Jacobi diagrams which remain connected after the Wilson loop is stripped off. It is natural to define the edge action of Γ on such a diagram P simply by edge multiplication of a graph $G \in \Gamma$ with P, using an internal edge of P. More specifically, we cut an arbitrary edge of the graph G. Then we cut an edge in P not lying on the Wilson loop and attach the new endpoints of P to those of G, like we did before for the algebra Γ itself. The graph that results is $G \cdot P$; it has one Wilson loop and is connected, hence it belongs to $\mathcal{P}$. Because of the connectivity of P and lemmas 7.2.3, 7.2.4 this result $G \cdot P$ does not depend on the choices of the edge in G and the internal edge in P. So this action is well-defined. The action is compatible with the gradings: $\Gamma_n \cdot \mathcal{P}_m \subset \mathcal{P}_{n+m}$. We shall use the edge action in the next chapter in conjunction with Lie-algebraic weight systems.

7.4.2. Proposition. *The vector space $\Gamma = \Gamma_0 \oplus \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \dots$ is isomorphic to the subspace $\mathcal{P}^2 = \mathcal{P}_1^2 \oplus \mathcal{P}_2^2 \oplus \mathcal{P}_3^2 \oplus \mathcal{P}_4^2 \oplus \dots \subset \mathcal{P}$ of closed diagrams generated by connected diagrams with 2 legs as a graded vector space: $G_n \cong \mathcal{P}_{n+1}^2$ for $n \geqslant 0$.*

Proof. The required mapping $\Gamma \to \mathcal{P}^2$ is given by the edge action of 3-graphs on the element $\Theta \in \mathcal{P}^2$ represented by the chord diagram with a single chord, $G \mapsto G \cdot \Theta$. The inverse mapping can be constructed in a natural way. For a closed diagram $P \in \mathcal{P}^2$ strip off the Wilson loop and glue together the two loose ends of the obtained diagram. The result will be a 3-graph of degree one less than P. Obviously, this mapping is well-defined and inverse to the previous one. $\square$

7.4.3. Vertex action of Γ on $\mathcal{P}$. To perform the vertex multiplication, we need at least one vertex in each of the factors. So the natural action we are speaking about is the action of the algebra $\Gamma_{\geqslant 1}$ (with the vertex product) on the space $\mathcal{P}_{>1}$ of primitive elements of degree strictly greater than 1. The action is defined in a natural way. Pick a vertex in $G \in \Gamma_{\geqslant 1}$, and an internal vertex in $P \in \mathcal{P}_{>1}$. Then the result $G \vee P$ will be the alternated average over all six ways to insert G with the removed vertex into P with the removed internal vertex. Again, because of connectivity of P and by Theorem 7.3.2, the action is well-defined. This action decreases the total grading by 1 and preserves the number of legs: $\Gamma_n \vee \mathcal{P}_m^k \subset \mathcal{P}_{n+m-1}^k$.

The simplest element of $\mathcal{P}$ on which one can act in this way is the "Mercedes diagram" $\bar{t}_1 =$ . This action has the following important property.

7.4.4. Lemma. (**a**) *The map $\Gamma_{\geqslant 1} \to \mathcal{P}$ defined as $G \mapsto G \vee \bar{t}_1$ is an inclusion, i. e. it has no kernel.*

(**b**) *The vertex and edge actions are related to each other via the formula:*

$$G \vee \bar{t}_1 = \frac{1}{2} G \cdot \Theta \ .$$

Proof. Indeed, $\bar{t}_1 = \frac{1}{2}\beta \cdot \Theta$. Therefore, $G \vee \bar{t}_1 = \frac{1}{2}(G \vee \beta) \cdot \Theta = \frac{1}{2} G \cdot \Theta$. Since the mapping $G \mapsto G \cdot \Theta$ is an isomorphism (proposition 7.4.2), the mapping $G \mapsto G \vee \bar{t}_1$ is also an isomorphism $\Gamma_{\geqslant 1} \cong \mathcal{P}_{>1}^2$. $\square$

7.4.5. Multiplication in the primitive space $\mathcal{P}$. By construction, the space of primitive elements $\mathcal{P}$ of the algebra $\mathcal{C}$ (see Sec. 5.5), considered by itself, does not possess any a priori defined multiplicative structure. Primitive elements only generate the algebra $\mathcal{C}$ in the same way as the variables $x_1, \ldots, x_n$ generate the polynomial algebra $\mathbb{R}[x_1, \ldots, x_n]$. However, the link between the space $\mathcal{P}$ and the algebra of 3-graphs Γ allows one to introduce in $\mathcal{P}$ a natural (non-commutative) multiplication. This can be done using two auxiliary maps between Γ and $\mathcal{P}$.

1. There is a linear inclusion $i : \Gamma_n \to \mathcal{P}_{n+1}$, defined on the generators as follows: we cut an arbitrary edge of the graph and attach a Wilson loop to the two resulting endpoints. The orientation of the Wilson loop can be chosen arbitrarily. This is precisely the inclusion from proposition 7.4.2: $G \mapsto G \cdot \Theta$.

2. There is a projection $\pi : \mathcal{P}_n \to \Gamma_n$, which consists in introducing the rotation in the vertices of the Wilson loop according to the rule 'forward–sideways–backwards' and then forgetting the fact that the Wilson loop was distinguished.

The composition homomorphism $\Gamma \to \Gamma$ in the sequence

$$\Gamma \xrightarrow{i} \mathcal{P} \xrightarrow{\pi} \Gamma$$

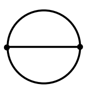
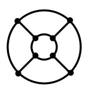
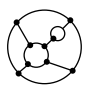
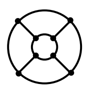
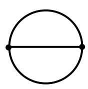
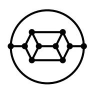
coincides with the multiplication by the bubble $G \mapsto \beta \cdot G$.

The edge action $\cdot : \Gamma \times \mathcal{P} \to \mathcal{P}$ gives rise to an operation $* : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ which can be defined by the rule

$$p * q = \pi(p) \cdot q,$$

where $\pi : \mathcal{P} \to \Gamma$ is the homomorphism of forgetting the Wilson loop defined above.

The operation $*$ is, in general, non-commutative:

 $*$  $=$ , but  $*$  $=$ .

These two elements of the space $\mathcal{P}$ are different; they can be distinguished, e. g., by the $\mathfrak{sl}_2$-invariant (see exercise 22 at the end of the next chapter). However, π projects these two elements into the same element $\beta \cdot \omega_4 \in \Gamma_5$.

7.4.6. Conjecture. *The algebra of primitive elements $\mathcal{P}$ has no divisors of zero with respect to multiplication $*$.*

7.5. Vogel's algebra Λ

In this section, we describe the relation between the algebra Γ and Vogel's algebra Λ.

Diagrams with 1- and 3-valent vertices can be considered with different additional structures on the set of univalent vertices (legs). If there is no structure, then we get the notion of an *open Jacobi diagram*. The space of open diagrams is considered modulo AS and IHX relations. If the legs are attached to a circle or a line, then we obtain *closed Jacobi diagrams*. In the space spanned by closed diagrams, AS, IHX and STU relations are used. Diagrams with a linear order (numbering) on the set of legs, considered modulo AS and IHX, but without STU relations, will be referred to as *fixed diagrams*.

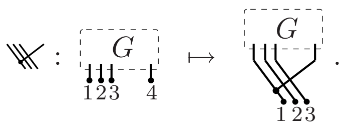
We will denote the set of all such diagrams equipped, as usual, with a rotation in the 3-valent vertices, by $\boldsymbol{X}$. By definition, speaking about the numbering means that an isomorphism between two such diagrams must preserve the numbering of the legs. In the set $\boldsymbol{X}$ there are two gradings: by the number of legs (denoted by the superscript) and by half the total number of vertices (denoted by the subscript).

7.5.1. Definition. The vector space spanned by connected fixed diagrams with k legs modulo the usual AS and IHX relations

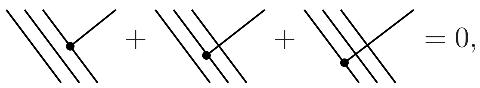
$$\mathcal{X}_n^k = \langle \boldsymbol{X}_n^k \rangle / \langle AS, IHX \rangle,$$

is called the *space of fixed diagrams* of degree n with k legs.

Remark. There is a lot of relations between the spaces $\mathcal{X}^k$. For example, one may think about the diagram  as of a linear operator from $\mathcal{X}^4$ to $\mathcal{X}^3$. Namely, it acts on an element G of $\mathcal{X}^4$ as follows

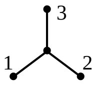


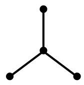
Exercise. Prove the following relation



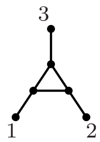
between the three linear operators from $\mathcal{X}^4$ to $\mathcal{X}^3$.

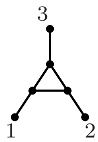
The space of open diagrams $\mathcal{B}^k$ studied in Chapter 5 is the quotient of $\mathcal{X}^k$ by the permutation group S_k acting by renumbering the legs. The mapping of taking the quotient $\mathcal{X} \to \mathcal{B}$ has a nontrivial kernel, for example, a tripod which is nonzero in $\mathcal{X}^3$ becomes zero in $\mathcal{B}$:

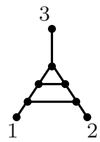
$0 \neq$ 

$\longmapsto$  $= 0.$

7.5.2. Definition. The algebra Λ is the subspace of all elements of $\mathcal{X}^3$, antisymmetric with respect to the permutations of legs. The multiplication is defined on the generators via a sort of vertex multiplication. Namely, choose a vertex in the first diagram, remove it together with 3 adjacent halfedges and insert the second diagram instead — in compliance with the rotation. The fact that this operation is well-defined, is proved in the same way as for the vertex multiplication in Γ. Since antisymmetry is presupposed, we do not need to take the alternated average over the six ways of insertion, like in Γ, — all the six summands will be equal to each other.

Example. 

$\vee$ 

$=$ 

.

Conjecturally, the antisymmetry requirement in this definition is superfluous:

7.5.3. Conjecture. $\Lambda = \mathcal{X}^3$, *i.e., any fixed diagram with 3 legs is antisymmetric with respect to leg permutations.*

Multiplication in the algebra Λ naturally generalizes to the action of Λ on different spaces generated by 1- and 3-valent diagrams, e. g. the space of open diagrams $\mathcal{B}$ and the space of 3-graphs Γ. The same argument as above leads to the following theorem.

7.5.4. Theorem. *The action of the algebra Λ on any space spanned by diagrams modulo AS and IHX relations is well-defined.*

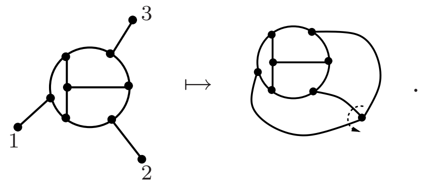
7.5.5. Relation between Λ and Γ. Recall that by $\Gamma_{\geqslant 1}$ we denoted the direct sum of all homogeneous components of Γ except for Γ_0:

$$\Gamma_{\geqslant 1} = \bigoplus_{n=1}^{\infty} \Gamma_n.$$

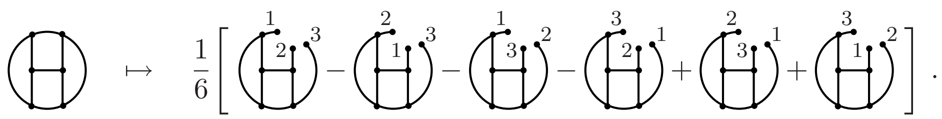
The vector space $\Gamma_{\geqslant 1}$ is an algebra with respect to vertex multiplication.

There are two naturally defined maps between Λ and $\Gamma_{\geqslant 1}$:

- $\Lambda \to \Gamma_{\geqslant 1}$. Add three half-edges and a vertex to an element of Λ in agreement with the leg numbering:



- $\Gamma_{\geqslant 1} \to \Lambda$. Choose an arbitrary vertex of a 3-graph, delete it together with the three adjacent half-edges and antisymmetrize:



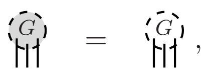
It is fairly evident that these maps are mutually inverse (in particular, this implies that the map $\Gamma_{\geqslant 1} \to \Lambda$ is well-defined). We thus arrive at the following result.

Proposition. *The vector spaces $\Gamma_{\geqslant 1}$ and Λ are isomorphic.*

It is evident by definition that under this isomorphism the multiplication in Λ corresponds to the vertex multiplication in Γ, therefore we have:

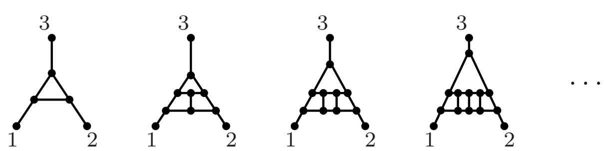
Corollary. *The algebra Λ is isomorphic to $(\Gamma_{\geqslant 1}, \vee)$.*

Remark. If conjecture 7.5.3 is true for $k = 3$ then all the six terms (together with their signs) in the definition of the map $\Gamma_{\geqslant 1} \to \Lambda$ are equal to each other. This means that there is no need to antisymmetrize. What we do is remove one vertex (with a small neighbourhood) and number the three legs obtained according to their cyclic ordering at the deleted vertex. Also this would simplify the definition of the vertex multiplication in section 7.3 because in this case



and we truly insert one graph in a vertex of another.

Conjecture. ([**Vo1**]) *The algebra Λ is generated by the elements t and x_k with odd $k = 3, 5, \ldots$:*



t $\qquad$ x_3 $\qquad$ x_4 $\qquad$ x_5 $\qquad$ $\ldots$

7.6. Lie algebra weight systems for the algebra Γ

The construction of the Lie algebra weight system $\eta_{\mathfrak{g}}(\cdot)$ for the algebra of 3-graphs proceeds in the same way as for the algebra $\mathcal{C}$ (Sec. 6.2), using the structure tensor J. Since 3-graphs have no univalent vertices, this weight system takes values in the ground field $\mathbb{C}$. For a graph $G \in \Gamma$ we put $\eta_{\mathfrak{g}}(G) := T_{\mathfrak{g}}(G) \in \mathfrak{g}^0 \cong \mathbb{C}$.

When computing the weight systems $\eta_{\mathfrak{sl}_N}(\cdot)$, and $\eta_{\mathfrak{so}_N}(\cdot)$ it is important to remember that their value on the circle (i.e. the unit of algebra Γ, or in other words, the 3-graph without vertices) is equal to the dimension of the corresponding Lie algebra, N^2-1, and $\frac{1}{2}N(N-1)$. However, when we apply state sum formulas form sections 6.2.4 and 6.2.6 for graphs of degree $\geqslant 1$ we replace every circle in the curve obtained by the resolutions just by N.

7.6.1. Changing the bilinear form. Tracing the construction of $\eta_{\mathfrak{g}}(\cdot)$, it is easy to see that the function $\eta_{\mathfrak{g},\lambda}(\cdot)$, corresponding to the form $\lambda\langle\cdot,\cdot\rangle$ is proportional to $\eta_{\mathfrak{g}}(\cdot)$:

$$\eta_{\mathfrak{g},\lambda}(G) = \lambda^{-n}\eta_{\mathfrak{g}}(G)$$

for $G \in \Gamma_n$.

7.6.2. Proposition. Multiplicativity with respect to the edge product in Γ. *For a simple Lie algebra $\mathfrak{g}$ and any choice of the ad-invariant non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ the function $\frac{1}{\dim\mathfrak{g}}\eta_{\mathfrak{g}} : \Gamma \to \mathbb{C}$ is multiplicative with respect to the edge product in Γ.*

Proof. This fact follows from the property that up to proportionality the quadratic Casimir tensor of a simple Lie algebra is the only ad-invariant, symmetric, non-degenerate tensor from $\mathfrak{g}\otimes\mathfrak{g}$.

Cut an arbitrary edge of the graph G_1 and consider the tensor that corresponds to the obtained graph with two univalent vertices. This tensor is proportional to the quadratic Casimir tensor $c \in \mathfrak{g}\otimes\mathfrak{g}$:

$$a\cdot c = a\sum_{i=1}^{\dim\mathfrak{g}} e_i\otimes e_i\ .$$

Now, $\eta_{\mathfrak{g},K}(G_1)$ can be obtained by contracting these two tensor factors. This gives $\eta_{\mathfrak{g},K}(G_1) = a\dim\mathfrak{g}$ because of orthonormality of the basis $\{e_i\}$. So we can find the coefficient $a = \frac{1}{\dim\mathfrak{g}}\eta_{\mathfrak{g}}(G_1)$. Similarly, for the second graph G_2 we get the tensor $\frac{1}{\dim\mathfrak{g}}\eta_{\mathfrak{g}}(G_2)\cdot c$. Now, if we put together one pair of univalent vertices of the graphs G_1 and G_2 thus cut, then the partial contraction of the element $c\otimes c \in \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ will give an element $\frac{1}{(\dim\mathfrak{g})^2}\eta_{\mathfrak{g}}(G_1)\cdot\eta_{\mathfrak{g}}(G_2)c \in \mathfrak{g}\otimes\mathfrak{g}$.

But, on the other hand, this tensor equals $\frac{1}{\dim \mathfrak{g}}\eta_{\mathfrak{g}}(G_1 \cdot G_2)c \in \mathfrak{g} \otimes \mathfrak{g}$. This proves the multiplicativity of the function $\frac{1}{\dim \mathfrak{g}}\eta_{\mathfrak{g}}$. $\square$

7.6.3. Compatibility with the edge action of Γ on $\mathcal{C}$. Recall the definition of the edge action of 3-graphs on closed diagrams (see Sec. 7.4.1). We choose an edge in $G \in \Gamma$ and an internal edge in $C \in \mathcal{C}$, and then take the connected sum of G and C along the chosen edges. In fact, this action depends on the choice of the connected component of $C \setminus \{\text{Wilson loop}\}$ containing the chosen edge. It is well defined only on the primitive subspace $\mathcal{P} \subset \mathbb{C}$. In spite of this indeterminacy we have the following lemma.

Lemma. *For any choice of the gluing edges, $\rho_{\mathfrak{g}}(G \cdot C) = \frac{\eta_{\mathfrak{g}}(G)}{\dim \mathfrak{g}}\rho_{\mathfrak{g}}(C)$.*

Indeed, in order to compute $\rho_{\mathfrak{g}}(C)$ we assemble the tensor $T_{\mathfrak{g}}(C)$ from elementary piece tensors gluing them along the edges by contraction with the quadratic Casimir tensor c. By the previous argument, to compute the tensor $T_{\mathfrak{g}}(G \cdot C)$ one must use the tensor $\frac{1}{\dim \mathfrak{g}}\eta_{\mathfrak{g}}(G) \cdot c$ instead of simply c for the chosen edge. This gives the coefficient $\frac{1}{\dim \mathfrak{g}}\eta_{\mathfrak{g}}(G)$ in the expression for $\rho_{\mathfrak{g}}(G \cdot C)$ as compared with $\rho_{\mathfrak{g}}(C)$.

One particular case of this action is especially interesting: when the graph G varies, while C is fixed and equal to Θ, the chord diagram with only one chord. In this case the action is an isomorphism of the vector space Γ with the subspace $\mathcal{P}^2$ of the primitive space $\mathcal{P}$ generated by connected closed diagrams with 2 legs (section 7.4.2).

Corollary. *For the weight systems associated to a simple Lie algebra $\mathfrak{g}$ and the Killing form $\langle \cdot, \cdot \rangle^K$:*

$$\eta_{\mathfrak{g},K}(G) = \rho^{ad}_{\mathfrak{g},K}(G \cdot \Theta) \ ,$$

where $\rho^{ad}_{\mathfrak{g},K}$ is the weight system corresponding to the adjoint representation of $\mathfrak{g}$.

Proof. Indeed, according to the Lemma, for the universal enveloping algebra invariants we have

$$\rho_{\mathfrak{g},K}(G \cdot \Theta) = \frac{1}{\dim \mathfrak{g}}\eta_{\mathfrak{g}}(G)\rho_{\mathfrak{g},K}(\Theta) = \frac{1}{\dim \mathfrak{g}}\eta_{\mathfrak{g}}(G)\sum_{i=1}^{\dim \mathfrak{g}} e_i e_i \ ,$$

where $\{e_i\}$ is a basis orthonormal with respect to the Killing form. Now to compute $\rho^{ad}_{\mathfrak{g},K}(G \cdot \Theta)$ we take the trace of the product of operators in the adjoint representation:

$$\rho^{ad}_{\mathfrak{g},K}(G \cdot \Theta) = \frac{1}{\dim \mathfrak{g}}\eta_{\mathfrak{g}}(G)\sum_{i=1}^{\dim \mathfrak{g}} \mathrm{Tr}(\mathrm{ad}_{e_i}\mathrm{ad}_{e_i}) = \eta_{\mathfrak{g}}(G)$$

by the definition of the Killing form. □

7.6.4. Proposition. Multiplicativity with respect to the vertex product in Γ. *Let $w:\Gamma\to\mathbb{C}$ be an edge-multiplicative weight system, and $w(\beta)\neq 0$. Then $\frac{1}{w(\beta)}w:\Gamma\to\mathbb{C}$ is multiplicative with respect to the vertex product. In particular, for a simple Lie algebra $\mathfrak{g}$, $\frac{1}{\eta_{\mathfrak{g}}(\beta)}\eta_{\mathfrak{g}}(\cdot)$ is vertex-multiplicative.*

Proof. According to 7.3.4 the edge product is related to the vertex product like this: $G_1\cdot G_2=\beta\cdot(G_1\vee G_2)$ because of edge-multiplicativity. Therefore,

$$w(G_1)\cdot w(G_2)=w(\beta\cdot(G_1\vee G_2))=w(\beta)\cdot w(G_1\vee G_2)\ .$$

This means that the weight system $\frac{1}{w(\beta)}w:\Gamma\to\mathbb{C}$ is multiplicative with respect to the vertex product. □

Corollary. *The weight systems $\frac{1}{2N(N^2-1)}\eta_{\mathfrak{sl}_N}$, $\frac{2}{N(N-1)(N-2)}\eta_{\mathfrak{so}_N}:\Gamma\to\mathbb{C}$ associated with the ad-invariant form $\langle x,y\rangle=\mathrm{Tr}(xy)$ are multiplicative with respect to the vertex product in Γ.*

This follows from the direct computation on a "bubble": $\eta_{\mathfrak{sl}_N}(\beta)=2N(N^2-1)$, and $\eta_{\mathfrak{so}_N}(\beta)=\frac{1}{2}N(N-1)(N-2)$.

7.6.5. Compatibility with the vertex action of Γ on $\mathcal{C}$. The vertex action $G\vee C$ of a 3-graph $G\in\Gamma$ on a closed digram $C\in\mathcal{C}$ (see Sec. 7.4.3) is defined as the alternating sum of 6 ways to glue the graph G with the closed diagram C along chosen internal vertices in C and G. Again this action is well defined only on the primitive space $\mathcal{P}$.

Lemma. *Let $\mathfrak{g}$ be a simple Lie algebra. Then for any choice of the gluing vertices in G and C:*

$$\rho_{\mathfrak{g}}(G\vee C)=\frac{\eta_{\mathfrak{g}}(G)}{\eta_{\mathfrak{g}}(\beta)}\rho_{\mathfrak{g}}(C)\ .$$

Proof. Using the edge action (section 7.6.3) and its relation to the vertex action (section 7.3.4) we can write

$$\frac{\eta_{\mathfrak{g}}(G)}{\dim\mathfrak{g}}\rho_{\mathfrak{g}}(C)=\rho_{\mathfrak{g}}(G\cdot C)=\rho_{\mathfrak{g}}(\beta\cdot(G_1\vee C))=\frac{\eta_{\mathfrak{g}}(\beta)}{\dim\mathfrak{g}}\rho_{\mathfrak{g}}(G_1\vee C)\ ,$$

which is equivalent to what we need. □

7.6.6. $\mathfrak{sl}_N$- and $\mathfrak{so}_N$-polynomials. In this section we consider weight systems $\eta_{\mathfrak{sl}_N}(\cdot)$ supplied with the bilinear form $\langle x,y\rangle = \mathrm{Tr}(xy)$, and $\eta_{\mathfrak{so}_N}(\cdot)$ supplied with the bilinear form $\langle x,y\rangle = \frac{1}{2}\mathrm{Tr}(xy)$. In the $\mathfrak{so}_N$ case this form is more convenient since it gives polynomials with integral coefficients. In particular, for such a form $\eta_{\mathfrak{so}_N}(\beta) = N(N-1)(N-2)$, and in the state sum formula from the Theorem of Sec. 6.2.6 the coefficient in front of the sum will be 1.

The polynomial $\eta_{\mathfrak{sl}_N}(G)$ $(= \eta_{\mathfrak{gl}_N}(G))$ is divisible by $2N(N^2-1)$ (exercise 9) and the quotient is a multiplicative function with respect to the vertex product. We call the quotient the *reduced* $\mathfrak{sl}$*-polynomial* and denote it by $\widetilde{\mathfrak{sl}}(G)$.

Dividing the $\mathfrak{so}$-polynomial $\eta_{\mathfrak{so}_N}(G)$ by $N(N-1)(N-2)$ (see exercise 10), we obtain the *reduced* $\mathfrak{so}$*-polynomial* $\widetilde{\mathfrak{so}}(G)$, which is also multiplicative with respect to the vertex product.

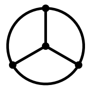
A. Kaishev [**Kai**] computed the values of $\widetilde{\mathfrak{sl}}$-, and $\widetilde{\mathfrak{so}}$-polynomials on the generators of Γ of small degrees (for $\widetilde{\mathfrak{so}}$-polynomial the substitution $M = N-2$ is used).

deg		$\widetilde{\mathfrak{sl}}$-polynomial	$\widetilde{\mathfrak{so}}$-polynomial
1	β	1	1
4	ω_4	N^3+12N	$M^3-3M^2+30M-24$
6	ω_6	N^5+32N^3+48N	$M^5-5M^4+80M^3-184M^2+408M-288$
7	ω_7	$N^6+64N^4+64N^2$	$M^6-6M^5+154M^4-408M^3+664M^2-384$
8	ω_8	$N^7+128N^5+128N^3$ $+192N$	$M^7-7M^6+294M^5-844M^4+1608M^3-2128M^2$ $+4576M-3456$
9	ω_9	$N^8+256N^6+256N^4$ $+256N^2$	$M^8-8M^7+564M^6-1688M^5+3552M^4-5600M^3$ $-5600M^3+6336M^2+6144M-9216$
10	ω_{10}	$N^9+512N^7+512N^5$ $+512N^3+768N$	$M^9-9M^8+1092M^7-3328M^6+7440M^5-13216M^4$ $+18048M^3-17920M^2+55680M-47616$
10	δ	$N^9+11N^7+114N^5$ $-116N^3$	$M^9-9M^8+44M^7-94M^6+627M^5+519M^4$ $-2474M^3-10916M^2+30072M-17760$
11	ω_{11}	$N^{10}+1024N^8+1024N^6$ $+1024N^4+1024N^2$	$M^{10}-10M^9+2134M^8-6536M^7+15120M^6$ $-29120M^5+45504M^4-55040M^3+48768M^2$ $+145408M-165888$

One may find a lot of recognizable patterns in this table. For example, wee see that

$\widetilde{\mathfrak{sl}}(\omega_n) = N^{n-1} + 2^{n-1}(N^{n-3} + \cdots + N^2)$, for odd $n > 5$;
$\widetilde{\mathfrak{sl}}(\omega_n) = N^{n-1} + 2^{n-1}(N^{n-3} + \cdots + N^3) + 2^{n-2}3N$, for even $n \geqslant 4$.

It would be interesting to know if these observations make part of some general theorems.

Later we will need the values of the $\mathfrak{sl}$- and $\mathfrak{so}$-polynomials on the “Mercedes element” $\tau =$  $= \frac{1}{2}\beta^2$. They are $\widetilde{\mathfrak{sl}}(\tau) = N$, and $\widetilde{\mathfrak{so}}(\tau) = M = N - 2$.

7.7. Weight systems not coming from Lie algebras

$\boxed{\text{Must be rewritten}}$

In this section we shall briefly describe the construction of P. Vogel [**Vo1**] which proves the existence of a weight system independent from Lie algebraic weight systems for all semisimple Lie algebras.

Instead of Vogel's Λ we will use its isomorphic copy, the algebra $\Gamma_{\geqslant 1}$ with vertex multiplication. It is enough to construct an element from $\mathcal{C}$ with the following properties:

- it is non-zero;
- all semisimple Lie algebra weight systems vanish on it.

The construction consists of two steps. At the first step we will find a non-zero element of $\Gamma_{\geqslant 1}$ on which all Lie algebra weight systems vanish. We represent this element as a polynomial in the generators of small degree with respect to the vertex product. At the second step we construct the required element of the algebra of closed diagrams $\mathcal{C}$.

Consider the following three elements

$$\begin{aligned}
\Gamma_7 \ni X_{\mathfrak{sl}} &:= 3\omega_6 \vee \tau - 6\omega_4 \vee \tau^{\vee 3} - \omega_4^{\vee 2} + 4\tau^{\vee 6}\ ; \\
\Gamma_{10} \ni X_{\mathfrak{so}} &:= -108\omega_{10} + 3267\omega_8 \vee \tau^{\vee 2} - 1920\omega_6 \vee \omega_4 \vee \tau \\
&\quad -20913\omega_6 \vee \tau^{\vee 4} + 372\omega_4^{\vee 3} + 8906\omega_4^{\vee 2} \vee \tau^{\vee 3} \\
&\quad +13748\omega_3 \vee \tau^{\vee 6} - 3352\tau^{\vee 9}\ ; \\
\Gamma_7 \ni X_{\mathfrak{ex}} &:= 45\omega_6 \vee \tau - 71\omega_4 \vee \tau^{\vee 3} - 18\omega_4^{\vee 2} + 32\tau^{\vee 6}\ .
\end{aligned}$$

Using the multiplicativity with respect to the vertex product and the tables from the previous subsection one can check that

$$\widetilde{\mathfrak{sl}}(X_{\mathfrak{sl}}) = 0, \qquad \text{and} \qquad \widetilde{\mathfrak{so}}(X_{\mathfrak{so}}) = 0.$$

Vogel [**Vo1**] shows that weight systems associated to the exceptional Lie algebras vanish on $X_{\mathfrak{ex}}$. Therefore, any semisimple Lie algebra weight system vanishes on the product

$$X := X_{\mathfrak{sl}} \vee X_{\mathfrak{so}} \vee X_{\mathfrak{ex}} \in \Gamma_{22}\ .$$

On the other hand, according to Vogel [**Vo1**], X is a non-zero element of Γ, because there is a Lie superalgebra $D(2,1,\alpha)$ weight system which takes a non-zero value on X.

The study of the Lie superalgebra weight systems is beyond the scope of this book, so we restrict ourselves with a reference to [**Kac**] for a general theory of Lie superalgebras and to [**FKV, Lieb**] as regards the weight systems coming from Lie superalgebras.

Now we are going to construct the required primitive element of $\mathcal{C}$. Consider the "Mercedes closed diagram" represented by the same graph

$\bar{t}_1 =$  ,

where the circle is understood as the Wilson loop and there is only one internal trivalent vertex. According to Sec. 7.4.3, the algebra of 3-graphs Γ acts on the primitive space $\mathcal{P}$ via vertex multiplication. By Lemma 7.4.4, the map $G \mapsto G \vee \bar{t}_1$ is an injection $\Gamma_n \to \mathcal{P}_{n+1}$. Therefore, the element

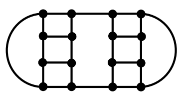
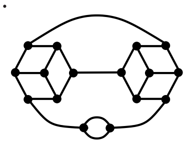
$$X \vee \bar{t}_1$$

is a non-zero primitive element of degree 23 in the algebra of closed diagrams.

By Lemma 7.6.5, all simple (and therefore semisimple) Lie algebra weight systems vanish on $X \vee \bar{t}_1$. Therefore a weight system which is non-zero on $X \vee \bar{t}_1$ cannot come from semisimple Lie algebras. In fact (see [**Vo1**]), even the Lie superalgebra weight system which is non-zero on X vanishes on $X \vee \bar{t}_1$. So even Lie superalgebra weight systems are not enough to generate all weight systems.

Exercises

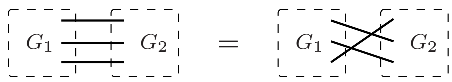
(1) Find explicitly a chain of IHX and AS relations that proves the following equality in the algebra Γ of 3-graphs:

 = 

(2) Let $\tau_2 : \mathcal{X}^2 \in \mathcal{X}^2$ denote the transposition of legs in a fixed diagram. Prove that τ_2 is an identity. *Hint*: (1) prove that a hole can be dragged through a trivalent vertex (as in Lemma 7.2.3, where x is empty), (2) to change the numbering of the two legs, use manoeuvres like in Lemma 7.2.4 with $y = \emptyset$).

(3) * Let Γ be the algebra of 3-graphs.

- Is it true that Γ is generated by plane graphs?
- Find generators and relations of the algebra Γ.
- Suppose that a graph $G \in \Gamma$ consists of two parts G_1 and G_2 connected by three edges. Is the following equality:



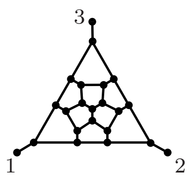
true?

(4) Let $\mathcal{X}^k$ be the space of 1- and 3-valent graphs with k numbered legs. Consider the transposition of two legs of an element of $\mathcal{X}^k$.
- Give a example of a non-zero element of $\mathcal{X}^k$ with even k which is changed under such a transposition.
- * Is it true that any such transposition changes the sign of the element if k is odd? (The first nontrivial case is when $k = 3$ — this is Conjecture 7.5.3.)

(5) * Let Λ be Vogel's algebra, i.e. the subspace of $\mathcal{X}^3$ consisting of all antisymmetric elements.
- Is it true that $\Lambda = \mathcal{X}^3$ (this is again Conjecture 7.5.3)?
- Is it true that Λ is generated by the elements t and x_k (this is the Conjecture 7.5.5; see also Exercises 6 and 7)?

(6) Let t, x_3, x_4, x_5, ... be the elements of the space $\mathcal{X}^3$ defined above.
- Prove that x_i's belong to Vogel's algebra Λ, i.e. that they are antisymmetric with respect to permutations of legs.
- Prove the relation $x_4 = -\frac{4}{3}t \vee x_3 - \frac{1}{3}t^{\vee 4}$.
- Prove that x_k with an arbitrary even k can be expressed through t, x_3, x_5, ...

(7) Prove that the dodecahedron

$d =$ 

belongs to Λ, and express it as a vertex polynomial in t, x_3, x_5, x_7, x_9.

(8) * The group S_3 acts in the space of fixed diagrams with 3 legs $\mathcal{X}^3$, splitting it into 3 subspaces:
- symmetric, which is isomorphic to $\mathcal{B}^3$ (open diagrams with 3 legs),
- totally antisymmetric, which is Vogel's Λ by definition, and
- some subspace Q, corresponding to a 2-dimensional irreducible representation of S_3.

Question: is it true that $Q = 0$?

(9) Show that $N = 0$, $N = -1$, and $N = 1$ are roots of the polynomial $\eta_{\mathfrak{gl}_N}(G)$ for any 3-graph $G \in \Gamma_n$ $(n > 1)$.

(10) Show that $N = 0$, $N = 1$ and $N = 2$ are roots of polynomial $\eta_{\mathfrak{so}_N}(G)$ for any 3-graph $G \in \Gamma_n$ $(n > 0)$.

Part 3

The Kontsevich Integral

Chapter 8

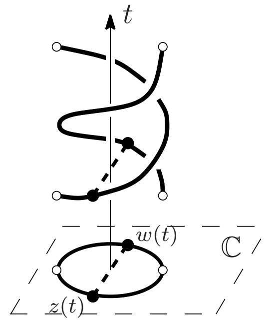
The Kontsevich integral

The Kontsevich integral appeared in the paper [**Kon1**] by M. Kontsevich as a tool to prove the Fundamental Theorem of the theory of Vassiliev invariants (that is, Theorem 4.2.1). Any Vassiliev knot invariant with coefficients in a field of characteristic 0 can be factored through the universal invariant defined by the Kontsevich integral.

Detailed (and different) expositions of the construction and properties of the Kontsevich integral can be found in [**BN1, CD3, Les**]. Other important references are [**Car1**], [**LM1**], [**LM2**].

About the notation: in this chapter we shall think of $\mathbb{R}^3$ as the product of a (horizontal) complex plane $\mathbb{C}$ with the complex coordinate z and a (vertical) real line $\mathbb{R}$ with the coordinate t. All Vassiliev invariants are always thought of having values in the complex numbers.

8.1. First examples



We start with two examples where the Kontsevich integral appears in a simplified form and with a clear geometric meaning.

8.1.1. The braiding number of a 2-braid.

A braid on two strands has a complete invariant: the number of full twists that one strand makes around the other.

Let us consider the horizontal coordinates of points on the strands, $z(t)$ and $w(t)$, as functions of the vertical coordinate

t, $0 \leqslant t \leqslant 1$, then the number of full twists can be computed by the integral formula

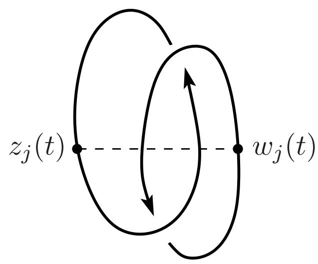
$$\frac{1}{2\pi i}\int_0^1 \frac{dz - dw}{z - w}.$$

Note that the number of full twists is not necessarily an integer; however, the number of *half-twists* always is.

8.1.2. Kontsevich type formula for the linking number. The Gauss integral formula for the linking number of two spatial curves $lk(K, L)$ (discussed in Section 2.2.2) involves integration over a torus (namely, the product of the two curves). Here we shall give a different integral formula for the same invariant, with the integration over an interval, rather than a torus. This formula generalizes the expression for the braiding number of a braid on two strands and, as we shall later see, gives the first term of the Kontsevich integral of a two-component link.

Definition. A link in $\mathbb{R}^3$ is a *Morse* link if the function t (the vertical coordinate) on it has only non-degenerate critical points. A Morse link is a *strict* Morse link if the critical values of the vertical coordinate are all distinct. Similarly one speaks of *Morse tangles* and *strict Morse tangles.*

Theorem. *Suppose that two disjoint connected curves K, L are embedded into $\mathbb{R}^3$ as a strict Morse link.*



Then

$$lk(K, L) = \frac{1}{2\pi i}\int \sum_j (-1)^{\downarrow_j} \frac{d(z_j(t) - w_j(t))}{z_j(t) - w_j(t)},$$

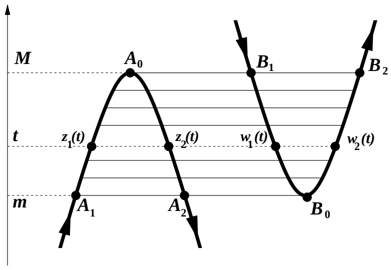
where the index j enumerates all possible choices of a pair of strands on the link as functions $z_j(t)$, $w_j(t)$ corresponding to K and L, respectively, and the integer $\downarrow_j$ is the number of strands in the pair which are oriented downwards.

Remark. In fact, the condition that the link in question is a strict Morse link can be relaxed. One may consider piecewise linear links with no horizontal segments, or smooth links whose vertical coordinate function has no flattening points (those where all the derivatives vanish).

Proof. The proof consists of three steps which — in a more elaborate setting — will also appear in the full construction of the Kontsevich integral.

Step 1. *The value of the sum in the right hand side is an integer.* Note that for a strict Morse link with two components K and L, the *configuration space* of all horizontal chords joining K and L is a closed one-dimensional manifold, that is, a disjoint union of several circles.

For example, assume that two adjacent critical values m and M (with $m < M$) of the vertical coordinate correspond to a minimum on the component K and a maximum on the component L respectively:



The space of all horizontal chords that join the shown parts of K and L consists of four intervals which join together to form a circle. The motion along this circle starts, say, at a chord A_1B_0 and proceeds as

$$A_1B_0 \to A_0B_1 \to A_2B_0 \to A_0B_2 \to A_1B_0.$$

Note that when the moving chord passes a critical level (either m or M), the direction of its motion changes, and so does the sign $(-1)^{\downarrow_j}$. Another example see in the exercise (1) on page 246.

It is now clear that our integral formula counts the number of complete turns made by the horizontal chord while running through the whole configuration space of chords with one end $(z_j(t), t)$ on K and the other end $(w_j(t), t)$ on L. This is, clearly, an integer.

Step 2. *The value of the right hand side remains unchanged under a continuous horizontal deformation of the link.* (By a horizontal deformation we mean a deformation of a link which moves every point in a horizontal plane $t = \text{const.}$) The assertion is evident, since the integral changes continuously while always remaining an integer. Note that this is true even if we allow self-intersections within each of the components; this does not influence the integral because $z_j(t)$ and $w_j(t)$ lie on the different components.

Step 3. *Reduction to the combinatorial formula for the linking number (Section 2.2).* Choose a vertical plane in $\mathbb{R}^3$ and represent the link by a generic projection to that plane. By a horizontal deformation, we can flatten the link so that it lies in the plane completely, save for the small fragments around the diagram crossings between K and L (as we noted above, *self-intersections* of each component are allowed). Now, the rotation of the horizontal chord for each crossing is by $\pm\pi$, and the signs are in agreement with the number of strands oriented downwards. The reader is invited to draw the two different possible crossings, then, for each picture, consider the four possibilities for the orientations of the strands and make sure that the sign of the half-turn of the moving horizontal chord always agrees with the factor $(-1)^{\downarrow_j}$. (Note that the integral in the theorem is computed over t, so that each specific term computes the angle of rotation of the chord as it moves from bottom to top.) □

The Kontsevich integral can be regarded as a generalization of this formula. Here we kept track of one horizontal chord moving along the two curves. The full Kontsevich integral keeps track of how *finite sets* of horizontal chords on the knot (or a tangle) rotate when moved in the vertical direction. This is the somewhat naïve approach that we use in the next section. Later, in Section 10.1, we shall adopt a more sophisticated point of view, interpreting the Kontsevich integral as the monodromy of the Knizhnik–Zamolodchikov connection in the complement to the union of diagonals in $\mathbb{C}^n$.

8.2. The construction

Let us recall some notation and terminology of the preceding section. For points of $\mathbb{R}^3$ we use coordinates (z, t) with z complex and t real; the planes $t = \text{const}$ are thought of being horizontal. Having chosen the coordinates, we can speak of *strict Morse knots*, namely, knots with the property that the coordinate t restricted to the knot has only non-degenerate critical points with distinct critical values.

We define the Kontsevich integral for strict Morse knots. Its values belong to the graded completion $\widehat{\mathcal{A}}$ of the algebra of chord diagrams with 1-term relations $\mathcal{A} = \mathcal{A}^{fr}/(\Theta)$. (By definition, the elements of a graded algebra are finite linear combinations of homogeneous elements. The *graded completion* consists of all infinite combinations of such elements.)

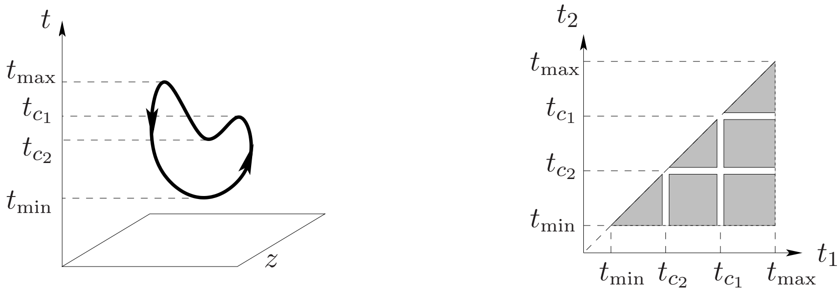
8.2.1. Definition. The *Kontsevich integral* $Z(K)$ of a strict Morse knot K is given by the following formula:

$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int\limits_{\substack{t_{\min}<t_m<\cdots<t_1<t_{\max} \\ t_j \text{ are noncritical}}} \sum_{P=\{(z_j,z'_j)\}} (-1)^{\downarrow_P} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j} .$$

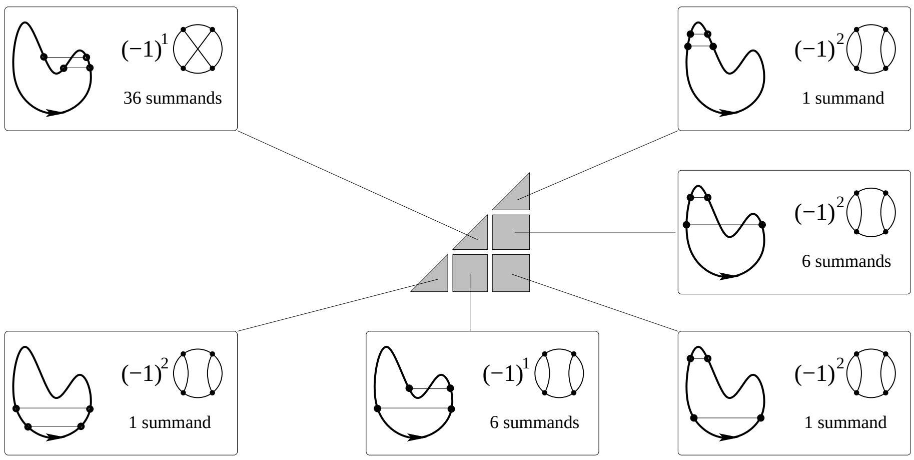
The ingredients of this formula have the following meaning.

The real numbers $t_{\min}$ and $t_{\max}$ are the minimum and the maximum of the function t on K.

The integration domain is the set of all points of the m-dimensional simplex $t_{\min} < t_m < \cdots < t_1 < t_{\max}$ none of whose coordinates t_i is a critical value of t. The m-simplex is divided by the critical values into several *connected components*. For example, for the following embedding of the unknot and $m = 2$ the corresponding integration domain has six connected components and looks like



The number of summands in the integrand is constant in each connected component of the integration domain, but can be different for different components. In each plane $\{t = t_j\} \subset \mathbb{R}^3$ choose an unordered pair of distinct points (z_j, t_j) and (z'_j, t_j) on K, so that $z_j(t_j)$ and $z'_j(t_j)$ are continuous functions. We denote by $P = \{(z_j, z'_j)\}$ the set of such pairs for $j = 1, \ldots, m$ and call it a *pairing*.



The integrand is the sum over all choices of the pairing P. In the example above for the component $\{t_{c_1} < t_1 < t_{\max},\ t_{\min} < t_2 < t_{c_2}\}$, at the bottom-right corner, we have only one possible pair of points on the levels $\{t = t_1\}$ and $\{t = t_2\}$. Therefore, the sum over P for this component consists of only one summand. In contrast, in the next to it component, $\{t_{c_2} < t_1 < t_{c_1},$

$t_{\min} < t_2 < t_{c_2}\}$, we still have only one possibility for the chord (z_2, z'_2) on the level $\{t = t_2\}$, but the plane $\{t = t_1\}$ intersects our knot K in four points. So we have $\binom{4}{2} = 6$ possible pairs (z_1, z'_1) and the total number of summands here is six (see the picture above).

For a pairing P the symbol '$\downarrow_P$' denotes the number of points (z_j, t_j) or (z'_j, t_j) in P where the coordinate t decreases as one goes along K.

Fix a pairing P. Consider the knot K as an oriented circle and connect the points (z_j, t_j) and (z'_j, t_j) by a chord. We obtain a chord diagram with m chords. (Thus, intuitively, one can think of a pairing as a way of inscribing a chord diagram into a knot in such a way that all chords are horizontal and are placed on different levels.) The corresponding element of the algebra $\mathcal{A}$ is denoted by D_P. In the picture below, for each connected component in our example, we show one of the possible pairings, the corresponding chord diagram with the sign $(-1)^{\downarrow_P}$ and the number of summands of the integrand (some of which are equal to zero in $\mathcal{A}$ due to the one-term relation).

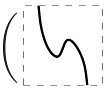
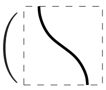
Over each connected component, z_j and z'_j are smooth functions in t_j. By $\bigwedge\limits_{j=1}^{m} \dfrac{dz_j - dz'_j}{z_j - z'_j}$ we mean the pullback of this form to the integration domain of the variables $t_1, \ldots, t_m$. The integration domain is considered with the orientation of the space $\mathbb{R}^m$ defined by the natural order of the coordinates $t_1, \ldots, t_m$.

By convention, the term in the Kontsevich integral corresponding to $m = 0$ is the (only) chord diagram of order 0 taken with coefficient one. It is the unit of the algebra $\widehat{\mathcal{A}}$.

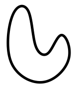
8.2.2. Basic properties. We shall see later in this chapter that the Kontsevich integral has the following basic properties:

- $Z(K)$ converges for any strict Morse knot K.
- It is invariant under the deformations of the knot in the class of (not necessarily strict) Morse knots.
- It behaves in a predictable way under the deformations that add a pair of new critical points to a Morse knot.

Let us explain the last item in more detail. While the Kontsevich integral is indeed an invariant of Morse knots, it is not preserved by deformations that change the number of critical points of t. However, the following formula shows how the integral changes when a new pair of critical points is added to the knot:

(8.2.2.1) $Z\Big($  $\Big) = Z(H) \cdot Z\Big($  $\Big).$

Here the first and the third pictures represent two embeddings of an arbitrary knot that coincide outside the fragment shown,

$H \ :=$ 

is the *hump* (an unknot with two maxima), and the product is the product in the completed algebra $\widehat{\mathcal{A}}$ of chord diagrams. The equality (8.2.2.1) allows to define a genuine knot invariant by the formula

$$I(K) = \frac{Z(K)}{Z(H)^{c/2}},$$

where c denotes the number of critical points of K and the ratio means the division in the algebra $\widehat{\mathcal{A}}$ according to the rule $(1+a)^{-1} = 1-a+a^2-a^3+\dots$ The knot invariant $I(K)$ is sometimes referred to as the *final* Kontsevich integral as opposed to the *preliminary* Kontsevich integral $Z(K)$.

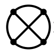
The central importance of the final Kontsevich integral in the theory of finite-type invariants is that it is a universal Vassiliev invariant in the following sense.

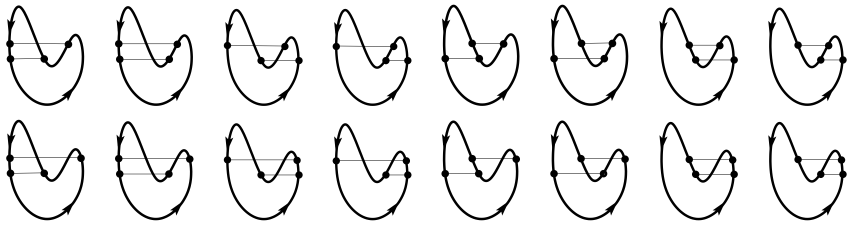
Consider an unframed weight system w of degree n (that is, a function on the set of chord diagrams with m chords satisfying one- and four-term relations). Applying w to the m-homogeneous part of the series $I(K)$, we get a numerical knot invariant $w(I(K))$. This invariant is a Vassiliev invariant of order m and any Vassiliev invariant can be obtained in this way. This argument will be used to prove the Fundamental Theorem on Vassiliev Invariants, see Section 8.8.

The Kontsevich integral has many interesting properties that we shall describe in this and in the subsequent chapters. Among these are its behaviour with respect to the connected sum of knots (Section 8.4 and 8.7.1) to the coproduct in the Hopf algebra of chord diagrams (Section 9.1), cablings (Section 9.7), taking satellites (Section ???), mutation (Section 9.5.4). We shall see that it can be computed combinatorially (Section 10.2) and has rational coefficients (Section ???).

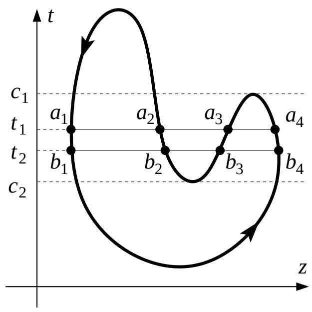
8.3. Example of calculation

Here we shall calculate the coefficient of the chord diagram  in $Z(H)$, where H is the hump (plane curve with 4 critical points, as in the previous section) directly from the definition of the Kontsevich integral. The following computation is valid for an arbitrary shape of the curve, provided that the length of the segments a_1a_2 and a_3a_4 (see picture below) decreases with t_1, while that of the segment a_2a_3 increases.

First of all, note that out of the total number of 51 pairings shown in the picture on page 227, the following 16 contribute to the coefficient of :



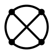
We are, therefore, interested only in the band between the critical values c_1 and c_2. Denote by a_1, a_2, a_3, a_4 (resp. b_1, b_2, b_3, b_4) the four points of intersection of the knot with the level $\{t = t_1\}$ (respectively, $\{t = t_2\}$):



The sixteen pairings shown in the picture above correspond to the differential forms

$$(-1)^{j+k+l+m} d\ln a_{jk} \wedge d\ln b_{lm},$$

where $a_{jk} = a_k - a_j$, $b_{lm} = b_m - b_l$, and the pairs (jk) and (lm) can take 4 different values each: $(jk) \in \{(13), (23), (14), (24)\} =: A$, $(lm) \in \{(12), (13), (24), (34)\} =: B$. The sign $(-1)^{j+k+l+m}$ is equal to $(-1)^{\downarrow_P}$, because in our case upward oriented strings have even numbers, while downward oriented strings have odd numbers.

The coefficient of  is therefore equal to

$$\begin{aligned}
&\frac{1}{(2\pi i)^2} \int\limits_{\Delta} \sum_{(jk)\in A} \sum_{(lm)\in B} (-1)^{j+k+l+m} d\ln a_{jk} \wedge d\ln b_{lm} \\
&= -\frac{1}{4\pi^2} \int\limits_{\Delta} \sum_{(jk)\in A} (-1)^{j+k+1} d\ln a_{jk} \wedge \sum_{(lm)\in B} (-1)^{l+m-1} d\ln b_{lm} \\
&= -\frac{1}{4\pi^2} \int\limits_{\Delta} d\ln \frac{a_{14}a_{23}}{a_{13}a_{24}} \wedge d\ln \frac{b_{12}b_{34}}{b_{13}b_{24}},
\end{aligned}$$

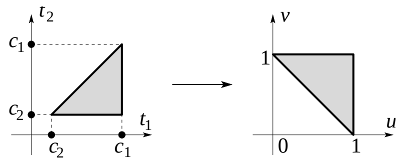
where the integration domain Δ is the triangle described by the inequalities $c_2 < t_1 < c_1$, $c_2 < t_2 < t_1$. Assume the following notation:

$$u = \frac{a_{14}a_{23}}{a_{13}a_{24}}, \qquad v = \frac{b_{12}b_{34}}{b_{13}b_{24}}.$$

It is easy to see that u is an increasing function of t_1 ranging from 0 to 1, while v is an decreasing function of t_2 ranging from 1 to 0. Therefore, the mapping $(t_1, t_2) \mapsto (u, v)$ is a diffeomorphism with a negative Jacobian, and after the change of variables the integral we are computing becomes

$$\frac{1}{4\pi^2} \int\limits_{\Delta'} d\ln u \wedge d\ln v$$

where Δ' is the image of Δ. It is obvious that the boundary of Δ' contains the segments $u = 1, 0 \leqslant v \leqslant 1$ and $v = 1, 0 \leqslant u \leqslant 1$ that correspond to $t_1 = c_1$ and $t_2 = c_2$. What is not immediately evident is that the third side of the triangle Δ also goes into a straight line, namely, $u + v = 1$. Indeed, if $t_1 = t_2$, then all b's are equal to the corresponding a's and the required fact follows from the identity $a_{12}a_{34} + a_{14}a_{23} = a_{13}a_{24}$.



Therefore,

$$\begin{aligned}\frac{1}{4\pi^2} \int\limits_{\Delta'} d\ln u \wedge d\ln v &= \frac{1}{4\pi^2} \int\limits_0^1 \left(\int\limits_{1-u}^1 d\ln v \right) \frac{du}{u} \\ &= -\frac{1}{4\pi^2} \int\limits_0^1 \ln(1-u)\frac{du}{u}\ .\end{aligned}$$

Taking the Taylor expansion of the logarithm we get

$$\frac{1}{4\pi^2} \sum_{k=1}^{\infty} \int\limits_0^1 \frac{u^k}{k}\frac{du}{u} = \frac{1}{4\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{4\pi^2}\zeta(2) = \frac{1}{24}.$$

Two things are quite remarkable in this answer: (1) that it is expressed via a value of the zeta function, and (2) that the answer is rational. In fact, for any knot K the coefficient of any chord diagram in $Z(K)$ is rational and can be computed through the values of multivariate ζ-functions:

$$\zeta(a_1, \ldots, a_n) = \sum_{0<k_1<k_2<\cdots<k_n} k_1^{-a_1} \ldots k_n^{-a_n}.$$

We shall speak about that in more detail in Section 10.3.

For a complete formula for $Z(H)$ see Section 11.4.

8.4. The Kontsevich integral for tangles

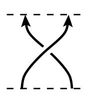
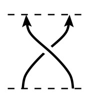
The definition of the preliminary Kontsevich integral for knots (see Section 8.2) makes sense for an arbitrary strict Morse tangle T. One only needs to replace the completed algebra $\widehat{\mathcal{A}}$ of chord diagrams by the graded completion of the vector space of tangle chord diagrams on the skeleton of T, and take $t_{\min}$ and $t_{\max}$ to correspond to the bottom and the top of T, respectively. In the section 8.5 we shall show that the coefficients of the chord diagrams in the Kontsevich integral of any (strict Morse) tangle actually converge.

In particular, one can speak of the Kontsevich integral of links or braids.

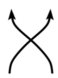
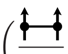
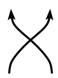
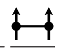
8.4.1. Exercise. For a two-component link, what is the coefficient in the Kontsevich integral of the chord diagram of degree 1 whose chord has ends on both components?

Hint: see Section 8.1.2.

8.4.2. Exercise. Compute the integrals

$R := Z\Big($  $\Big)$ and $R^{-1} := Z\Big($  $\Big)$.

Answer:

$R =$  $\cdot \exp\Big(\frac{1}{2}$  $\Big)$, $R^{-1} =$  $\cdot \exp\Big(-\frac{1}{2}$  $\Big)$

where $\exp a$ is the series $1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \dots$.

Strictly speaking, before describing the properties of the Kontsevich integral we need to show that it is always well-defined. This will be done in the following section. Meanwhile, we shall assume that this is indeed the case for all the tangles in question.

8.4.3. Proposition. *The Kontsevich integral for tangles is multiplicative:*

$$Z(T_1 \cdot T_2) = Z(T_1) \cdot Z(T_2)$$

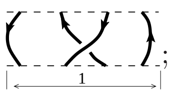
whenever the product $T_1 \cdot T_2$ is defined.

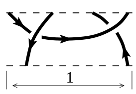
Proof. Let $t_{\min}$ and $t_{\max}$ correspond to the bottom and the top of $T_1 \cdot T_2$, respectively, and let t_{mid} be the level of the top of T_2 (or the bottom of T_1, which is the same). In the expression for the Kontsevich integral of the tangle $T_1 \cdot T_2$ let us remove from the domain of integration all points with at least one coordinate t equal to t_{mid}. This set is of codimension one, so the value of integral remains unchanged. On the other hand, the connected components of the new domain of integration are precisely all products of the connected components for T_1 and T_2, and the integrand for $T_1 \cdot T_2$ is the exterior product of the integrands for T_1 and T_2. The Fubini theorem on multiple integrals implies that $Z(T_1 \cdot T_2) = Z(T_1) \cdot Z(T_2)$. $\square$

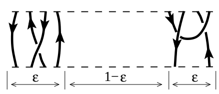
The behaviour of the Kontsevich integral under the tensor product of tangles is more complicated. In the expression for $Z(T_1 \otimes T_2)$ indeed there are terms that add up to the tensor product $Z(T_1) \otimes Z(T_2)$: they involve pairings without chords that connect T_1 with T_2. However, the terms with pairings that do have such chords are not necessarily zero and we have no effective way of describing them. Still, there is something we can say but we need a new definition for this.

8.4.4. Parameterized tensor products. By a (horizontal) *ε-rescaling* of $\mathbb{R}^3$ we mean the map sending (z,t) to $(\varepsilon z, t)$. For $\varepsilon > 0$ it induces an operation on tangles; we denote by εT the result of an ε-rescaling applied to T. Note that ε-rescaling of a tangle does not change its Kontsevich integral.

Let T_1 and T_2 be two tangles such that $T_1 \otimes T_2$ is defined. For $0 < \varepsilon \leqslant 1$ we define the *ε-parameterized tensor product* $T_1 \otimes_\varepsilon T_2$ as the result of placing εT_1 next to εT_2 on the left, with the distance of $1-\varepsilon$ between the two tangles:

$T_1 =$ 

; $T_2 =$ 

; $T_1 \otimes_\varepsilon T_2 =$ 

.

More precisely, let $\mathbf{0}_{1-\varepsilon}$ be the empty tangle of width $1-\varepsilon$ and the same height and depth as εT_1 and εT_2. Then

$$T_1 \otimes_\varepsilon T_2 = \varepsilon T_1 \otimes \mathbf{0}_{1-\varepsilon} \otimes \varepsilon T_2.$$

When $\varepsilon = 1$ we get the usual tensor product. Note that when $\varepsilon < 1$, the parameterized tensor product is, in general, not associative.

8.4.5. Proposition. *The Kontsevich integral for tangles is asymptotically multiplicative with respect to the parameterized tensor product:*

$$\lim_{\varepsilon \to 0} Z(T_1 \otimes_\varepsilon T_2) = Z(T_1) \otimes Z(T_2)$$

whenever the product $T_1 \otimes T_2$ is defined. Moreover, the difference $Z(T_1 \otimes_\varepsilon T_2) - Z(T_1) \otimes Z(T_2)$ as ε tends to 0 is of the same or smaller order of magnitude as ε.

Proof. As we have already noted before, $Z(T_1 \otimes_\varepsilon T_2)$ consists of two parts: the terms that do not involve chords that connect εT_1 with εT_2, and the terms that do. The first part does not depend on ε and is equal to $Z(T_1) \otimes Z(T_2)$, and the second part tends to 0 as $\varepsilon \to 0$.

Indeed, each pairing $P = \{(z_j, z'_j)\}$ for $T_1 \otimes T_2$ give rise to a continuous family of pairings $P_\varepsilon = \{(z_j(\varepsilon), z'_j(\varepsilon))\}$ for $T_1 \otimes_\varepsilon T_2$. Consider one such family P_ε. For all k

$$dz_k(\varepsilon) - dz'_k(\varepsilon) = \varepsilon(dz_k - dz'_k).$$

If the kth chord has has both ends on εT_1 or on εT_2, we have

$$z_k(\varepsilon) - z'_k(\varepsilon) = \varepsilon(z_k - z'_k)$$

for all ε. Therefore the limit of the first part is equal to $Z(T_1) \otimes Z(T_2)$.

On the other hand, if P_ε has at least one chord connecting the two factors, we have $|z_k(\varepsilon) - z'_k(\varepsilon)| \to 1$ as $\varepsilon \to 0$. Thus the integral corresponding to the pairing P_ε tends to zero as ε get smaller, and we see that the whole second part of the Kontsevich integral of $T_1 \otimes_\varepsilon T_2$ vanishes in the limit at least as fast as ε,

$$Z(T_1 \otimes_\varepsilon T_2) = Z(T_1) \otimes Z(T_2) + O(\varepsilon)\ .$$

□

8.5. Convergence of the integral

8.5.1. Proposition. *For any strict Morse tangle T, the Kontsevich integral $Z(T)$ converges.*

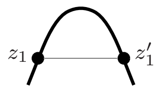
Proof. The integrand of the Kontsevich integral may have singularities near the boundaries of the connected components. This happens near a critical point of a tangle when the pairing includes a "short" chord whose ends are on the branches of the knot that come together at a critical point.

Let us assume that the tangle T has at most one critical point. This is sufficient since any strict Morse tangle can be decomposed as a product of such tangles. The argument in the proof of Proposition 8.4.3 shows that the Kontsevich integral of a product converges whenever the integral of the factors does.

Suppose, without loss of generality, that T has a critical point which is a maximum with the value t_c. Then we only need to consider pairings with no chords above t_c. Indeed, for any pairing its coefficient in the Kontsevich integral of T is a product of two integrals: one corresponding to the chords above t_c, and the other - to the chords below t_c. The first integral obviously converges since the integrand has no singularities, so it is sufficient to consider the factor with chords below t_c.

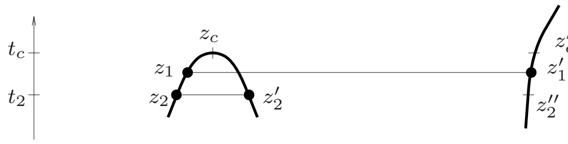
Essentially, there are two cases.

1) An isolated chord (z_1, z_1') tends to zero:



In this case the corresponding chord diagram D_P is equal to zero in $\mathcal{A}$ by the one-term relation.

2) A chord (z_j, z_j') tends to zero near a critical point but is separated from that point by one or more other chords:



Consider, for example, the case shown on the figure, where the "short" chord (z_2, z_2') is separated from the critical point by another, "long" chord (z_1, z_1'). We have:

$$\left| \int_{t_2}^{t_c} \frac{dz_1 - dz_1'}{z_1 - z_1'} \right| \leqslant C \left| \int_{t_2}^{t_c} d(z_1 - z_1') \right|$$
$$= C \left| (z_c - z_2) - (z_c' - z_2'') \right| \leqslant C' |z_2 - z_2'|$$

for some positive constants C and C'. This integral is of the same order as $z_2 - z_2'$ and this compensates the denominator corresponding to the second chord.

More generally, one shows by induction that if a "short" chord (z_j, z_j') is separated from the maximum by $j-1$ chords, the first of which is "long", the integral

$$\int_{t_j < t_{j-1} < \cdots < t_1 < t_c} \bigwedge_{i=1}^{j-1} \frac{dz_i - dz_i'}{z_i - z_i'}$$

is of the same order as $z_j - z_j'$. This implies the convergence of the Kontsevich integral. □

8.6. Invariance of the integral

8.6.1. Theorem. *The Kontsevich integral is invariant under the deformations in the class of (not necessarily strict) Morse knots.*

The proof of this theorem spans the whole of this section.

Any deformation of a knot within the class of Morse knots can be approximated by a sequence of deformations of three types: orientation-preserving re-parameterizations, *horizontal deformations* and *movements of critical points.*

The invariance of the Kontsevich integral under orientation-preserving re-parameterizations is immediate since the parameter plays no role in the definition of the integral apart from determining the orientation of the knot.

A horizontal deformation is an isotopy of a knot in $\mathbb{R}^3$ which preserves all horizontal planes $\{t = \text{const}\}$ and leaves all the critical points (together with some small neighbourhoods) fixed. The invariance under horizontal deformations is the most essential point of the theory. We prove it in the next subsection.

A movement of a critical point C is an isotopy which is identical everywhere outside a small neighborhood of C and does not introduce new critical points on the knot. In subsection 8.6.3 we consider invariance under the movements of critical points.

As we mentioned before, the Kontsevich integral is not invariant under isotopies that change the number of critical points. Its behavior under such deformations will be discussed in section 8.7.

8.6.2. Invariance under horizontal deformations. Let us decompose the given knot into a product of tangles without critical points of the function t and very thin tangles containing the critical levels. A horizontal deformation keeps fixed the neighbourhoods of the critical points, so, due to multiplicativity, it is enough to prove that the Kontsevich integral for a tangle without critical points is invariant under horizontal deformations that preserve the boundary pointwise.

Proposition. *Let T_0 be a tangle without critical points and T_λ, a horizontal deformation of T_0 to T_1 (preserving the top and the bottom of the tangle). Then $Z(T_0) = Z(T_1)$.*

Proof. Denote by ω the integrand form in the mth term of the Kontsevich integral:

$$\omega = \sum_{P=\{(z_j, z'_j)\}} (-1)^{\downarrow} D_P \quad \bigwedge_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j} \ .$$

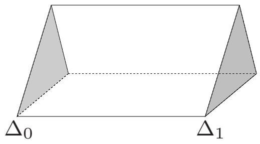
Here the functions z_j, z'_j depend not only on t_1, ..., t_m, but also on λ, and *all differentials are understood as complete differentials with respect to all these variables.* This means that the form ω is not exactly the form which appears in the Kontsevich's integral (it has some additional $d\lambda$'s), but this does not change the integrals over the simplices

$$\Delta_\lambda = \{t_{\min} < t_m < \cdots < t_1 < t_{\max}\} \times \{\lambda\},$$

because the value of λ on such a simplex is fixed.

We must prove that the integral of ω over Δ_0 is equal to its integral over Δ_1.

Consider the product polytope

$$\Delta = \Delta_0 \times [0,1] =$$  .

By Stokes' theorem, we have $\displaystyle\int_{\partial\Delta} \omega \quad = \quad \int_{\Delta} d\omega$.

The form ω is exact: $d\omega = 0$. The boundary of the integration domain is $\partial\Delta = \Delta_0 - \Delta_1 + \sum\{faces\}$. The theorem will follow from the fact that $\omega|_{\{face\}} = 0$. To show this, consider two types of faces.

The first type corresponds to $t_m = t_{\min}$ or $t_1 = t_{\max}$. In this situation, $dz_j = dz'_j = 0$ for $j = 1$ or m, since z_j and z'_j do not depend on λ.

The faces of the second type are those where we have $t_k = t_{k+1}$ for some k. In this case we have to choose the kth and $(k+1)$st chords on the same level $\{t = t_k\}$. In general, the endpoints of these chords may coincide and we do not get a chord diagram at all. Strictly speaking, ω and D_P do not extend to such a face so we have to be careful. It is natural to extend D_P to this face as a locally constant function. This means that for the case in which some endpoints of kth and $(k+1)$st chords belong to the same string (and therefore coincide) we place kth chord a little higher than $(k+1)$st chord, so that its endpoint differs from the endpoint of $(k+1)$st chord. This trick yields a well-defined prolongation of D_P and ω to the face, and we use it here.

All summands of ω are divided into three parts:

1) kth and $(k+1)$st chords connect the same two strings;

2) kth and $(k+1)$st chords are chosen in such a way that their endpoints belong to four different strings;

3) kth and $(k+1)$st chords are chosen in such a way that there exist exactly three different strings containing their endpoints.

Consider all these cases one by one.

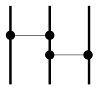
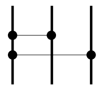
1) We have $z_k = z_{k+1}$ and $z'_k = z'_{k+1}$ or vice versa. So $d(z_k - z'_k) \wedge d(z_{k+1} - z'_{k+1}) = 0$ and therefore the restriction of ω to the face is zero.

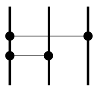
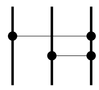
2) All choices of chords in this part of ω appear in mutually canceling pairs. Fix four strings and number them by 1, 2, 3, 4. Suppose that for a certain choice of the pairing, the kth chord connects the first two strings and $(k+1)$st chord connects the last two strings. Then there exists another choice for which on the contrary the kth chord connects the last two strings and $(k+1)$st chord connects the first two strings. These two choices give

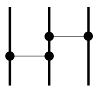
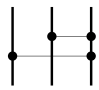
two summands of ω differing by a sign:

$$\cdots d(z_k - z'_k) \wedge d(z_{k+1} - z'_{k+1}) \cdots + \cdots d(z_{k+1} - z'_{k+1}) \wedge d(z_k - z'_k) \cdots = 0.$$

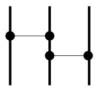
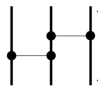
3) This is the most difficult case. The endpoints of kth and $(k+1)$st chords have exactly one string in common. Call the three relevant strings 1, 2, 3 and denote by ω_{ij} the 1-form $\dfrac{dz_i - dz_j}{z_i - z_j}$. Then ω is the product of a certain $(m-2)$-form and the sum of the following six 2-forms:

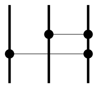
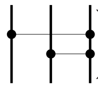
$(-1)^{\downarrow}$  $\omega_{12} \wedge \omega_{23}$ $+$ $(-1)^{\downarrow}$  $\omega_{12} \wedge \omega_{13}$

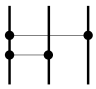
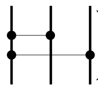
$+(-1)^{\downarrow}$  $\omega_{13} \wedge \omega_{12}$ $+$ $(-1)^{\downarrow}$  $\omega_{13} \wedge \omega_{23}$

$+(-1)^{\downarrow}$  $\omega_{23} \wedge \omega_{12}$ $+$ $(-1)^{\downarrow}$  $\omega_{23} \wedge \omega_{13}$.

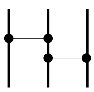
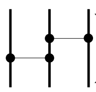
Using the fact that $\omega_{ij} = \omega_{ji}$, we can rewrite this as follows:

$\Big($ $(-1)^{\downarrow}$  $-$ $(-1)^{\downarrow}$  $\Big)$ $\omega_{12} \wedge \omega_{23}$

$+\Big($ $(-1)^{\downarrow}$  $-$ $(-1)^{\downarrow}$  $\Big)$ $\omega_{23} \wedge \omega_{31}$

$+\Big($ $(-1)^{\downarrow}$  $-$ $(-1)^{\downarrow}$  $\Big)$ $\omega_{31} \wedge \omega_{12}$.

The four-term relations in horizontal form (page 99) say that the expressions in parentheses are one and the same element of $\mathcal{A}_T$, hence the whole sum is equal to

$\Big($ $(-1)^{\downarrow}$  $-$ $(-1)^{\downarrow}$  $\Big)$ $(\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12}).$

The 2-form that appears here is actually zero! This simple, but remarkable fact, known as *Arnold's identity* (see [**Ar1**]) can be put into the following form:

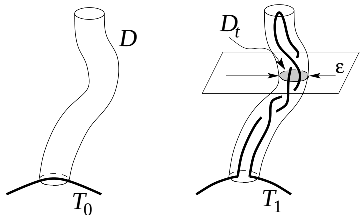
$$f + g + h = 0 \implies \frac{df}{f} \wedge \frac{dg}{g} + \frac{dg}{g} \wedge \frac{dh}{h} + \frac{dh}{h} \wedge \frac{df}{f} = 0$$

(in our case $f = z_1 - z_2$, $g = z_2 - z_3$, $h = z_3 - z_1$) and verified by a direct computation.

This finishes the proof. $\square$

Remark. The Kontsevich integral of a tangle may change, if the boundary points are moved. Examples may be found below in Exercises **??** and 6—9.

8.6.3. Moving the critical points. Let T_0 and T_1 be two tangles which are identical except a sharp "tail" of width ε, which may be twisted:



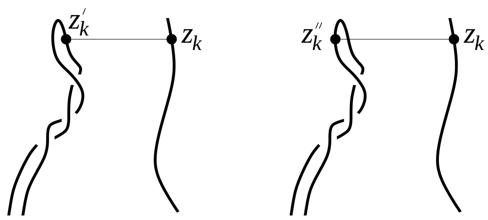
More exactly, we assume that (1) T_1 is different from T_0 only inside a region D which is the union of disks D_t of diameter ε lying in horizontal planes with fixed $t \in [t_1, t_2]$, (2) each tangle T_0 and T_1 has exactly one critical point in D, and (3) each tangle T_0 and T_1 intersects every disk D_t at most in two points. We call the passage from T_0 to T_1 a *special movement* of the critical point. To prove Theorem 8.6.1 it is sufficient to show the invariance of the Kontsevich integral under such movements. Note that special movements of critical points may take a Morse knot out of the class of strict Morse knots.

Proposition. *The Kontsevich integral remains unchanged under a special movement of the critical point:* $Z(T_0) = Z(T_1)$.

Proof. The difference between $Z(T_0)$ and $Z(T_1)$ can come only from the terms with a chord ending on the tail.

If the highest of such chords connects the two sides of the tail, then the corresponding tangle chord diagram is zero by a one-term relation.

So we can assume that the highest, say, the kth, chord is a "long" chord, which means that it connects the tail with another part of T_1. Suppose the endpoint of the chord belonging to the tail is (z'_k, t_k). Then there exists another choice for kth chord which is almost the same but ends at another point of the tail (z''_k, t_k) on the same horizontal level:



The corresponding two terms appear in $Z(T_1)$ with the opposite signs due to the sign $(-1)^{\downarrow}$.

Let us estimate the difference of the integrals corresponding to such kth chords:

$$\left| \int\limits_{t_{k+1}}^{t_c} d(\ln(z'_k - z_k)) \; - \int\limits_{t_{k+1}}^{t_c} d(\ln(z''_k - z_k)) \right| = \left| \ln\left(\frac{z''_{k+1} - z_{k+1}}{z'_{k+1} - z_{k+1}} \right) \right|$$

$$= \left| \ln\left(1 + \frac{z''_{k+1} - z'_{k+1}}{z'_{k+1} - z_{k+1}} \right) \right| \sim \left| z''_{k+1} - z'_{k+1} \right| \leqslant \varepsilon$$

(here t_c is the value of t at the uppermost point of the tail).

Now, if the next $(k+1)$st chord is also long, then, similarly, it can be paired with another long chord so that they give a contribution to the integral proportional to $\left| z''_{k+2} - z'_{k+2} \right| \leqslant \varepsilon$.

In the case the $(k+1)$st chord is short (that is, it connects two points z''_{k+1}, z'_{k+1} of the tail) we have the following estimate for the double integral corresponding to kth and $(k+1)$st chords:

$$\left| \int\limits_{t_{k+2}}^{t_c} \left(\int\limits_{t_{k+1}}^{t_c} d(\ln(z'_k - z_k)) \; - \int\limits_{t_{k+1}}^{t_c} d(\ln(z''_k - z_k)) \right) \frac{dz''_{k+1} - dz'_{k+1}}{z''_{k+1} - z'_{k+1}} \right|$$

$$\leqslant \text{const} \cdot \left| \int\limits_{t_{k+2}}^{t_c} \left| z''_{k+1} - z'_{k+1} \right| \frac{dz''_{k+1} - dz'_{k+1}}{\left| z''_{k+1} - z'_{k+1} \right|} \right|$$

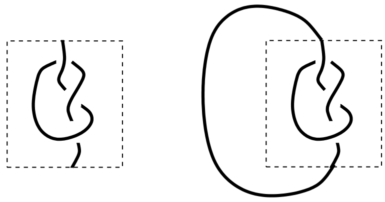
$$= \text{const} \cdot \left| \int\limits_{t_{k+2}}^{t_c} d(z''_{k+1} - z'_{k+1}) \right| \sim \left| z''_{k+2} - z'_{k+2} \right| \leqslant \varepsilon \; .$$

Continuing this argument, we see that the difference between $Z(T_0)$ and $Z(T_1)$ is $O(\varepsilon)$. Now, by horizontal deformations we can make ε tend to zero. This proves the theorem and completes the proof of the Kontsevich integral's invariance in the class of knots with nondegenerate critical points. □

8.7. Changing the number of critical points

The multiplicativity of the Kontsevich integral for tangles (Propositions 8.4.3 and 8.4.5) have several immediate consequences for knots.

8.7.1. From long knots to usual knots. A long (Morse) knot can be closed up so as to produce a usual (Morse) knot:



Recall that the algebras of chord diagrams for long knots and for usual knots are essentially the same; the isomorphism is given by closing up a linear chord diagram.

Proposition. *The Kontsevich integral of a long knot T coincides with that of its closure K_T.*

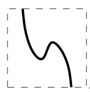
Proof. Denote by $\mathbf{1}$ the tangle consisting of one vertical strand. Then K_T can be written as $T_{max} \cdot (T \otimes_\varepsilon \mathbf{1}) \cdot T_{min}$ where T_{max} and T_{min} are a maximum and a minimum respectively, and $0 < \varepsilon \leqslant 1$.

Since the Kontsevich integral of K_T does not depend on ε, we can take $\varepsilon \to 0$. Therefore,

$$Z(K_T) = Z(T_{max}) \cdot (Z(T) \otimes Z(\mathbf{1})) \cdot Z(T_{min}).$$

However, the Kontsevich integrals of T_{max}, T_{min} and $\mathbf{1}$ consist of one diagram with no chords, and the Proposition follows. □

A corollary of this is the formula (8.2.2.1) (page 229) which describes the behavior of the Kontsevich integral under the addition of a pair of critical points. Indeed, adding a pair of critical points to a long knot T is the same as multiplying it by

,

and (8.2.2.1) then follows from the multiplicativity of the Kontsevich integral for tangles.

8.7.2. The universal Vassiliev invariant. The formula (8.2.2.1) allows one to define the *universal Vassiliev invariant* by either

$$I(K) = \frac{Z(K)}{Z(H)^{c/2}}$$

or

$$I'(K) = \frac{Z(K)}{Z(H)^{c/2-1}},$$

where c denotes the number of critical points of K in an arbitrary Morse representation, and the quotient means division in the algebra $\widehat{\mathcal{A}}$: $(1+a)^{-1} = 1 - a + a^2 - a^3 + \dots$.

Any isotopy of a knot in $\mathbb{R}^3$ can be approximated by a sequence consisting of isotopies within the class of (not necessarily strict) Morse knots and insertions/deletions of "humps", that is, pairs of adjacent maxima and minima. Hence, the invariance of $Z(K)$ in the class of Morse knots and the formula (8.2.2.1) imply that both $I(K)$ and $I'(K)$ are invariant under an arbitrary deformation of K. (The meaning of the "universality" will be explained in the next section.)

The version $I'(K)$ has the advantage of being multiplicative with respect to the connected sum of knots; in particular, it vanishes (more precisely, takes the value 1) on the unknot. However, the version $I(K)$ is also used as it has a direct relationship with the quantum invariants (see ???). In particular, we shall use the term "Kontsevich integral of the unknot"; this, of course, refers to I, and not I'.

8.8. Proof of the Kontsevich theorem

First of all we reformulate the Kontsevich theorem (or, more exactly, the Kontsevich part of the Vassiliev–Kontsevich theorem 4.2.1) as follows.

8.8.1. Theorem. *Let w be an unframed weight system of order n. Then there exists a Vassiliev invariant of order n whose symbol is w.*

Proof. The desired knot invariant is given by the formula

$$K \longmapsto w(I(K)).$$

Let D be a chord diagram of order n and let K_D be a singular knot with chord diagram D. The theorem follows from the fact that $I(K_D) = D +$ (terms of order $> n$). Since the denominator of $I(K)$ starts with the unit of the algebra $\mathcal{A}$, it is sufficient to prove that

$$Z(K_D) = D + (\text{terms of order} > n). \tag{8.8.1.1}$$

In fact, we shall establish (8.8.1.1) for D an arbitrary *tangle* chord diagram and $K_D = T_D$ - a singular tangle with the diagram D.

If $n = 0$, the diagram D has no chords and T_D is non-singular. For a non-singular tangle the Kontsevich integral starts with a tangle chord diagram with no chords, and (8.8.1.1) clearly holds. Note that the Kontsevich integral of any singular tangle (with at least one double point) necessarily starts with terms of degree at least 1.

Consider now the case $n = 1$. If T_D is a singular 2-braid, there is only one possible term of degree 1, namely the chord diagram with the chord connecting the two strands. The coefficients of this diagram in $Z(T_+)$ and $Z(T_-)$, where $T_+ - T_-$ is a resolution of the double point of T_D, simply measure the number of full twists in T_+ and T_- respectively. The difference of these numbers is 1, so in this case (8.8.1.1) is also true.

Now, let T_D be an arbitrary singular tangle with exactly one double point, and V_ε be the ε-neighbourhood of the singularity. We can assume that the intersection of T_D with V_ε is a singular 2-braid, and that the double point of T_D is resolved as $T_D = T_+^\varepsilon - T_-^\varepsilon$ where T_+^ε and T_-^ε coincide with T outside V_ε.

Let us write the degree 1 part of $Z(T_\pm^\varepsilon)$ as a sum $Z'_\pm + Z''_\pm$ where $Z'_\pm$ is the integral over all chords whose both ends are contained in V_ε and $Z''_\pm$ is the rest, that is, the integral over the chords with at least one end outside V_ε. As ε tends to 0, $Z''_+ - Z''_-$ vanishes. On the other hand, for all ε we have that $Z''_+ - Z''_-$ equals to the diagram D with the coefficient 1. This settles the case $n = 1$.

Finally, if $n > 1$, using a suitable deformation, if necessary, we can always achieve that T_D is a product of n singular tangles with one double point each. Now (8.8.1.1) follows from the multiplicativity of the Kontsevich integral for tangles.

□

8.8.2. Universality of $I(K)$. In the proof of the Kontsevich Theorem we have seen that for a singular knot K with n double points, $I(K)$ starts with terms of degree n. This means that if $I_n(K)$ denotes the nth graded component of the series $I(K)$, then the function $K \mapsto I_n(K)$ is a Vassiliev invariant of order n.

In some sense, all Vassiliev invariants are of this type:

8.8.3. Proposition. *Any Vassiliev invariant can be factored through I: for any $v \in \mathcal{V}$ there exists a linear function f on $\widehat{\mathcal{A}}$ such that $v = f \circ I$.*

Proof. Let $v \in \mathcal{V}_n$. By the Kontsevich theorem we know that there is a function f_0 such that v and $f_0 \circ I_n$ have the same symbol. Therefore, the difference $v - f_0 \circ I_n$ belongs to $\mathcal{V}_{n-1}$ and is thus representable as $f_1 \circ I_{n-1}$. Proceeding in this way, we shall finally obtain:

$$v = \sum_{i=1}^{n} f_i \circ I_{n-i}.$$

□

Remark. The construction of the foregoing proof shows that the universal Vassiliev invariant induces a splitting of the filtered space $\mathcal{V}$ into a direct sum with summands isomorphic to the factors $\mathcal{V}_n/\mathcal{V}_{n-1}$. Elements of these subspaces are referred to as *canonical Vassiliev invariants.* We shall speak about them in more detail later in Section 11.2.

As a corollary, we get the following statement:

8.8.4. Theorem. *The universal Vassiliev invariant I is exactly as strong as the set of all Vassiliev invariants: for any two knots K_1 and K_2 we have*

$$I(K_1) = I(K_2) \quad \Longleftrightarrow \quad \forall v \in \mathcal{V} \quad v(K_1) = v(K_2).$$

8.9. Towards the combinatorial Kontsevich integral

Since the Kontsevich integral comprises all Vassiliev invariants, calculating it explicitly is a very important problem. Knots are, essentially, combinatorial objects so it is not surprising that the Kontsevich integral, which we have defined analytically, can be calculated combinatorially from the knot diagram. Different versions of such combinatorial definition were proposed in several papers ([**BN2, Car1, LM1, LM2, Piu**]) and treated in several books ([**Kas, Oht1**]). Such a definition will be given in Chapter 10; here we shall explain the idea behind it.

The multiplicativity of the Kontsevich integral hints at the following method of computing it: present a knot as a product of several standard tangles whose Kontsevich integral is known and then multiply the corresponding values of the integral. This method works well for the quantum invariants, see Sections 2.6.5 and 2.6.6; however, for the Kontsevich integral it turns out to be too naïve to be of direct use.

Indeed, in the case of quantum invariants we decompose the knot into elementary tangles, that is, crossings, max/min events and pieces of vertical strands using both the usual product and the tensor product of tangles. While the Kontsevich integral behaves well with respect to the usual product of tangles, there is no simple expression for the integral of the tensor product of two tangles, even if one of the factors is a trivial tangle. As a consequence, the Kontsevich integral is hard to calculate even for the generators of the braid group, not to mention other possible candidates for "standard" tangles.

Still, we know that the Kontsevich integral is *asymptotically* multiplicative with respect to the parameterized tensor product. This suggests the following procedure.

Write a knot K as a product of tangles $K = T_1 \cdot \ldots \cdot T_n$ where each T_i is a tensor product of several elementary tangles. Let us think of each T_i as of an ε-parameterized tensor product of elementary tangles with $\varepsilon = 1$. We want to vary this ε to make it very small. There are two issues here that should be taken care of.

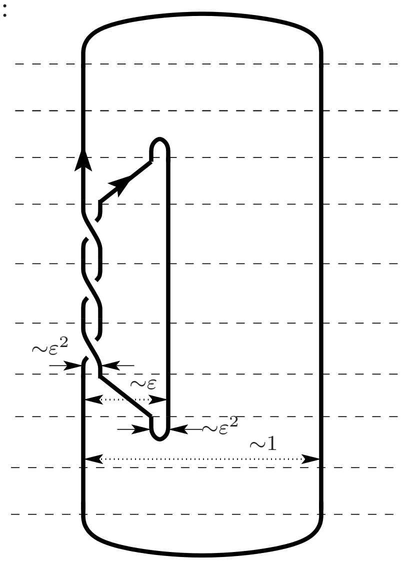
Firstly, the ε-parameterized tensor product is not associative for $\varepsilon \neq 1$, so we need a *parenthesizing* on the factors in T_i. We choose the parenthesizing arbitrarily on each T_i and denote by T_i^ε the tangle obtained from T_i by replacing $\varepsilon = 1$ by an arbitrary positive $\varepsilon \leqslant 1$.

Secondly, even though the tangles T_i and T_{i+1} are composable, the tangles T_i^ε and T_{i+1}^ε may fail to be composable for $\varepsilon < 1$. Therefore, for each i we have to choose a family of *associating* tangles without crossings Q_i^ε which connect the bottom endpoints of T_i^ε with the corresponding top endpoints of T_{i+1}^ε.

Now we can define a family of knots K^ε as

$$K^\varepsilon = T_1^\varepsilon \cdot Q_1^\varepsilon \cdot T_2^\varepsilon \cdot \ldots \cdot Q_{n-1}^\varepsilon \cdot T_n^\varepsilon.$$

The following picture illustrates this construction on the example of a trefoil knot:



T_1^ε	$\overrightarrow{\max}$
Q_1^ε	
T_2^ε	$(\mathrm{id} \otimes_\varepsilon \overrightarrow{\max}) \otimes_\varepsilon \mathrm{id}^*$
Q_2^ε	
T_3^ε	$(X_- \otimes_\varepsilon \mathrm{id}^*) \otimes_\varepsilon \mathrm{id}^*$
T_4^ε	$(X_- \otimes_\varepsilon \mathrm{id}^*) \otimes_\varepsilon \mathrm{id}^*$
T_5^ε	$(X_- \otimes_\varepsilon \mathrm{id}^*) \otimes_\varepsilon \mathrm{id}^*$
Q_5^ε	
T_6^ε	$(\mathrm{id} \otimes_\varepsilon \underleftarrow{\min}) \otimes_\varepsilon \mathrm{id}^*$
Q_6^ε	
T_7^ε	$\underleftarrow{\min}$

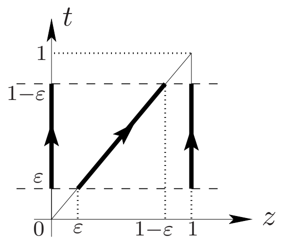
Figure 8.9.0.1. A decomposition of the trefoil into associating tangles and ε-parameterized tensor products of elementary tangles, with the notations from Section 1.7.7. The associating tangles between T_3^ε, T_4^ε and T_5^ε are omitted since these tangles are composable for all ε.

Since for each ε the knot K^ε is isotopic to K it is tempting to take $\varepsilon \to 0$, calculate the limits of the Kontsevich integrals of the factors and then take their product. The limit

$$\lim_{\varepsilon \to 0} Z(T_i^\varepsilon)$$

is easily evaluated, so it only remains to calculate the limit of $Z(Q_i^\varepsilon)$ as ε tends to zero.

Calculating this last limit is not a straightforward task, to say the least. In particular, if Q^ε is the simplest associating tangle



we shall see in the next chapter that asymptotically, as $\varepsilon \to 0$ we have

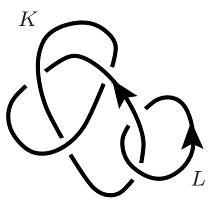
$$Z(\uparrow\!\nearrow\!\uparrow) \simeq \varepsilon^{\frac{1}{2\pi i}\uparrow\,\overline{\uparrow\uparrow}} \cdot \Phi_{\mathrm{KZ}} \cdot \varepsilon^{-\frac{1}{2\pi i}\overline{\uparrow\uparrow}\,\uparrow},$$

where ε^x is defined as the formal power series $\exp(x \log \varepsilon)$ and Φ_{KZ} is the power series known as the *Knizhnik-Zamolodchikov associator*. Similar formulae can be written for other associating tangles.

There are two difficulties here. One is that the integral $Z(Q^\varepsilon)$ does not converge as ε tends to 0. However, all the divergence is hidden in the terms $\varepsilon^{\frac{1}{2\pi i}\uparrow\,\overline{\uparrow\uparrow}}$ and $\varepsilon^{-\frac{1}{2\pi i}\overline{\uparrow\uparrow}\,\uparrow}$ and careful analysis shows that all such terms from all associating tangles cancel each other out in the limit, and can be omitted. The second problem is to calculate the associator. This a highly non-trivial task, and is the main subject of the next chapter.

Exercises

(1) For the link with two components K and L shown on the right draw the configuration space of horizontal chords joining K and L as in the proof of the linking number theorem from Section 8.1.2 (see page 224). Compute the linking number of K and L using this theorem.



(2) Is it true that $Z(\overline{H}) = Z(H)$, where H is the hump as shown in page 229 and $\overline{H}$ is the same hump reflected in a horizontal line?

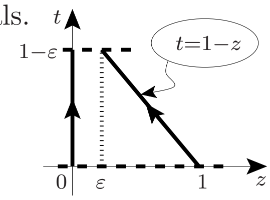
(3) M. Kontsevich in his pioneering paper [**Kon1**] and some of his followers (for example, [**BN1, CD3**]) defined the Kontsevich integral slightly

differently, numbering the chords upwards. Namely, $\quad Z_{Kont}(K) =$

$$= \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int\limits_{\substack{t_{\min}<t_1<\dots<t_m<t_{\max} \\ t_j \text{ are noncritical}}} \sum_{P=\{(z_j,z'_j)\}} (-1)^{\downarrow_P} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j} \ .$$

Prove that for any tangle T, $Z_{Kont}(T) = Z(T)$, as series of tangle chord diagrams.

Hint. Change of variables in multiple integrals.

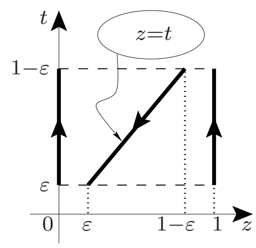


(4) Prove that for the tangle $\uparrow\nwarrow$ shown on the right $\quad Z(\uparrow\nwarrow) = \exp\Bigl(\frac{\text{chord}}{2\pi i} \cdot \ln\varepsilon\Bigr)$.

(5) The *Euler dilogarithm* is defined by the power series $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ for $|z| \leqslant 1$. Prove the following identities

$$\mathrm{Li}_2(0) = 0; \qquad \mathrm{Li}_2(1) = \tfrac{\pi^2}{6}; \qquad \mathrm{Li}'_2(z) = -\tfrac{\ln(1-z)}{z} \ ;$$
$$\tfrac{d}{dz}\Bigl(\mathrm{Li}_2(1-z) + \mathrm{Li}_2(z) + \ln z \ln(1-z)\Bigr) = 0 \ ;$$
$$\mathrm{Li}_2(1-z) + \mathrm{Li}_2(z) + \ln z \ln(1-z) = \tfrac{\pi^2}{6} \ .$$

About these and other remarkable properties of $\mathrm{Li}_2(z)$ see [**Lew, Kir, Zag2**].



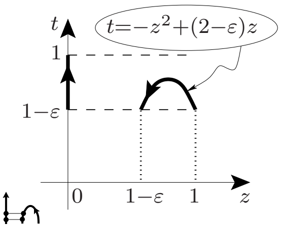
(6) Consider the associating tangle $\uparrow\nearrow\uparrow$ shown on the right. Compute $Z(\uparrow\nearrow\uparrow)$ up to the second order.

Answer. $\uparrow\downarrow\uparrow \ - \ \frac{1}{2\pi i}\ln\left(\frac{1-\varepsilon}{\varepsilon}\right)\Bigl(\ \cdots\ - \ \cdots\ \Bigr)$
$- \frac{1}{8\pi^2}\ln^2\left(\frac{1-\varepsilon}{\varepsilon}\right)\Bigl(\ \cdots\ + \ \cdots\ \Bigr)$
$+\frac{1}{4\pi^2}\Bigl(\ln(1-\varepsilon)\ln\left(\frac{1-\varepsilon}{\varepsilon}\right) + \mathrm{Li}_2(1-\varepsilon) - \mathrm{Li}_2(\varepsilon)\Bigr)\ \cdots$
$-\frac{1}{4\pi^2}\Bigl(\ln(\varepsilon)\ln\left(\frac{1-\varepsilon}{\varepsilon}\right) + \mathrm{Li}_2(1-\varepsilon) - \mathrm{Li}_2(\varepsilon)\Bigr)\ \cdots$

The calculation here uses the dilogarithm function defined in problem (5). Note that the Kontsevich integral diverges as $\varepsilon \to 0$.

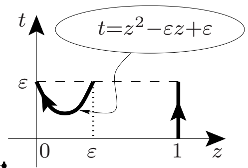
(7) Make the similar computation $Z(\uparrow\nwarrow\uparrow)$ for the reflected tangle. Describe the difference with the answer to the previous problem.

(8) Compute the Kontsevich integral $Z(\;)$ of the maximum tangle shown on the right.



Answer. $\; + \frac{1}{2\pi i}\ln(1-\varepsilon)\;$
$+\frac{1}{4\pi^2}\Big(\mathrm{Li}_2\big(\frac{\varepsilon}{2-\varepsilon}\big)-\mathrm{Li}_2\big(\frac{-\varepsilon}{2-\varepsilon}\big)\Big)\;$
$+\frac{1}{8\pi^2}\Big(\ln^2 2-\ln^2\Big(\frac{1-\varepsilon}{2-\varepsilon}\Big)+2\mathrm{Li}_2\big(\frac{1}{2}\big)-2\mathrm{Li}_2\big(\frac{1-\varepsilon}{2-\varepsilon}\big)\Big)\;$
$+\frac{1}{8\pi^2}\Big(\ln^2 2-\ln^2(2-\varepsilon)+2\mathrm{Li}_2\big(\frac{1}{2}\big)-2\mathrm{Li}_2\big(\frac{1}{2-\varepsilon}\big)\Big)\;$

(9) Compute the Kontsevich integral $Z(\;)$ of the minimum tangle shown on the right.



Answer. $\; - \frac{1}{2\pi i}\ln(1-\varepsilon)\;$
$+\frac{1}{4\pi^2}\Big(\mathrm{Li}_2\big(\frac{\varepsilon}{2-\varepsilon}\big)-\mathrm{Li}_2\big(\frac{-\varepsilon}{2-\varepsilon}\big)\Big)\;$
$+\frac{1}{8\pi^2}\Big(\ln^2 2-\ln^2\Big(\frac{1-\varepsilon}{2-\varepsilon}\Big)+2\mathrm{Li}_2\big(\frac{1}{2}\big)-2\mathrm{Li}_2\big(\frac{1-\varepsilon}{2-\varepsilon}\big)\Big)\;$
$+\frac{1}{8\pi^2}\Big(\ln^2 2-\ln^2(2-\varepsilon)+2\mathrm{Li}_2\big(\frac{1}{2}\big)-2\mathrm{Li}_2\big(\frac{1}{2-\varepsilon}\big)\Big)\;$

Note that all nontrivial terms in the last two problems tend to zero as $\varepsilon\to 0$.

(10) Express the Kontsevich integral of the hump as the product of tangle chord diagrams from problems 6, 8, 9:

$$Z(\;)=Z(\;)\cdot Z(\;)\cdot Z(\;)\ .$$

To do this introduce shorthand notation for the coefficients:

$Z(\;)=\; + A\; + B\; + C\; + D\;$

$Z(\;)=\; + E\,(\;-\;) + F\,(\;+\;) + G\; + H\;$

$Z(\;)=\; + I\; + J\; + K\; + L\;.$

Show that the order 1 terms of the product vanish.

The only nonzero chord diagram of order 2 on the hump is the cross (diagram without isolated chords). The coefficient of this diagram is $B+D+G+J+L-AE+AI+EI$. Show that it is equal to

$$\frac{\mathrm{Li}_2\big(\frac{\varepsilon}{2-\varepsilon}\big)-\mathrm{Li}_2\big(\frac{-\varepsilon}{2-\varepsilon}\big)+\mathrm{Li}_2\big(\frac{1}{2}\big)-\mathrm{Li}_2\big(\frac{1}{2-\varepsilon}\big)-\mathrm{Li}_2(\varepsilon)}{2\pi^2}+\frac{\ln^2 2-\ln^2(2-\varepsilon)}{4\pi^2}+\frac{1}{24}\ .$$

Using the properties of the dilogarithm mentioned in problem 5 prove that the last expression equals $\frac{1}{24}$. This is also a consequence of the remarkable *Roger five-term relation* (see, for example, [**Kir**])

$$\mathrm{Li}_2 x+\mathrm{Li}_2 y-\mathrm{Li}_2 xy=\mathrm{Li}_2\frac{x(1-y)}{1-xy}+\mathrm{Li}_2\frac{y(1-x)}{1-xy}+\ln\frac{(1-x)}{1-xy}\ln\frac{(1-y)}{1-xy}$$

and the *Landen connection formula* (see, for example, [**Roos**])

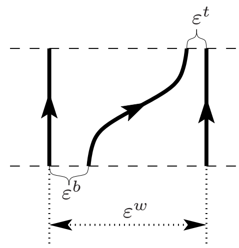
$$\mathrm{Li}_2 z+\mathrm{Li}_2\frac{-z}{1-z}=-\frac{1}{2}\ln^2(1-z)\ .$$

(11) Let S_i be the operation of reversing the orientation of the ith string of a tangle T. Denote by the same symbol S_i the operation on tangle chord diagrams which multiplies a chord diagram by (-1) raised to the power equal to the number of chord endpoints lying on the ith string (see the formal definition in **??**). Prove that

$$Z(S_i(T)) = S_i(Z(T))$$

We shall use this operation in the next chapter.

(12) Compute the Kontsevich integral $Z(AT^t_{b,w})$ up to the order 2. Here ε is a small parameter, and w, t, b are natural numbers subject to inequalities $w < b$ and $w < t$.

$AT^t_{b,w} =$ 

Answer. $Z(AT^t_{b,w}) = \; +$

$+\frac{1}{2\pi i}\ln\left(\frac{\varepsilon^w-\varepsilon^t}{\varepsilon^b}\right) \; - \; \frac{1}{2\pi i}\ln\left(\frac{\varepsilon^w-\varepsilon^b}{\varepsilon^t}\right)$

$-\frac{1}{8\pi^2}\ln^2\left(\frac{\varepsilon^w-\varepsilon^t}{\varepsilon^b}\right) \; - \; \frac{1}{8\pi^2}\ln^2\left(\frac{\varepsilon^w-\varepsilon^b}{\varepsilon^t}\right)$

$-\frac{1}{4\pi^2}\left(\ln(\varepsilon^{b-w})\ln\left(\frac{\varepsilon^w-\varepsilon^b}{\varepsilon^t}\right) + \mathrm{Li}_2(1-\varepsilon^{b-w}) - \mathrm{Li}_2(\varepsilon^{t-w})\right)$

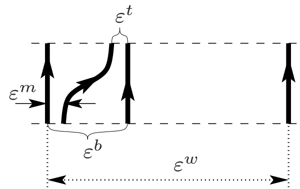
$+\frac{1}{4\pi^2}\left(\ln(1-\varepsilon^{t-w})\ln\left(\frac{\varepsilon^w-\varepsilon^b}{\varepsilon^t}\right) + \mathrm{Li}_2(1-\varepsilon^{b-w}) - \mathrm{Li}_2(\varepsilon^{t-w})\right)$.

(13) How will the Kontsevich's integral change, if in its definition integration over simplices is replaced by integration over cubes?

(14) Prove that

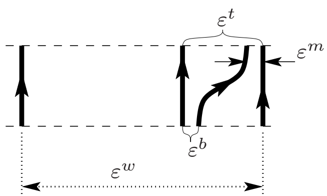
$$\Phi = \lim_{\varepsilon\to 0} \varepsilon^{-\frac{w}{2\pi i}\cdot\left(\; + \; \right)} \varepsilon^{-\frac{t}{2\pi i}\cdot} \cdot Z(AT^t_{b,w}) \cdot \varepsilon^{\frac{b}{2\pi i}\cdot} \cdot \varepsilon^{\frac{w}{2\pi i}\cdot\left(\; + \; \right)} .$$

(15) Prove that for the tangle $T^t_{m,b,w}$ on the right picture



$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{b-w}{2\pi i}\left(\; + \; \right)} \cdot \varepsilon^{-\frac{t-w}{2\pi i}} \cdot$$
$$\cdot Z(T^t_{m,b,w}) \cdot$$
$$\cdot \varepsilon^{\frac{m-w}{2\pi i}} \cdot \varepsilon^{\frac{b-w}{2\pi i}\left(\; + \; \right)} = \Phi \otimes \mathrm{id}.$$

(16) Prove that for the tangle $T^{t,m}_{b,w}$ on the right picture



$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{t-w}{2\pi i}\left(\; + \; \right)} \cdot \varepsilon^{-\frac{m-w}{2\pi i}} \cdot$$
$$\cdot Z(T^{t,m}_{b,w}) \cdot$$
$$\cdot \varepsilon^{\frac{b-w}{2\pi i}} \cdot \varepsilon^{\frac{t-w}{2\pi i}\left(\; + \; \right)} = \mathrm{id} \otimes \Phi.$$

Chapter 9

Operations on knots and the Kontsevich integral

We saw in Section 8.6.2 that the Kontsevich integral is multiplicative with respect to the connected sum of knots. Using this we prove a group-like property of the Kontsevich integral in Section 9.1. Then we consider other operations on knots and describe their effect on the Kontsevich integral. The operations in question are:

- σ — mirror reflection (changing the orientation of the ambient space),
- τ — changing the orientation of a knot,
- M_T — mutation of a knot with respect to a distinguished tangle T,
- $\mathbb{C}_n$ — n-th disconnected cabling of a knot,
- $\hat{\mathbb{C}}_n$ — n-th connected cabling of a knot .

9.1. The group-like property

9.1.1. Theorem. *For any Morse knot K the Kontsevich integral $Z(K)$ is a group-like element of the Hopf algebra $\widehat{\mathcal{A}}$:*

$$\delta(Z(K)) = Z(K) \otimes Z(K)\,,$$

where δ is the comultiplication defined in Section 4.4.4.

Proof. By multiplicativity, it is sufficient to prove this property for the Kontsevich integral of a tangle. Let T be a tangle without maximum and minimum points embedded into a slice $\mathbb{C}_z \times [a,b] \subset \mathbb{R}^3$. We will prove that

$$\delta(Z(T)) = Z(T) \otimes Z(T)\,, \tag{9.1.1.1}$$

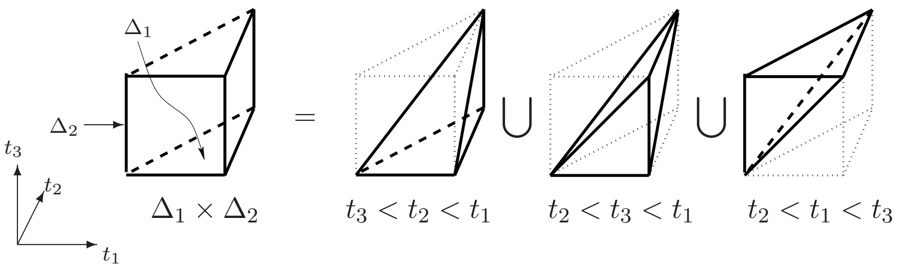
On the right-hand side of (9.1.1.1), consider the coefficient of the tensor product of two chord diagrams $D_1 \otimes D_2$ with m and n chords respectively. It comes from a particular choice of the pairing P_1 for m chords of D_1 on the levels $t_1, \dots t_m$, and a pairing P_2 for n chords of D_2 on the levels $t_{m+1}, \dots, t_{m+n}$. Denote by Δ_1 the simplex $a < t_m < \dots < t_1 < b$, and by Δ_2 the simplex $a < t_{m+n} < \dots < t_{m+1} < b$. Then the coefficient at $D_1 \otimes D_2$ on the right-hand side of (9.1.1.1) is the product of two integrals

$$\frac{(-1)^{\downarrow_1+\downarrow_2}}{(2\pi i)^{m+n}} \left(\int\limits_{\Delta_1} \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'} \right) \cdot \left(\int\limits_{\Delta_2} \bigwedge_{j=m+1}^{m+n} \frac{dz_j - dz_j'}{z_j - z_j'} \right)\,,$$

which can be written as a single integral over the product of simplices:

$$\frac{(-1)^{\downarrow_1+\downarrow_2}}{(2\pi i)^{m+n}} \int\limits_{\Delta_1 \times \Delta_2} \bigwedge_{j=1}^{m+n} \frac{dz_j - dz_j'}{z_j - z_j'}\,.$$

Now we split the product $\Delta_1 \times \Delta_2$ into the union of mutually disjoint simplices corresponding to all possible shuffles of two linearly ordered words $t_m < \dots < t_1$ and $t_{m+n} < \dots < t_{m+1}$. A *shuffle* of two words $t_m \dots t_1$ and $t_{m+n} \dots t_{m+1}$ is a word consisting of the letters $t_{m+n}, \dots, t_1$ and such that its subwords consisting of letters $t_m \dots t_1$ and $t_{m+n} \dots t_{m+1}$ preserve their linear orders. Here is an example ($m = 2$, $n = 1$) of such splitting:



The integral over the product of simplices is equal to the sum of integrals corresponding to all possible shuffles. But the integral over one particular simplex is precisely the coefficient in $Z(K)$ of the chord diagram obtained by merging the chord diagrams D_1 and D_2 according to the shuffle. This is equal to one term of the coefficient of $D_1 \otimes D_2$ in the left-hand side of (9.1.1.1). It is easy to see that the terms in the coefficient of $D_1 \otimes D_2$ in $\delta(Z(K))$ are in one to one correspondence with all ways to merge D_1 and D_2, that is, with all possible shuffles of the words $t_m \dots t_1$ and $t_{m+n} \dots t_{m+1}$. $\square$

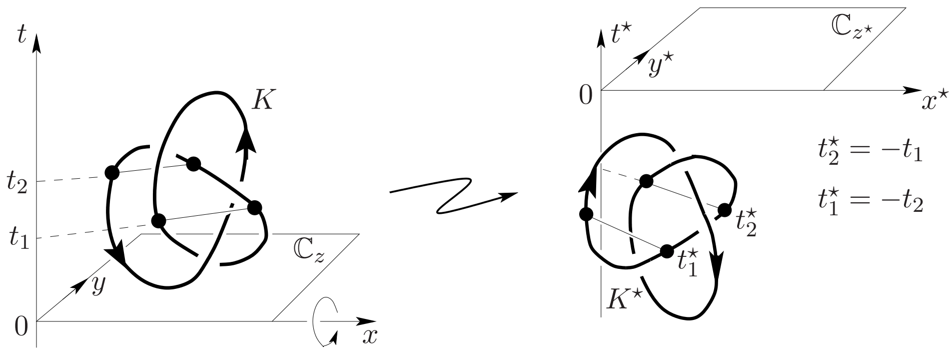
9.2. Reality of the integral

Choose a countable family of chord diagrams $\{D_i\}$ that constitute a basis in the vector space $\mathcal{A}$. The Kontsevich integral of a knot K can be written as an infinite series $Z(K) = \sum c_i D_i$ whose coefficients c_i depend on the knot K. A priori these coefficients are complex numbers.

9.2.1. Theorem. *All coefficients c_i of the Kontsevich integral $Z(K) = \sum c_i D_i$ are real numbers.*

Remark. Of course, this fact is a consequence of the Le–Murakami theorem 10.4.13 stating that these coefficients are rational. We state the previous theorem because it has a simple independent proof which is quite instructive.

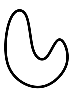
Proof. Rotate the knot K around the real axis x by 180°:



and denote the obtained knot by $K^\star$. To distinguish the objects (space, coordinates etc.) related to the knot $K^\star$, we will tag the corresponding symbols with a star. Coordinates $t^\star$, $z^\star$ in the space that contains $K^\star$ are related to the coordinates t, z in the ambient space of K as follows: $t^\star = -t$, $z^\star = \overline{z}$.

The rotation can be realized by a smooth isotopy, hence the universal Vassiliev invariants of K and $K^\star$ coincide: $\widetilde{Z}(K) = \widetilde{Z}(K^\star)$. The two knots have the same number c of critical points of the height function. Therefore

$$Z(K) = \widetilde{Z}(K) \cdot Z(h)^{c/2} = \widetilde{Z}(K^\star) \cdot Z(h)^{c/2} = Z(K^\star),$$

where $h =$  is the plane unknot with one hump. Therefore, to prove the reality of $Z(K)$, it is enough to prove that each coefficient of $Z(K^\star)$ is the complex conjugate to the corresponding coefficient of $Z(K)$.

Fix the number of chords m. Then each pairing $P = \{(z_j, z'_j)\}$, $1 \leqslant j \leqslant m$, for the knot K corresponds to a pairing $P^\star = \{(z_j^\star, z_j^{\star\prime})\}$ for the knot

$K^\star$, where $z_j^\star = \overline{z}_{m-j+1}$ and $z_j^{\star\prime} = \overline{z'}_{m-j+1}$. Note that the corresponding chord diagrams are equal: $D_P = D_{P^\star}$. Moreover, $\downarrow^\star = 2m - \downarrow$ and hence $(-1)^{\downarrow^\star} = (-1)^{\downarrow}$. The simplex $\Delta = t_{\min} < t_1 < \cdots < t_m < t_{\max}$ for the variables t_i corresponds to the simplex $\Delta^\star = -t_{\max} < t_m^\star < \cdots < t_1^\star < -t_{\min}$ for the variables $t_i^\star$. The coefficient of $D_{P^\star}$ in $Z(K^\star)$ is

$$c(D_{P^\star}) \;=\; \frac{(-1)^{\downarrow}}{(2\pi i)^m} \int \bigwedge_{j=1}^{m} d\ln(z_j^\star - z_j^{\star\prime}) \;,$$

where $z_j^\star$ and $z_j^{\star\prime}$ are understood as functions in $t_1^\star, \ldots, t_m^\star$ and the integral is taken over a connected component in the simplex $\Delta^\star$. In the last integral we make the change of variables according to the formula $t_j^\star = -t_{m-j+1}$. The Jacobian of this transformation is equal to $(-1)^{m(m+1)/2}$. Therefore,

$$c(D_{P^\star}) \;=\; \frac{(-1)^{\downarrow}}{(2\pi i)^m} \int (-1)^{m(m+1)/2} \bigwedge_{j=1}^{m} d\ln(\overline{z}_{m-j+1} - \overline{z'}_{m-j+1})$$

(integral over the corresponding connected component in the simplex Δ). Now permute the differentials to arrange the subscripts in the increasing order. The sign of this permutation is $(-1)^{m(m-1)/2}$. Note that $(-1)^{m(m+1)/2} \cdot (-1)^{m(m-1)/2} = (-1)^m$. Hence,

$$\begin{aligned} c(D_{P^\star}) \;&=\; \frac{(-1)^{\downarrow}}{(2\pi i)^m}(-1)^m \int \bigwedge_{j=1}^{m} d\ln(\overline{z}_j - \overline{z'}_j) \\ &=\; \frac{(-1)^{\downarrow}}{(2\pi i)^m} \int \bigwedge_{j=1}^{m} \overline{d\ln(z_j - z_j')} \;=\; \overline{c(D_P)}. \end{aligned}$$

The theorem is proved. □

9.3. Change of orientation

Let τ be the operation on knots which inverts their orientation (see 1.4). The same letter will also denote the analogous operation on chord diagrams (inverting the orientation of the outer circle or, which is the same thing, axial symmetry of the diagram).

9.3.1. Theorem. *The Kontsevich integral commutes with the operation τ:*

$$Z(\tau(K)) = \tau(Z(K)).$$

Proof. The required identity follows directly from the definition of the Kontsevich integral in Sec. 8.2. Indeed, the number $\downarrow$ for the knot $\tau(K)$ with inverse orientation is equal to the number $\uparrow$ for the initial knot K. Here by $\uparrow$ we of course mean the number of points (z_j, t_j) or (z_j', t_j) in a pairing P where the coordinate t grows along the orientation of K. Since the number of points in a pairing is always even, $(-1)^{\downarrow} = (-1)^{\uparrow}$. Hence the corresponding chord diagrams appear in $Z(\tau(K))$ and in $\tau(Z(K))$ with the same sign. The theorem is proved. $\square$

Corollary. The following two assertions are equivalent:

- *Vassiliev invariants do not distinguish the orientation of knots,*
- *all chord diagrams are symmetric: $D = \tau(D)$ modulo one- and four-term relations.*

The calculations of [**Kn0**] show that up to order 12 all chord diagrams are symmetric. For bigger orders the problem is still open.

9.3.2. Exercise. Prove the equivalence of the two claims:

- all chord diagrams are symmetric modulo one- and four-term relations.
- all chord diagrams are symmetric modulo only four-term relations.

9.4. Mirror reflection

Let σ be the operation sending a knot to its mirror image (see 1.4). Define the corresponding operation σ on chord diagrams as identity on the diagrams of even order and as multiplication by (-1) on the diagrams of odd order.

9.4.1. Theorem. *The Kontsevich integral commutes with the operation σ:*

$$Z(\sigma(K)) = \sigma(Z(K)),$$

where by $\sigma(Z(K))$ we mean simultaneous application of σ to all the chord diagrams participating in $Z(K)$.

Proof. Let us realize the operation σ on knots by the reflection of $\mathbb{R}^3 = \mathbb{R}_t \times \mathbb{C}_z$ coming from the complex conjugation in $\mathbb{C}_z$: $(t, z) \mapsto (t, \overline{z})$.

Consider the Kontsevich integral of K:

$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int \sum_P (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} d\ln(z_j - z_j') .$$

Then the Kontsevich integral for $\sigma(K)$ will look as

$$
\begin{aligned}
Z(\sigma(K)) &= \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int \sum_P (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} d\ln(\overline{z_j - z_j'}) \\
&= \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \overline{\int \sum_P (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} d\ln(z_j - z_j')} \\
&= \overline{\sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int \sum_P (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} d\ln(z_j - z_j')} \ .
\end{aligned}
$$

We see that the terms of $Z(\sigma(K))$ with an even number of chords coincide with those of $\overline{Z(K)}$ and terms of $Z(\sigma(K))$ with an odd number of chords differ from the corresponding terms of $\overline{Z(K)}$ by a sign. Since $Z(K)$ is real, this implies the theorem. □

Remark. Taking into account the fact that the Kontsevich integral is equivalent to the totality of all finite type invariants, and finite type invariants are defined by weight systems, Theorem 9.4.1 can be restated as follows: *Let v be an invariant of finite degree n, let K be a singular knot with n double points and $\overline{K} = \sigma(K)$ its mirror image. Then $v(K) = v(\overline{K})$ for even n and $v(K) = -v(\overline{K})$ for odd n.* This fact was first noticed by V. Vassiliev in [**Va1**].

Recall (see p. 23) that a knot K is called *plus-amphicheiral*, if it is equivalent to its mirror image as an oriented knot: $K = \sigma(K)$, and *minus-amphicheiral* if it is equivalent to the inverse of the mirror image: $K = \tau\sigma K$. Here τ is the axial symmetry on chord diagrams, and we call a chord diagram (or a linear combination of diagrams) *symmetric*, resp. *antisymmetric*, if τ acts on it as identity, resp. as multiplication by -1.

9.4.2. Corollary. *The Kontsevich integral $Z(K)$ (and hence the universal Vassiliev invariant $I(K)$) of a plus-amphicheiral knot K consist only of even order terms. For a minus-amphicheiral knot K the Kontsevich integral $Z(K)$ and the universal Vassiliev invariant $I(K)$ have the following property: their even-degree part consists only of symmetric chord diagrams, while their odd-degree part consists only of anti-symmetric diagrams.*

Proof. For a plus-amphicheiral knot, the theorem implies that $Z(K) = \sigma(Z(K))$, hence all the odd order terms in the series $Z(K)$ vanish. The quotient of two even series in the graded completion $\widehat{\mathcal{A}}$ is obviously even, therefore the same property holds for $I(K) = Z(K)/Z(H)^{c/2}$, too.

For a minus-amphicheiral knot K, we have $Z(K) = \tau(\sigma(Z(K)))$, which implies the second assertion. $\square$

Note that it is an open question whether non-symmetric chord diagrams exist. If they don't, then, of course, both assertions of the Theorem, for plus- and minus-amphicheiral knots, coincide.

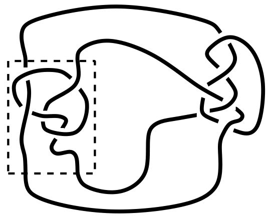
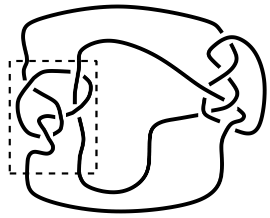
9.5. Mutation

The purpose of this subsection is to show that the Kontsevich integral commutes with the operation of mutation (this fact was first noticed by T. Le). As an application, we will construct a counterexample to the original intersection graph conjecture (p. 117) and describe all Vassiliev invariants which do not distinguish mutants following [**ChL**].

9.5.1. Mutation of knots.

9.5.2. Definition. Suppose we have a knot K with a distinguished tangle T which has two strings at the bottom and two strings at the top. Let us cut out the tangle, rotate it through 180° around a vertical axis and insert it back. This operation M_T is called *mutation* and the knot $M_T(K)$ thus obtained is called a *mutant* of K.

Here is a widely known pair of mutant knots, $11n34$ and $11n42$, which are mirrors of the *Conway* and *Kinoshita–Terasaka* knots respectively:

$11n34 = \overline{C} =$  $11n42 = \overline{KT} =$ 

9.5.3. Theorem ([**MC**]). . *There exists a Vassiliev invariant v of order 11 such that $v(C) \neq v(KT)$.*

Morton and Cromwell manufactured the invariant v using the Lie algebra $\mathfrak{gl}_N$ with a nonstandard representation (or, in other words, the HOMFLY polynomial of certain cablings of the knots).

J. Murakami [**Mu**] showed that any invariant or order at most 10 does not distinguish mutants. So order 11 is the first place where distinguishing mutants invariants occur.

9.5.4. Mutation of the Kontsevich integral. Let K be a knot with a distinguished tangle T that has two strings at the bottom and two strings at the top. Let $M_T(K)$ be the mutant of K obtained by the rotation of T. We define the mutation $M_T(Z(K))$ of the Kontsevich integral as the simultaneous mutation of all chord diagrams participating in $Z(K)$ (see the definition below). Then the following theorem holds.

9.5.5. Theorem ([**Le**]). *The Kontsevich integral commutes with the mutation operation M_T:*

$$Z(M_T(K)) = M_T(Z(K)).$$

By lemma **??** $Z(K)$ does not contain any chord diagram with a chord connecting T with the remaining part of K. This means that all chord diagrams which appear in $Z(K)$ are products (in the sense of section **??**) of chord diagrams on T and chord diagrams on $K \setminus T$. A chord diagram on the tangle, considered as a part of the whole chord diagram for the knot, was called *share* in Section 4.8.4.

The choice of a tangle T in a knot K distinguishes the T-shares (which correspond to the tangle chord diagrams of $Z(T)$) in all chord diagrams that appear in $Z(K)$.

9.5.6. Definition. The mutation $M_T(Z(K))$ of $Z(K)$ is the simultaneous rotation of the T-shares in all diagrams of $Z(K)$.

9.5.7. Proof of the theorem. Let T_r be the tangle obtained from T by rotation under the mutation M_T. Then $Z(K) = Z(T) \cdot Z(K \setminus T)$ and $Z(M_T(K)) = Z(T_r) \cdot Z(K \setminus T)$. Hence

$$\begin{aligned} M_T(Z(K)) &= M_T(Z(T) \cdot Z(K \setminus T)) = M_T(Z(T)) \cdot Z(K \setminus T) \\ &= Z(T_r) \cdot Z(K \setminus T) = Z(M_T(K)) \ . \end{aligned}$$

The theorem is proved. □

9.5.8. Counterexample to the Intersection Graph Conjecture. It is easy to see that the mutation of chord diagrams does not change the intersection graph. Thus, if the intersection graph conjecture (see 4.8.3) were true, the Kontsevich integrals of mutant knots would coincide. Hence all Vassiliev invariants would take the same value on mutant knots. But this contradicts Theorem 9.5.3. Therefore the intersection graph conjecture is false.

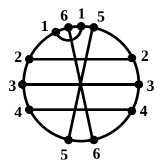
9.5.9. Now we can prove a theorem announced on page 118.

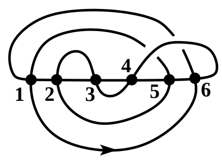
Theorem ([**ChL**]). *The symbol of a Vassiliev invariant that does not distinguish mutant knots depends on the intersection graph only.*

Proof. Suppose we have a Vassiliev knot invariant v of order at most n that does not distinguish mutant knots. Let D_1 and D_2 be chord diagrams with n chords whose intersection graphs coincide. We are going to prove that the values of the weight system of v on D_1 and D_2 are equal.

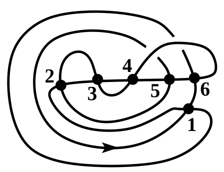
According to the theorem from page 118, D_2 can be obtained from D_1 by a sequence of mutations. It is enough to consider the case when D_1 and D_2 differ by a single mutation in a share S. Let K_1 be a singular knot with n double points whose chord diagram is D_1. Consider the collection of double points of K_1 corresponding to the chords occurring in the share S. By the definition of a share, K_1 has two arcs containing all these double points and no others. By sliding the double points along one of these arcs and shrinking the other arc we may enclose these arcs into a ball whose interior does not intersect the rest of the knot. In other words, we may isotope the knot K_1 to a singular knot so as to collect all the double points corresponding to S in a tangle T_S. Performing an appropriate rotation of T_S we obtain a singular knot K_2 with the chord diagram D_2. Since v does not distinguish mutants, its values on K_1 and K_2 are equal. The theorem is proved. □

To illustrate the proof, consider the chord diagram D_1 below. Pick a singular knot K_1 representing D_1.

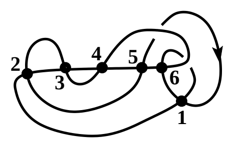
$D_1 =$ 

$K_1 =$ 

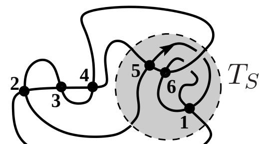
To perform a mutation in the share containing the chords 1,5,6, we must slide the double point 1 close to the double points 5 and 6, and then shrink the corresponding arcs:



Sliding the double point 1



Shrinking the arcs



Forming the tangle T_S

Now doing an appropriate rotation of the tangle T_S we obtain a singular knot K_2 representing the chord diagram D_2.

Combing the last theorem with 9.5.5 we get the following corollary.

9.5.10. Corollary. *Let w be a weight system on chord diagrams with n chords. Consider a Vassiliev invariant $v(K) := w \circ I(K)$. (In Section 11.2 they are called canonical.) Then v does not distinguish mutants if and only if the weight system w depends only on the intersection graph.*

9.6. Framed version of the Kontsevich integral

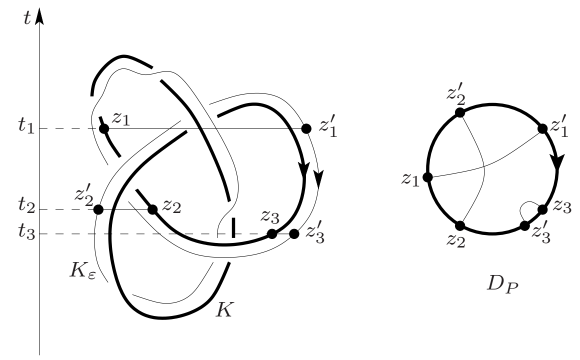
For framed knots and links, the Kontsevich integral was first defined by Le and Murakami [**LM2**] who gave a combinatorial construction of it. Later V. Goryunov defined [**Gor1**] the Kontsevich integral in an analytic fashion, his definition is a modification of the original definition from Section 8.2 and it works equally well for any framing.

The main difference of this theory from the one studied in Chapter 8 is that the framed version of the Kontsevich integral takes values in the algebra $\widehat{\mathcal{A}}^{fr}$ of chord diagrams modulo 4-term relations only.

9.6.1. The approach of Goryunov. Let K_ε be a copy of K shifted a small distance ε in the direction of the framing. We assume that both K and K_ε are in general position with respect to a height function t as in Section 8.2. Then we construct the integral $Z(K, K_\varepsilon)$ defined by the very same formula

$$Z(K,K_\varepsilon) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int\limits_{\substack{t_{\min}<t_m<\dots<t_1<t_{\max} \\ t_j \text{ are noncritical}}} \sum_{P=\{(z_j,z_j')\}} (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'} ,$$

where the pairings $P = \{(z_j, z_j')\}$ and the corresponding chord diagrams D_P are understood in a different way. Namely, we consider only those pairings where z_j lies on K while z_j' lies on K_ε. To obtain the chord diagram D_P we consider K as an oriented circle and connect a point $(z_j, t_j) \in K$ on it with a neighbor of the point $(z_j', t_j) \in K_\varepsilon$ lying on K. Every time when we choose neighboring points (z_j, z_j') in a pairing P, we get an isolated chord of D_P. The next picture illustrates these notions.



Now the framed Kontsevich integral can be defined as

$$\boxed{Z^{fr}(K) = \lim_{\varepsilon \to 0} Z(K, K_\varepsilon)} \,.$$

In [**Gor1**] V. Goryunov proved that the limit does exist and is invariant under the deformations of the framed knot K in the class of framed Morse knots. He used [**Gor2**] this construction to study Arnold's J^+-theory of plane curves (or, equivalently, Legendrian knots in a solid torus).

Having the definition of $Z^{fr}(K)$ we can define the final framed Kontsevich integral in a usual way

$$I^{fr}(K) = \frac{Z^{fr}(K)}{Z^{fr}(H)^{c/2}} ,$$

where H is the hump unknot (see page 229) with the blackboard framing.

The importance of the framed final Kontsevich integral $I^{fr}(K)$ is that it establishes the framed version of Theorem 8.8.1 (the Kontsevich part of the Vassiliev-Kontsevich theorem 4.2.1):

9.6.2. Theorem. *Let w be a framed weight system of order n (i. e. vanishing on chord diagrams whose number of chords is different from n). Then there exists a framed Vassiliev invariant of order $\leqslant n$ whose symbol is w.*

9.6.3. The approach of Le and Murakami. For the integral expression $Z^{fr}(K) = \lim\limits_{\varepsilon\to 0} Z(K, K_\varepsilon)$, one may try to do the combinatorial constructions similar to those of Sec.**??**. It is more convenient to use the blackboard framing. This will lead to the original combinatorial definition of the framed Kontsevich integral of [**LM2**]. It turns out that the basic elementary ingredients of the construction, the tangle chord diagrams max/min, R, and Φ remain literally the same as in the unframed case. The whole framed integral is constructed from them using operations $\otimes$, S_i, Δ_i from Sec.**??**, and the multiplication of tangle chord diagrams. The only difference in the framed case is that in the final expression we do not treat the diagrams with isolated chords as zero elements of the algebra $\widehat{\mathcal{A}}^{fr}$.

9.6.4. BGRT approach. There is one more way to define the final framed Kontsevich integral [**BGRT**]:

$$I^{fr}(K) = e^{\frac{w(K)}{2}\Theta} \cdot I(\overline{K}) \in \widehat{\mathcal{A}}^{fr},$$

where K is a framed knot, $\overline{K}$ is the same knot without framing, $w(K)$ is the writhe of K (the linking number of K and a copy of K shifted slightly in the direction of the framing), Θ is the chord diagram with one chord, and the exponent means the usual power series in the algebra $\widehat{\mathcal{A}}^{fr}$. Here, the universal Vassiliev invariant $I(\overline{K}) \in \widehat{\mathcal{A}}$ is understood as an element of the completed algebra $\widehat{\mathcal{A}}^{fr}$ (without 1-term relations) by virtue of a natural inclusion $\bar{p} : \mathcal{A} \to \mathcal{A}^{fr}$ (see page 110) defined as identity on the primitive subspace of $\mathcal{A}$.

9.6.5. The case of framed tangles. The above methods produce the Kontsevich integral not just for framed knots, but, more generally, for framed tangles. The preliminary integral $Z^{fr}(T)$ of a tangle T can be constructed as in the approaches of Goryunov or Le-Murakami, and the final integral $I^{fr}(T)$ is defined as

$$I^{fr}(T) = Z^{fr}(H)^{-m_1}\# \ldots \#Z^{fr}(H)^{-m_k}\#Z^{fr}(T),$$

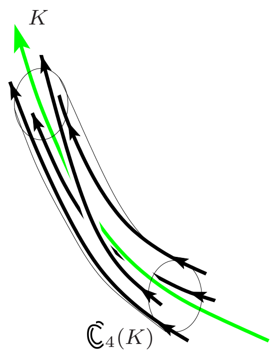
where m_i is the number of maxima on the ith component of T and $Z^{fr}(H)^{-m_i}$ acts on the ith component of $Z^{fr}(T)$, see 5.10.4. Here k is the number of components of T.

9.6.6. Group-like property. The framed Kontsevich integral $Z^{fr}(K)$ is a group-like element of the Hopf algebra $\widehat{\mathcal{A}}^{fr}$. This fact is proved in exactly the same way as the corresponding property of the unframed Kontsevich integral, see Section 9.1. The same statement holds for tangles.

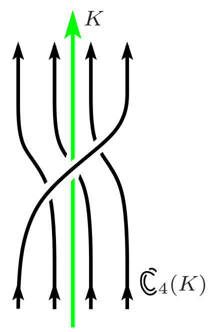
9.7. Cablings

9.7.1. Cablings of knots. Cablings are defined for framed knots in the following way. The framing, as a section of the normal bundle of a knot K, determines a trivialization of the bundle and supplies it with a complex structure, where the framing vector at every point ends at 1 of the corresponding normal plane considered as the plane of complex numbers.

The *n-th connected cabling* $\widehat{\mathbb{C}}_n(K)$ is the knot consisting of a bunch of n strings that follow the knot K in a narrow toroidal neighborhood rotating with respect to the framing so that after one full turn around K they close up with a clockwise rotation by $2\pi/n$.



4-th connected cabling of K



4-th connected cabling with respect to the blackboard framing

In formulas this definition can be expressed as follows. Consider K as the image of a map $K\colon S^1 \to \mathbb{R}^3$, where S^1 is the unit circle in the complex plane $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let $f(z)$ be the framing. Then we set

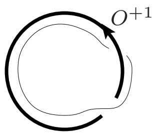
$$\widehat{\mathbb{C}}_n(K)(z) := K(z^n) + \varepsilon z f(z^n)\,,$$

where ε is a small positive number and operations are understood in terms of the complex structure on normal planes.

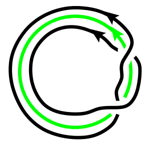
Similarly, the *n-th parallel*, or *n-th disconnected cabling* $\mathbb{C}_n(K)$ is a link consisting of n disjoint parallel copies of the knot K lying on the boundary of a thin toroidal neighbourhood of K, such that one of the copies is obtained by shifting points of K along the framing.

In formulas, the j-th component of the link $\mathbb{C}_n(K)(z)$ is given by

$$K(z) + \varepsilon e^{\frac{2\pi i}{n} j} f(z)\,.$$



unknot with framing $+1$



2-d disconnecting cabling of O^{+1} is a Hopf link, $\mathbb{C}_2(O^{+1}) =$

The links $\widehat{\mathbb{C}}_n(K)$ and $\mathbb{C}_n(K)$ inherit the framing of K.

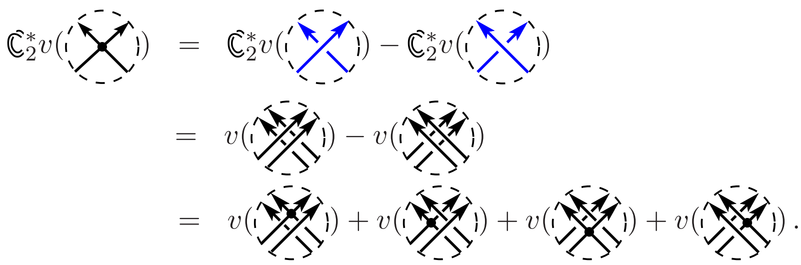
9.7.2. Cablings of knot invariants. Let v be a link invariant. Its *n-th cablings* $\widehat{\mathbb{C}}^*_n v$ and $\mathbb{C}^*_n v$ are defined in a natural way by setting

$$\widehat{\mathbb{C}}^*_n v(K) := v(\widehat{\mathbb{C}}_n(K))\,, \qquad \mathbb{C}^*_n v(K) := v(\mathbb{C}_n(K))\,.$$

9.7.3. Proposition. *If v is a framed Vassiliev invariant of order $\leqslant m$, then $\widehat{\mathbb{C}}^*_n v$ and $\mathbb{C}^*_n v$ are also framed Vassiliev invariants of order $\leqslant m$.*

Proof. We prove the proposition only for connected cablings and $n = 2$; for disconnected cablings and arbitrary n the proof is similar and we leave it to the reader.

According to Vassiliev's skein relation (Equation (3.1.2.1) on page 72), the extension of $\widehat{\mathbb{C}}^*_2 v$ to singular links satisfies

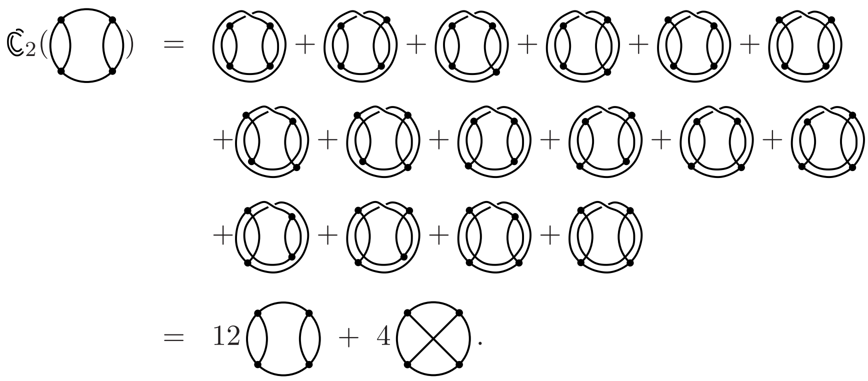


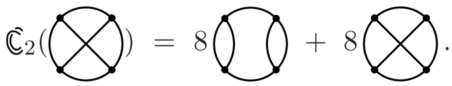
Therefore, vanishing of v on knots with more than m double points implies vanishing of $\widehat{\mathbb{C}}^*_2 v$ on knots with more than m double points. $\square$

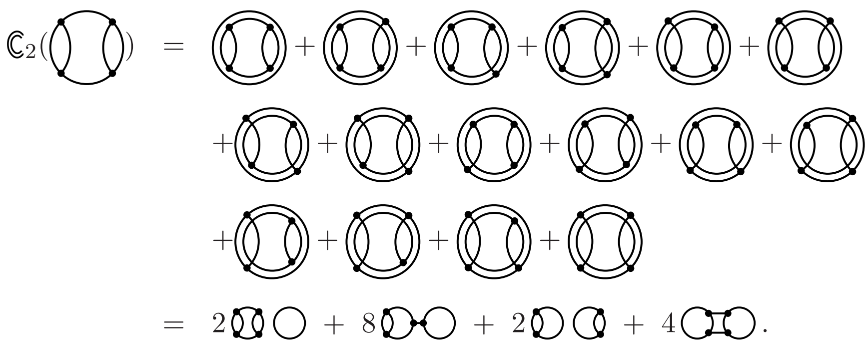
9.7.4. Cablings of chord diagrams. In this section we explore how the symbols of Vassiliev invariants v and $\hat{\mathbb{C}}_n^* v$ (resp. $\mathbb{C}_n^* v$) are related to each other.

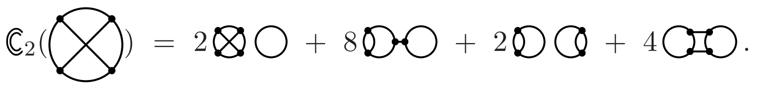
For a chord diagram D define $\hat{\mathbb{C}}_n(D)$ (resp. $\mathbb{C}_n(D)$) to be the sum of chord diagrams obtained by all possible ways of lifting the ends of the chords to the n-sheeted connected (resp. disconnected) covering of the circle of D. We extend $\hat{\mathbb{C}}_n$ (resp. $\mathbb{C}_n$) to the space spanned by chord diagrams by linearity.

Examples.









It is easy to see that the mapping $\hat{\mathbb{C}}_n$ satisfies the 4-term relation (Exercise 4 at the end of the Chapter), so it descends to the graded space $\mathcal{A}$. The examples above show that it is not an algebra homomorphism. However, it is a coalgebra automorphism according to Exercise 10 below.

Theorem. *Let v be a Vassiliev invariant. Then*

$$\mathrm{symb}(\hat{\mathbb{C}}_n^* v)(D) = \mathrm{symb}(v)(\hat{\mathbb{C}}_n(D))\,.$$

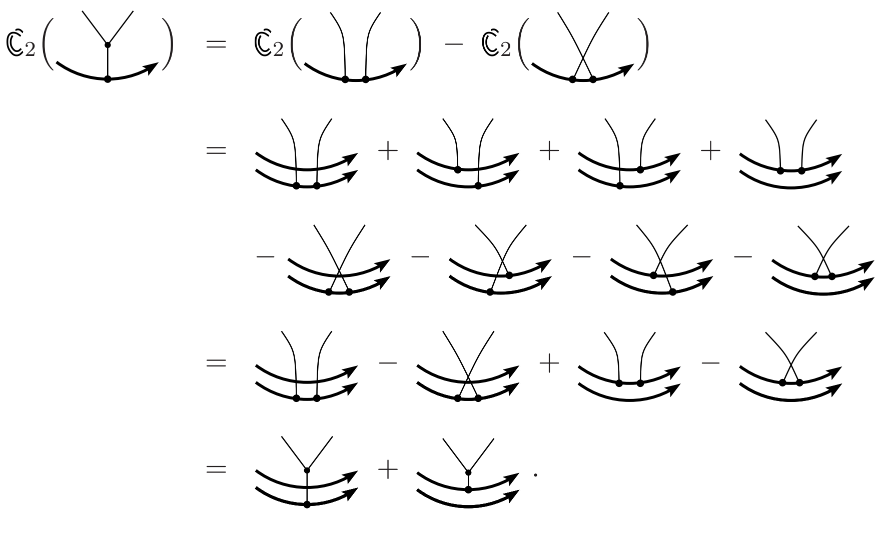
The proof follows directly from a careful analysis of the proof of Proposition 9.7.3.

9.7.5. Cablings of Jacobi diagrams. Since the algebra $\mathcal{A}$ is isomorphic both to the algebra of closed diagrams $\mathcal{C}$ (Theorem 5.3.1) and to the algebra of open diagrams $\mathcal{B}$ (Section 5.7), the operations $\hat{\mathbb{C}}_n$ and $\mathbb{C}_n$ transfers to these spaces, too. Below, we give an explicit description of the resulting operators.

Proposition. *For a closed diagram C, $\hat{\mathbb{C}}_n(C)$ (resp. $\mathbb{C}_n(D)$) is equal to the sum of closed diagrams obtained by all possible ways of lifting the univalent vertices of C to the n sheeted connected (resp. disconnected) covering of the Wilson loop of C.*

Proof. Induction on the number of internal vertices of C. If there are no internal vertices, then C is a chord diagram and the claim to be proved coincides with the initial definition of $\hat{\mathbb{C}}_n$ (resp. $\mathbb{C}_n$).

In general we may use STU relation to decrease the number of internal vertices and then use the induction hypothesis. The only thing one should check is that $\hat{\mathbb{C}}_n$ and $\mathbb{C}_n$ are compatible with the STU relation. We show how it works for $n = 2$:



□

9.7.6. Corollary. [**KSA**] *Every open diagram B with k univalent vertices (legs) is an eigenvector of the linear operator $\hat{\mathbb{C}}_n$ with eigenvalue n^k.*

Proof. The isomorphism $\chi : \mathcal{B} \cong \mathcal{C}$ takes an open diagram $B \in \mathcal{B}$ with k legs into the average of the $k!$ closed diagrams obtained by all possible numberings of the legs and attaching the Wilson loop according to the numbering. The value of $\mathfrak{C}_n$ on each closed diagram is equal to the sum of the n^k diagrams obtained by all possible lifts of its legs to the n-sheeted covering of the Wilson loop. Therefore, $\mathfrak{C}_n(B)$ is $1/k!$ times the sum of $n^k k!$ closed diagrams arranged in a 2-dimensional array with n^k columns and $k!$ rows. Each column, corresponding to a specific lift to the covering, contains the diagrams that differ from one another by all possible renumberings of their legs. The sum over each column divided by $k!$ is thus equal to B. Since the number of columns is n^k, we obtain $\mathfrak{C}_n(B) = n^k B$. □

9.7.7. Cablings of the Lie algebra weight systems. Given a semisimple Lie algebra $\mathfrak{g}$, in Chapter 6.1 we constructed the universal Lie algebra weight system $\varphi_{\mathfrak{g}} : \mathcal{A} \to U(\mathfrak{g})$. If, additionally, a representation V of $\mathfrak{g}$ is specified, then we have a numeric weight system $\varphi_{\mathfrak{g}}^V : \mathcal{A} \to \mathbb{C}$. The cabling operation $\mathfrak{C}_n$ acts on these weight systems as follows:

$$\mathfrak{C}_n^* \varphi_{\mathfrak{g}}(D) := \varphi_{\mathfrak{g}}(\mathfrak{C}_n(D)), \quad \mathfrak{C}_n^* \varphi_{\mathfrak{g}}^V(D) := \varphi_{\mathfrak{g}}^V(\mathfrak{C}_n(D))$$

for any chord diagram D.

The construction of the universal Lie algebra weight system (Section 6.1.1) rests on the assignment of basic vectors $e_i \in \mathfrak{g}$ to the endpoints of i'th chord, then taking their product along the Wilson loop and summing up over each index i. For the weight system $\mathfrak{C}_n^* \varphi_{\mathfrak{g}}$, to each endpoint of a chord we not only assign a basic vector, but we also indicate the level of the covering to which that particular point is lifted. To form an element of the universal enveloping algebra we must read the letters e_i along the circle n times. On the first pass we read only those letters which are related to the first sheet of the covering, omitting all the others. Then read the circle for the second time and now collect only the letters from the second sheet, etc. up to the n-th reading. The products of e_i's thus formed are summed up over all assignments and over all ways of lifting the endpoints to the covering.

The weight system $\mathfrak{C}_n^* \varphi_{\mathfrak{g}}$ can be expressed through $\varphi_{\mathfrak{g}}$ by means of a compact formula. To state it, we need two auxiliary operations in the tensor algebra of $U(\mathfrak{g})$. Let $\mu : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$ and $\delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ be the multiplication and comultiplication in $U(\mathfrak{g})$ (see Section A.1.7). Define the operations $\mu^n : U(\mathfrak{g})^{\otimes n} \to U(\mathfrak{g})$ and $\delta^n : U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes n}$ as compositions

$$\begin{aligned}
\mu^n &:= \mu \circ \big(\mathrm{id} \otimes \mu\big) \circ \cdots \circ \Big((\mathrm{id})^{\otimes(n-2)} \otimes \mu\Big)\,, \\
\delta^n &:= \Big((\mathrm{id})^{\otimes(n-2)} \otimes \delta\Big) \circ \cdots \circ \big(\mathrm{id} \otimes \delta\big) \circ \delta\,.
\end{aligned}$$

In other words, μ^n converts tensor products of the elements of $U(\mathfrak{g})$ into ordinary products, while δ^n sends each element $g \in \mathfrak{g}$ into

$$\delta^n(g) = g \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes g \otimes \ldots \otimes 1 + \cdots + 1 \otimes 1 \otimes \ldots \otimes g\,.$$

9.7.8. Proposition. *For a chord diagram D we have*

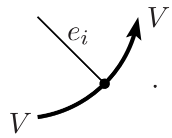
$$\mathbb{C}_n^* \varphi_{\mathfrak{g}}(D) = \mu^n \circ \delta^n(\varphi_{\mathfrak{g}}(D))\,.$$

We leave the proof of this proposition to the reader as an exercise (no. 14 at the end of the chapter).

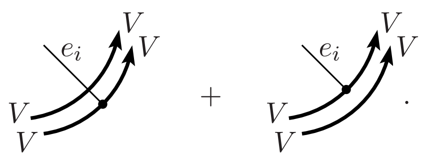
The operation $\mu^n \circ \delta^n$ expressed in terms of characters of the corresponding compact Lie group is called the *Adams operation* [**FH**] (see more details in [**BN1**]).

9.7.9. Cablings of the weight system $\varphi_{\mathfrak{g}}^V$. Recall that the weight system $\varphi_{\mathfrak{g}}^V$ associated with a representation $T : \mathfrak{g} \to V$ is obtained from $\varphi_{\mathfrak{g}}$ as follows (see Section 6.1.4 for details). Replace each occurence of e_i in $\varphi_{\mathfrak{g}}(D)$ by the corresponding linear operator $T(e_i) \in \mathrm{End}(V)$ and compute the sum of products of all such operators according to the arrangement of indices along the circle. The trace of the obtained operator is $\varphi_{\mathfrak{g}}^V(D)$.

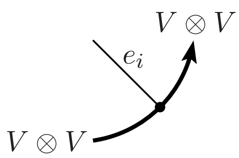
Informally speaking, with each end of a chord of D and with an element $e_i \in \mathfrak{g}$ assigned to it we accociate a linear operator $V \to V$ vizualised by the picture



To pass to the n-th cabling of $\varphi_{\mathfrak{g}}^V$ we must take the sum of n such operators, one per each sheet of the covering:



Instead of this sum, we may just as well consider a single operator $T^{\otimes n} : V^{\otimes n} \to V^{\otimes n}$:



acting according to the rule

$$T^{\otimes 2}(e_i)(v_1 \otimes v_2) = T(e_i)(v_1) \otimes v_2 + v_1 \otimes T(e_i)(v_2)$$

used to define the n-th tensor power $T^{\otimes n} : V^{\otimes n} \to V^{\otimes n}$ of the representation V (in our example $n = 2$). We think about the tensor factor number i as attached to the i-th sheet of the covering. Multiplying all the operators $T^{\otimes n}(e_i)$ along the Wilson loop of D we will get an operator (see 6.1.4)

$$U(T^{\otimes n}) \circ \varphi_{\mathfrak{g}}(D) : V^{\otimes n} \to V^{\otimes n}\,.$$

This operator may be considered as a tensor in the vector space

$$(V^{\otimes n})^* \otimes (V^{\otimes n}) \cong \underbrace{V^* \otimes \ldots \otimes V^*}_{n \text{ factors}} \otimes \underbrace{V \otimes \ldots \otimes V}_{n \text{ factors}}\,.$$

Taking the trace of this operator corresponds to contraction of the tensor by pairing the first factor V^* with the first factor V, the second V^* with the second V, etc. Such a contraction correspond to the disconnecting cabling of D. To get a formula for the connected cabling we should modify the weight system $\varphi_{\mathfrak{g}}^{V^{\otimes n}}$ by changing the operation of taking trace Tr. Namely we will contract the $(j+1)$-st factor V^* with the j-th factor V, and the first V^* with the n-th V. The number we obtain at the end is denoted by $\widetilde{\varphi}_{\mathfrak{g}}^{V^{\otimes n}}(D)$. Thus we get

9.7.10. Proposition.

$$\mathbb{C}_n^* \varphi_{\mathfrak{g}}^V = \widetilde{\varphi}_{\mathfrak{g}}^{V^{\otimes n}}\,.$$

9.7.11. The case of framed tangles. The definition of disconnected cablings from page 263 makes sense in the wider context of framed tangles. If $\boldsymbol{X} \cup \boldsymbol{y}$ is a framed tangle and $\boldsymbol{y}$ is a closed component, we define the *n-th disconnected cabling of $\boldsymbol{X} \cup \boldsymbol{y}$ along $\boldsymbol{y}$,* denoted by $\mathbb{C}_{n,\boldsymbol{y}}(\boldsymbol{X} \cup \boldsymbol{y})$, by replacing $\boldsymbol{y}$ with n parallel copies of $\boldsymbol{y}$ as above.

Disconnected cabling induces maps of corresponding chord diagram spaces, defined as follows. The projection in the normal bundle to $\boldsymbol{y}$ gives a map $\boldsymbol{y}_1 \cup \ldots \cup \boldsymbol{y}_n \to \boldsymbol{y}$. Extending this a map by the identity map on $\boldsymbol{X}$ we obtain a map from from the skeleton of $\mathbb{C}_{n,\boldsymbol{y}}(\boldsymbol{X} \cup \boldsymbol{y})$ to that of $\boldsymbol{X} \cup \boldsymbol{y}$. The induced map of diagrams

$$\mathbb{C}_{n,\boldsymbol{y}} : \mathcal{C}(\boldsymbol{X}, \boldsymbol{y}) \to \mathcal{C}(\boldsymbol{X}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_n)$$

assigning to a diagram in $\mathcal{C}(\boldsymbol{X}, \boldsymbol{y})$ the sum of all possible ways of lifting it to a diagram on $\mathcal{C}(\boldsymbol{X}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_n)$. A diagram which has exactly m ends of its chords on the component $\boldsymbol{y}$ is mapped by $\mathbb{C}_{n,\boldsymbol{y}}$ to a sum of n^m diagrams.

The notion of a connecting cabling from page 262 also can be generalized to framed tangles $\boldsymbol{X} \cup \boldsymbol{y}$, where $\boldsymbol{y}$ is a closed component. It is denoted by $\hat{\mathbb{C}}_{n,\boldsymbol{y}}(\boldsymbol{X} \cup \boldsymbol{y})$ and obtained by replacing $\boldsymbol{y}$ with its n-sheeted connected covering $\boldsymbol{y}^n$. It also induces a map of diagrams

$$\hat{\mathbb{C}}_{n,\boldsymbol{y}} : \mathcal{C}(\boldsymbol{X}, \boldsymbol{y}) \to \mathcal{C}(\boldsymbol{X}, \boldsymbol{y}^n)$$

assigning to a diagram in $\mathcal{C}(\boldsymbol{X},\boldsymbol{y})$ the sum of all possible ways of lifting it to a diagram on $\mathcal{C}(\boldsymbol{X},\boldsymbol{y}^n)$.

9.8. Cablings of the Kontsevich integral

The Kontsevich integral is very well-behaved with respect to both kinds of cablings.

Theorem ([**LM5**])**.** *For a framed tangle K with a closed component $\boldsymbol{y}$*

$$I^{fr}(\mathbb{C}_{n,\boldsymbol{y}}(K)) = \mathbb{C}_{n,\boldsymbol{y}}(I^{fr}(K))\,,$$

where I^{fr} is the framed versions of the final Kontsevich integrals from 9.6, and the cabling operation $\mathbb{C}_{n,\boldsymbol{y}}$ on the right-hand side is applied to every chord diagram of the series $I^{fr}(K)$.

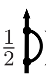
Proof. It will be sufficient to prove the theorem with Z^{fr} in the place of I^{fr}.

Let us follow Goryunov's approach to the framed Kontsevich integral (Section 9.6.1). It will be convenient to assume (this involves no loss of generality) that the framing vector on $\boldsymbol{y}$ has no vertical component.

In order to calculate the framed Kontsevich integral of $\mathbb{C}_{n,\boldsymbol{y}}(K)$ we have to draw a parallel in the direction of the framing to each of the components $\boldsymbol{y}_i$ obtained by splitting $\boldsymbol{y}$. Let us suppose that each of the $\boldsymbol{y}_i$ is obtained by shifting $\boldsymbol{y}$ by ε and that, in turn, the parallel of $\boldsymbol{y}_i$ is obtained by shifting $\boldsymbol{y}$ by ε^2.

Take an arbitrary diagram D that participates in $Z^{fr}(K)$, and let $\widetilde{D}$ be a lifting of D from $\mathcal{C}(\boldsymbol{X},\boldsymbol{y})$ to $\mathcal{C}(\boldsymbol{X},\boldsymbol{y}_1,\dots,\boldsymbol{y}_n)$. Then, as ε tends to 0 the coefficient of $\widetilde{D}$ in the integral $Z^{fr}(\mathbb{C}_{n,\boldsymbol{y}}(K))$ tends to the coefficient of D in $Z^{fr}(K)$. Since the Kontsevich integral is an invariant, this gives the statement of the theorem. $\square$

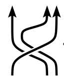
Theorem ([**BLT**])**.** *For a framed tangle K with a closed component $\boldsymbol{y}$*

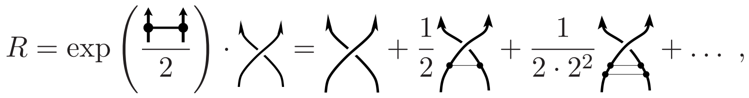
$$I^{fr}(\mathbb{C}_{n,\boldsymbol{y}}(K)) = \Big[\mathbb{C}_{n,\boldsymbol{y}}\Big(I^{fr}(K)\#_{\boldsymbol{y}}\exp\big(\tfrac{1}{2n}$$  $$\big)\Big)\Big]\#_{\boldsymbol{y}^n}\exp\big(-\tfrac{1}{2}$$  $$\big)\,.$$

Proof. The proof is a twist on the proof of the previous theorem.

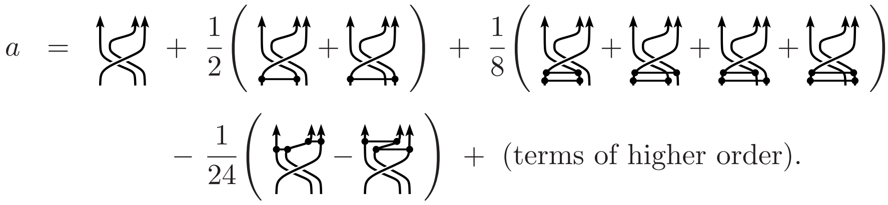
...

To prove the theorem we first introduced a twisted operation $\widetilde{\mathbb{C}}_{n,\boldsymbol{y}}$ on chord diagrams with the property $I^{fr}(\mathbb{C}_{n,\boldsymbol{y}}(K)) = \widetilde{\mathbb{C}}_{n,\boldsymbol{y}}(I^{fr}(K))$. Then we show that $\widetilde{\mathbb{C}}_{n,\boldsymbol{y}}(I^{fr}(K))$ is equal to the right-hand side of the theorem.

A connected cable differs from disconnected one by a small tangle A which represents the pattern how the n strings of the cable close up into a single circle $\boldsymbol{y}^n$ (see the pictures on page 262). To compute the Kontsevich integral of it we would like to represent it in a parenthesized form. For example, for $n = 3$ We can choose $A =$ . Denote the combinatorial Kontsevich integral (section **??**) of the tangle by $a = Z^{fr}_{comb}(A)$. According to the section **??** $a := \Delta_2^{n-2}(R)\Delta_2^{n-3}(\Phi)$, where R is the chord diagram on two braided strings introduced in Secs. 8.4.3 and **??** and used throughout Sec. 10.4,

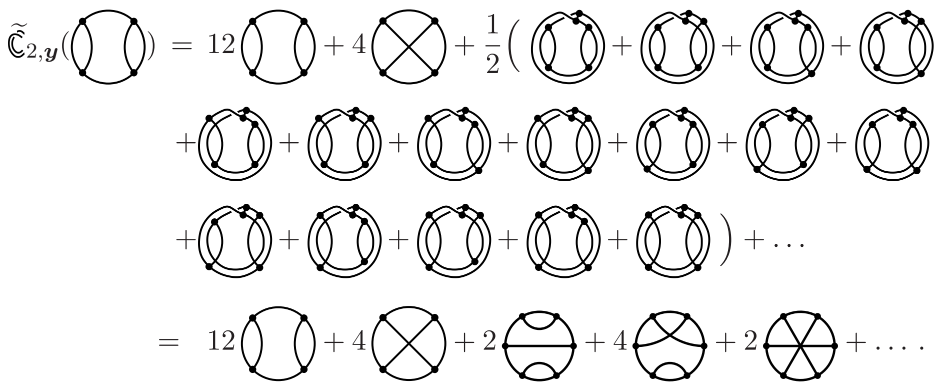


Φ is the associator (Sec. **??**), and Δ_2 is the operation of doubling the second string on tangle chord diagrams introduced in Sec. **??**. Here is an example with $n = 3$.



Note that in the case $n = 2$ the formula for a is simpler, $a = R$.

Now we modify the definition of $\mathfrak{C}_{n,\boldsymbol{y}}(D)$ by insertion of the chord diagram series a instead of empty (without chords) portion at the place where strings go from one sheet of the covering to another one. The result will be denoted by $\widetilde{\widetilde{\mathfrak{C}}}_{n,\boldsymbol{y}}$. The next picture illustrates this notion.



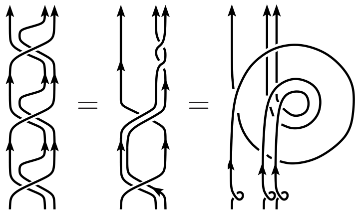
Here, inside the parentheses with common coefficient $1/2$ in each of the 16 diagrams you can see one and the same fragment with one chord near the crossing which is precisely the second term in the expansion of R.

The operator $\widetilde{\mathbb{C}}_{n,\boldsymbol{y}}$ maps an individual chord diagram D not into a finite combination of chord diagrams of the same degree as $\mathbb{C}_{n,\boldsymbol{y}}$ does, but rather into an infinite series of chord diagrams whose lowest part is just $\mathbb{C}_{n,\boldsymbol{y}}(D)$.

Obviously we have

$$I^{fr}(\mathbb{C}_{n,\boldsymbol{y}}(K)) = \widetilde{\mathbb{C}}_{n,\boldsymbol{y}}(I^{fr}(K))\,.$$

Now, to figure out the effect of $\widetilde{\mathbb{C}}_{n,\boldsymbol{y}}$ let us consider the tangle A^n:

$A^3 =$  .

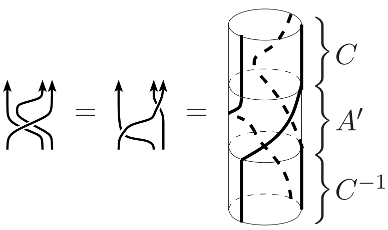
It can be considered as a disconnected parenthesized tangle cabling of a positive kink with additional negative small kinks at the bottom of each string. According to the remark **??** its combinatorial Kontsevich integral is given by the formula

$$a^n = I^{fr}_{comb}(A^n) = \mathbb{C}_n\Big(\exp\big(\tfrac{1}{2}\big)\Big)\cdot\exp\big(-\tfrac{1}{2}\big)^{\otimes n} =: b\,. \tag{9.8.0.1}$$

Here the dot means tangle multiplication and $\exp\big(\tfrac{1}{2}\big)^{\otimes n}$ means the framing changing factors given by the small kinks at the bottom of every string.

We are going to take the n-th root of b from the equation (9.8.0.1). However the problem is that b is a series of chord diagrams on n vertical strings while a is not; it contains also a cyclic permutation of strings of the skeleton.

To go around this problem we represent the tangle A as a conjugate of a tangle (not parenthesized) A' symmetric under the rotation by the angle $\frac{2\pi}{n}$ and whose boundary points are $e^{\frac{2\pi i}{n}j}$ for $j = 0, 1, \ldots, n-1$:

$A =$ 

$= C \cdot A' \cdot C^{-1}$

Set $a' := I^{fr}(A')$ and $c := I^{fr}(C)$. Note that here we use the ordinary Kontsevich integral, not combinatorial one, because the tangles A' and C are not parenthesized. However, $a = c\cdot a'\cdot c^{-1}$.

We may think about c and c^{-1} as elements of the algebra $\mathcal{A}(n)$ of chord diagrams on n vertical strings. Meanwhile A' permutes the end points of the strings. So if we push down all the chords of a', we represent a' as a chord diagram a'_v on n vertical strings supplied by a cyclic permutation of strings of the skeleton atop of it. In other words $a' = s \cdot a'_v$, where $a'_v \in \mathcal{A}(n)$ and s is a tangle chord diagram without any chord but permuting the end points of the skeleton in cyclic order. Invariance of A' under the cyclic rotation implies that its Kontsevich integral a' is invariants under the rotation. Hence a'_v commutes with s: $a'_v \cdot s = s \cdot a'_v$. Therefore we have

$$a^n = c \cdot (a')^n \cdot c^{-1} = c \cdot (s \cdot a'_v)^n \cdot c^{-1} = c \cdot (a'_v)^n \cdot c^{-1} = (c \cdot a'_v \cdot c^{-1})^n = b\,.$$

But this is an equation in the algebra $\mathcal{A}(n)$, and thus we can take the n-th root now.

$$c \cdot a'_v \cdot c^{-1} = b^{1/n} = \mathbb{C}_n\Big(\exp\big(\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)\Big) \cdot \exp\big(-\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)^{\otimes n}\,.$$

As a consequence

$$a = c \cdot s \cdot c^{-1} \cdot \mathbb{C}_n\Big(\exp\big(\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)\Big) \cdot \exp\big(-\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)^{\otimes n}\,.$$

To perform the operation $\widetilde{\widehat{\mathbb{C}}}_{n,\boldsymbol{y}}$ we have to insert the chord diagram series a into every chord diagram of the disconnected cabling $\mathbb{C}_{n,\boldsymbol{y}}(I^{fr}(K))$. In other words, we should consider the tangle multiplication

$$a \cdot \mathbb{C}_{n,\boldsymbol{y}}(K) = c \cdot s \cdot c^{-1} \cdot \mathbb{C}_n\Big(\exp\big(\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)\Big) \cdot \exp\big(-\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)^{\otimes n} \cdot \mathbb{C}_{n,\boldsymbol{y}-\{pt\}}(I^{fr}(K))$$

and close up every chord diagram in it.

The factor $\exp\big(-\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)^{\otimes n}$ belongs to the center of $\mathcal{A}(n)$ so it commutes with everything. The tangle product

$$\mathbb{C}_n\Big(\exp\big(\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)\Big) \cdot \mathbb{C}_{n,\boldsymbol{y}-\{pt\}}(I^{fr}(K)) = \mathbb{C}_{n,\boldsymbol{y}-\{pt\}}\Big(I^{fr}(K)\#_{\boldsymbol{y}}\exp\big(\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)\Big)\,.$$

By the sliding property (Section 5.10.5), c^{-1} commutes with $\mathbb{C}_{n,\boldsymbol{y}-\{pt\}}\Big(I^{fr}(K)\#_{\boldsymbol{y}}\exp\big(\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)\Big)$ and with $\exp\big(-\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)^{\otimes n}$ so it can be swept over the component $\boldsymbol{y}$ and canceled with c from the other side. Thus we end up with a closure of the tangle

$$s \cdot \mathbb{C}_{n,\boldsymbol{y}-\{pt\}}\Big(I^{fr}(K)\#_{\boldsymbol{y}}\exp\big(\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)\Big) \cdot \exp\big(-\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)^{\otimes n}\,.$$

The combination of the permutation of strings tangle S and the disconnected cabling $\mathbb{C}_{n,\boldsymbol{y}-\{pt\}}$ leads to the the connected cabling $\widehat{\mathbb{C}}_{n,\boldsymbol{y}}\Big(I^{fr}(K)\#_{\boldsymbol{y}}\exp\big(\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)\Big)$ in which we have to insert n diagrams $\exp\big(-\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)$. They all can be slid to one place and combined into $\exp\big(-\tfrac{1}{2}\uparrow\!\!\mathsf{D}\big)$. This gives that

$$\widetilde{\widehat{\mathbb{C}}}_{n,\boldsymbol{y}}(I^{fr}(K)) = \Big[\widehat{\mathbb{C}}_{n,\boldsymbol{y}}\Big(I^{fr}(K)\#_{\boldsymbol{y}}\exp\big(\tfrac{1}{2n}\uparrow\!\!\mathsf{D}\big)\Big)\Big]\#_{\boldsymbol{y}^n}\exp\big(-\tfrac{1}{2}\uparrow\!\!\mathsf{D}\big)$$

and finishes the proof of the theorem. $\square$

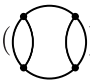
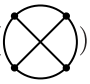
Exercises

(1) * Find two chord diagrams with 11 chords which have the same intersection graph but unequal modulo four- and one-term relations. According to Section 9.5.1, eleven is the least number of chords for such diagrams. Their existence is known, but no explicit examples were found yet.

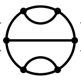
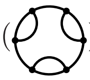
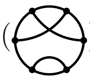
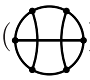
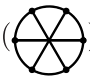
(2) * In the space $\mathcal{A}$ generated by chord diagram modulo four- and one-term relations consider a subspace $\mathcal{A}^M$ generated by those chord diagrams whose class modulo four- and one-term relations is determined by their intersection graphs only. It is naturally to regard the quotient space $\mathcal{A}/\mathcal{A}^M$ as the space of chord diagram *distinguishing mutants*. Find the dimension of $\mathcal{A}_n/\mathcal{A}_n^M$. It is known that it is zero for $n \leqslant 10$ and greater than zero for $n = 1$. Is it true that $\dim(\mathcal{A}_{11}/\mathcal{A}_{11}^M) = 1$?

(3) Carry out a proof that the framed Kontsevich integral $Z^{fr}(K)$ is a group-like element of the Hopf algebra $\widehat{\mathcal{A}}^{fr}$.

(4) (*D. Bar-Natan* [**BN1**]). Check that the mappings $\widehat{\mathbb{C}}_n$ and $\mathbb{C}_n$ are compatible with the four-term relation.

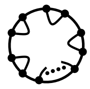
(5) Compute $\widehat{\mathbb{C}}_3($$)$ and $\widehat{\mathbb{C}}_3($$)$.

(6) Compute the eigenvalues and eigenvectors of $\widehat{\mathbb{C}}_3|_{\mathcal{A}_2}$.

(7) Compute $\widehat{\mathbb{C}}_2($$)$, $\widehat{\mathbb{C}}_2($$)$, $\widehat{\mathbb{C}}_2($$)$, $\widehat{\mathbb{C}}_2($$)$, and $\widehat{\mathbb{C}}_2($$)$.

(8) Compute the eigenvalues and eigenvectors of $\widehat{\mathbb{C}}_2|_{\mathcal{A}_3}$.

(9) Compute $\widehat{\mathbb{C}}_2(\Theta^m)$, where Θ^m is a chord diagram with m isolated chords, e.g., the one shown on the right.

$\Theta^m =$ 

m chords

(10) (*D. Bar-Natan* [**BN1**]). Prove that the mapping $\widehat{\mathbb{C}}_n$ commutes with the comultiplication of chord diagrams, i.e., the identity

$$\delta(\widehat{\mathbb{C}}_n(D)) = \sum_{J\subseteq[D]} \widehat{\mathbb{C}}_n(D_J) \otimes \widehat{\mathbb{C}}_n(D_{\overline{J}})$$

holds for any chord diagram D.

(11) (*D. Bar-Natan* [**BN1**]). Prove that $\mathbb{C}_m \circ \mathbb{C}_n = \mathbb{C}_{mn}$.

(12) Prove the theorem from Section 9.7.4:

$$\mathrm{symb}(\widehat{\mathbb{C}}_n^* v)(D) = \mathrm{symb}(v)(\widehat{\mathbb{C}}_n(D))\,.$$

(13) Show that the linear mapping $\hat{\mathbb{C}}_n$ of the space $\mathcal{B}$ preserves the product in $\mathcal{B}$ defined in 5.6 as a disjoint union of graphs. Thus $\hat{\mathbb{C}}_n$ is an automorphism of the algebra $\mathcal{B}$ and hence an automorphism of one of the two Hopf algebra structures on $\mathcal{B}$ (see 5.8).

(14) Prove the proposition from Sec.9.7.7:

$$\hat{\mathbb{C}}_n^* \varphi_{\mathfrak{g}}(D) = \mu^n \circ \delta^n(\varphi_{\mathfrak{g}}(D)) .$$

Chapter 10

The Drinfeld associator

In this chapter we give the details of the combinatorial construction for the Kontsevich integral. The main ingredient of this construction is the Kontsevich integral for the associating tangles, which can be expressed with the help of a very special power series, known as the *Drinfeld associator* Φ_{KZ}. Here the subscript "KZ" indicates that the associator comes from the solutions to the Knizhnik-Zamolodchikov equation.

The associator Φ_{KZ} is an infinite series in two non-commuting variables whose coefficients are combinations of multiple zeta values. In the construction of the Kontsevich integral only some properties of Φ_{KZ} are used; adopting them as axioms, we arrive at the general notion of an associator that appeared in Drinfeld's papers [**Dr1, Dr2**] in his study of quasi-Hopf algebras. These axioms actually describe a large collection of associators belonging to the completed algebra of chord diagrams on three strands. Among the associators, there are some with rational coefficients, which implies the rationality of the Kontsevich integral.

10.1. The KZ equation and iterated integrals

In this section, we give the original Drinfeld's definition of the associator in terms of the solutions of the simplest Knizhnik–Zamolodchikov equation.

The Knizhnik—Zamolodchikov (KZ) equation is a part of the Wess–Zumino–Witten model of conformal field theory [**KnZa**]. The theory of KZ type equations has been developed in the contexts of mathematical physics, representation theory and topology [**EFK, Var, Kas, Koh4, Oht1**]. Our exposition follows the topological approach and is close to that of the last three books.

10.1.1. General theory. The Knizhnik-Zamolodchikov equation arises in the following general setting. Let $\mathcal{H} = \bigcup_{j=1}^{p} H_j$ be a collection of affine hyperplanes in $\mathbb{C}^n$. Each hyperplane H_j is defined by a (non-homogeneous) linear equation $L_j = 0$. Let $\mathcal{A}$ be a graded $\mathbb{C}$-algebra with a unit which is supposed to be completed (4.5.2), connected (A.2.22), associative (but in general non-commutative). Fixing a set $\{c_j\}$ of homogeneous elements of $\mathcal{A}$ of degree 1, one for each hyperplane, we can consider a differential 1-form $\omega = \sum_{j=1}^{p} c_j d\log(L_j)$ on the complex variety $X = \mathbb{C}^n \setminus \mathcal{H}$ with values in $\mathcal{A}$.

Definition. A *Kinizhnik–Zamolodchikov*, or simply *KZ*, equation is an equation of the form

$$dI = \omega \cdot I, \tag{10.1.1.1}$$

where $I : X \to \mathcal{A}$ is the unknown function.

If $x_1, ..., x_n$ are complex coordinates on X and $L_k = \lambda_{k0} + \sum \lambda_{kj} x_j$ for each $k = 1, ..., n$, then the equation (10.1.1.1) takes the form

$$\frac{\partial I}{\partial x_k} = \sum_{j=1}^{n} \frac{\lambda_{kj} c_j}{L_j} I$$

which is a system of first order linear partial differential equations on a vector-valued function I of several complex variables; we are interested in complex-analytic solutions of this equation. The main difficulty is that the coefficients c_j as well as the values of I belong to a non-commutative domain $\mathcal{A}$.

Exercise. One may be tempted to solve the KZ equation as follows: $d\log(I) = \omega$, therefore $I = \exp \int \omega$. Explain why this is wrong.

The form ω must satisfy certain conditions so that the equation (10.1.1.1) may have non-zero solutions. Indeed, taking the differential of both sides of (10.1.1.1), we get that $0 = d(\omega I)$. Applying the Leibniz rule, using the fact that $d\omega = 0$ and substituting $dI = \omega I$, we see that a necessary condition for integrability can be written as

$$\omega \wedge \omega = 0 \tag{10.1.1.2}$$

It turns out that this condition is not only necessary, but also sufficient: if it holds, then the KZ equation has a unique local solution I_0 for any initial value setting $I_0(x_0) = a_0$ (here $x_0 \in X$ and $a_0 \in \mathcal{A}$). This fact is standard in differential geometry where it is called the *integrability of flat connections* (see, for instance, [**KN**]). A direct *ad hoc* proof can be found in [**Oht1**], Proposition 5.2.

10.1.2. Monodromy. Assume that the integrability condition (10.1.1.2) for the KZ equation holds. Given a (local) solution I of the KZ equation and $a \in \mathcal{A}$, the product Ia is also a (local) solution. Therefore, germs of local solutions for the KZ equation at a point x_0 form an $\mathcal{A}$-module. This module is free of rank 1; it is generated by the germ of a local solution taking value $1 \in \mathcal{A}$ at x_0.

The reason to consider germs rather that global solutions is that the global solutions of the KZ equation are generally multivalued. The reason why the solutions of Equation (10.1.1.1) are multivalued is the analytic continuation along closed paths in the base space X. Indeed, let I_0 be the (single-valued) local solution defined in a neighborhood of the point $x_0 \in X$ and satisfying $I_0(x_0) = a_0$. Let $\gamma : [0,1] \to X$ be a closed loop in X, i.e., a continuous mapping such that $\gamma(0) = \gamma(1) = x_0$. Since our equation is linear, the solution I_0 can be analytically continued along γ without running into singularities and thus lead to another local solution I_γ, also defined in a neighborhood of x_0.

Let $I_\gamma(x_0) = a_\gamma$. Suppose a_0 is an invertible element of $\mathcal{A}$. The fact that $\mathcal{S}(x_0) \cong \mathcal{A}$ implies that these two solutions are proportional to each other: $I_\gamma = I a_0^{-1} a_\gamma$. The coefficient $a_0^{-1} a_\gamma$ does not depend on a particular choice of the invertible element $a_0 \in \mathcal{A}$ and the loop γ within a fixed homotopy class. So we get a homomorphism $\pi_1(X) \to \mathcal{A}^*$ from the fundamental group of X into the multiplicative group of the algebra $\mathcal{A}$, called the *monodromy representation*.

10.1.3. Iterated integrals. Both the analytic continuation of the solutions and the monodromy representation can be expressed in terms of the 1-form ω. Choose a path γ, not necessarily closed, and consider the composition $I \circ \gamma$. This is a function $[0,1] \to \mathcal{A}$ which we denote by the same letter I; it satisfies the ordinary differential equation

$$\frac{d}{dt} I(t) = \omega(\dot\gamma(t)) \cdot I(t), \qquad I(0) = 1 \,. \tag{10.1.3.1}$$

The function I takes values in the completed graded algebra $\mathcal{A}$, and it can be expanded in an infinite series according to the grading:

$$I(t) = I_0(t) + I_1(t) + I_2(t) + \dots \ ,$$

where each term $I_m(t)$ is the homogeneous degree m part of $I(t)$.

The form ω is homogeneous of degree 1 (recall that $\omega = \sum c_j \omega_j$, where ω_j are $\mathbb{C}$-valued 1-forms and c_j's are elements of $\mathcal{A}_1$). Therefore Equation

(10.1.3.1) is equivalent to an infinite system of ordinary differential equations

$$\begin{array}{ll} I_0'(t) = 0, & I_0(0) = 1, \\ I_1'(t) = \omega(t) I_0(t), & I_1(0) = 0, \\ I_2'(t) = \omega(t) I_1(t), & I_2(0) = 0, \\ \ldots\ldots & \ldots\ldots \end{array}$$

where $\omega(t) = \gamma^*\omega$ is the pull-back of the 1-form to the interval $[0,1]$.

These equations can be solved iteratively, one by one. The first one gives: $I_0 = \text{const}$, so that for a basic solution we can take $I_0(t) = 1$. Then, $I_1(t) = \displaystyle\int_0^t \omega(t_1)$. Here t_1 is an auxiliary variable that ranges from 0 to t. Coming to the next equation, we now get:

$$I_2(t) = \int_0^t \omega(t_2)\cdot I_1(t_2) = \int_0^t \omega(t_2)\left(\int_0^{t_2}\omega(t_1)\right) = \int\limits_{0<t_1<t_2<t} \omega(t_2)\wedge\omega(t_1),$$

Proceeding in the same way, for an arbitrary m we obtain

$$I_m(t) = \int\limits_{0<t_1<t_2<\cdots<t_m<t} \omega(t_m)\wedge\omega(t_{m-1})\wedge\cdots\wedge\omega(t_1)$$

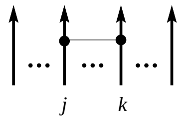
Renumbering the variables, we can write the result as follows:

$$I(t) = 1 + \sum_{m=1}^{\infty} \int\limits_{0<t_m<t_{m-1}<\cdots<t_1<1} \omega(t_1)\wedge\omega(t_2)\wedge\cdots\wedge\omega(t_m) \tag{10.1.3.2}$$

The value $I(1)$ represents the monodromy of the solution over the loop γ. Each iterated integral $I_m(1)$ is a homotopy invariant (of "order m") of γ. Note the resemblance of these expressions to the Kontsevich integral — we'll come back to that again later.

10.1.4. The formal KZ equation. The general scheme of the previous section allows for various specializations. The case of KZ equations related to Lie algebras and their representations is the most elaborated one (see, e.g., [**KnZa, Oht1, Koh4**]). We are especially interested in the following situation.

Suppose that, in the notation of the previous section, $X = \mathbb{C}^n \setminus \mathcal{H}$ where $\mathcal{H}$ is the union of the diagonal hyperplanes $\{z_j = z_k\}$, $1 \leqslant j < k \leqslant n$, and the algebra $\mathcal{A}$ is the completed algebra of *horizontal chord diagrams* $\widehat{\mathcal{A}}^h(n)$ for the tangle T consisting of n vertical strings modulo the horizontal four-term relation (see Equation 4.1.1.4 in Section 4.1). In the pictures, we will always suppose that the strings are oriented upwards. The multiplication in the algebra $\mathcal{A}^h(n)$ is defined by the vertical concatenation of tangle chord diagrams (to obtain the product xy, one puts x on top of y). The unit is the diagram without chords. The algebra is graded by the number of chords. It is generated by the diagrams

$u_{jk} =$  $, \qquad 1 \leqslant j, k \leqslant n,$

subject to relations (*infinitesimal pure braid relations*, first appeared in [**Koh2**])

$$\begin{aligned} &[u_{jk}, u_{jl} + u_{kl}] = 0, \quad \text{ if } j, k, l \text{ are different,} \\ &[u_{jk}, u_{lm}] = 0, \quad \text{ if } j, k, l, m \text{ are different,} \end{aligned}$$

where by definition $u_{jk} = u_{kj}$.

Consider an $\mathcal{A}^h(n)$-valued 1-form $\omega = \dfrac{1}{2\pi i} \displaystyle\sum_{1 \leqslant j < k \leqslant n} u_{jk} \frac{dz_j - dz_k}{z_j - z_k}$ and the corresponding KZ equation

$$dI = \frac{1}{2\pi i} \Big(\sum_{1 \leqslant j < k \leqslant n} u_{jk} \frac{dz_j - dz_k}{z_j - z_k} \Big) \cdot I \ . \tag{10.1.4.1}$$

This specialization of Equation 10.1.1.1 is referred to as the *formal KZ equation*.

The integrability condition (10.1.1.2) for the formal KZ equation is the following identity for a 2-form on X with values in the algebra $\mathcal{A}^h(n)$:

$$\omega \wedge \omega = \sum_{\substack{1 \leqslant j < k \leqslant n \\ 1 \leqslant l < m \leqslant n}} u_{jk} u_{lm} \frac{dz_j - dz_k}{z_j - z_k} \wedge \frac{dz_l - dz_m}{z_l - z_m} \quad = \quad 0 \ .$$

This identity, in a slightly different notation, was actually proved in Section **??** when we checked the horizontal invariance of the Kontsevich integral.

The space $X = \mathbb{C}^n \setminus \mathcal{H}$ is the configuration space of n different (and distinguishable) points in $\mathbb{C}$. A loop γ in this space may be identified with a pure braid (that is a braid that does not permute the endpoints of the strings), and the iterated integral formula (10.1.3.2) yields

$$I(1) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int\limits_{0 < t_m < \cdots < t_1 < 1} \sum_{P = \{(z_j, z'_j)\}} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j} \ ,$$

where P (a *pairing*) is a choice of m pairs of points on the braid, with j-th pair lying on the level $t = t_j$, and D_P is the product of m T-chord diagrams of type $u_{jj'}$ corresponding to the pairing P. We can see that the monodromy of the KZ equation over γ coincides with the Kontsevich integral of the corresponding braid (see **??**).

10.1.5. The case $n = 2$. There are some simplifications in the treatment of Equation 10.1.4.1 for small values of n. In the case $n = 2$ the algebra $\widehat{\mathcal{A}}^h(2)$ is free commutative with one generator and everything is very simple, as the following exercise shows.

Exercise. Solve explicitly Equation 10.1.4.1 and find the monodromy representation in the case $n = 2$.

10.1.6. The case $n = 3$. The formal KZ equation for $n = 3$ has the form

$$dI = \frac{1}{2\pi i}\Big(u_{12} d\log(z_2 - z_1) + u_{13} d\log(z_3 - z_1) + u_{23} d\log(z_3 - z_2)\Big) \cdot I\ ,$$

which is a partial differential equation in 3 variables. It turns out that it can be reduced to an ordinary differential equation.

Indeed, make the substitution

$$I = (z_3 - z_1)^{\frac{u}{2\pi i}} \cdot G\ ,$$

where $u := u_{12} + u_{13} + u_{23}$ and we understand the multiplier as a power series in the algebra $\widehat{\mathcal{A}}^h(3)$:

$$\begin{aligned}(z_3 - z_1)^{\frac{u}{2\pi i}} &= \exp\left(\frac{\log(z_3 - z_1)}{2\pi i} u\right)\\ &= 1 + \frac{\log(z_3 - z_1)}{2\pi i} u + \frac{1}{2!}\frac{\log^2(z_3 - z_1)}{(2\pi i)^2} u^2 + \frac{1}{3!}\frac{\log^3(z_3 - z_1)}{(2\pi i)^3} u^3 + \dots\end{aligned}$$

The element u and therefore the values of the function $(z_3 - z_1)^{\frac{u}{2\pi i}}$ commute with all elements of the algebra $\widehat{\mathcal{A}}^h(3)$ because of the following lemma.

Lemma. *The element $u = u_{12} + u_{13} + u_{23}$ belongs to the center of the algebra $\widehat{\mathcal{A}}^h(3)$.*

Proof. The algebra $\widehat{\mathcal{A}}^h(3)$ is generated by the elements u_{12}, u_{13}, and u_{23}, therefore it is enough to show that each of these three elements commutes with their sum. But the four-term relations mean precisely that

$$[u_{12}, u_{13} + u_{23}] = 0, \quad [u_{13}, u_{12} + u_{23}] = 0, \quad [u_{23}, u_{12} + u_{13}] = 0\ .$$

□

The algebra $\widehat{\mathcal{A}}^h(3)$ can thus be considered as a free algebra of mixed type, with two noncommutative generators u_{12}, u_{23} and one commutative generator u.

After the substitution and simplifications the differential equation for G becomes

$$dG = \frac{1}{2\pi i}\left(u_{12} d\log\Big(\frac{z_2 - z_1}{z_3 - z_1}\Big) + u_{23} d\log\Big(1 - \frac{z_2 - z_1}{z_3 - z_1}\Big)\right) G\ .$$

Denoting $\frac{z_2-z_1}{z_3-z_1}$ simply by z, we see that the function G depends only on z and as such satisfies the following ordinary differential equation (the *reduced KZ equation*)

(10.1.6.1) $$\frac{dG}{dz} = \left(\frac{A}{z} + \frac{B}{z-1}\right) G$$

where $A := \frac{u_{12}}{2\pi i}$, $B := \frac{u_{23}}{2\pi i}$. Initially, G was taking values in the algebra $\widehat{\mathcal{A}}^h(3)$ with three generators A, B, u. However, the space of local solutions of this equation is a free module over $\widehat{\mathcal{A}}^h(3)$ of rank 1, so the knowledge of just one solution is enough. Since the coefficients of Equation 10.1.6.1 do not involve u, the equation does have a solution with values in the ring of formal power series $\mathbb{C}\langle\langle A, B\rangle\rangle$ in two non-commuting variables A and B.

10.1.7. The reduced KZ equation. The reduced KZ equation (10.1.6.1) is a particular case of the general KZ equation (10.1.1.1), defined by the data $n = 1$, $X = \mathbb{C} \setminus \{0, 1\}$, $\mathcal{A} = \mathbb{C}\langle\langle A, B\rangle\rangle$, $c_1 = A$, $c_2 = B$.

Equation (10.1.6.1), although it is a first order ordinary differential equation, is not easy to solve, because the solutions take values in a non-commutative infinite-dimensional algebra. Two approaches at solving this equation immediately come to mind. In the following exercises we invite the reader to try them.

10.1.8. Exercise. Try to find the general solution of Equation (10.1.6.1) by representing it as a series $G = G_0 + G_1A + G_2B + G_{11}A^2 + G_{12}AB + G_{21}BA + \ldots$, where the G's with subscripts are complex-valued functions of z.

10.1.9. Exercise. Try to find the general solution of Equation (10.1.6.1) in the form of a Taylor series $G = \sum_k G_k(z - \frac{1}{2})^k$, where the G_k's are elements of the algebra $\mathbb{C}\langle\langle A, B\rangle\rangle$. (Note that it is not possible to expand the solutions at $z = 0$ or $z = 1$, because they have essential singularities at these points.)

These exercises show that direct approaches do not give much insight into the nature of solutions of the KZ equation (10.1.6.1). However, one good thing about this equation is that any solution can be obtained from one basic solution via multiplication by an element of the algebra $\mathbb{C}\langle\langle A, B\rangle\rangle$. The Drinfeld associator appears as a coefficient between two remarkable solutions.

Definition. The (Knizhnik-Zamolodchikov) *Drinfeld associator* Φ_{KZ} is the ratio $\Phi_{\mathrm{KZ}} = G_1^{-1}(z) \cdot G_0(z)$ of two special solutions $G_0(z)$ and $G_1(z)$ of this equation described in the following Lemma.

10.1.10. Lemma. ([**Dr1, Dr2**]) *There exist unique solutions* $G_0(z)$ *and* $G_1(z)$ *of equation (10.1.6.1), analytic in the domain* $\{z \in \mathbb{C} \mid |z| < 1, |z -$

$1| < 1\}$ *and having the following asymptotic behavior:*

$$G_0(z) \sim z^A \text{ as } z \to 0 \qquad \text{and} \qquad G_1(z) \sim (1-z)^B \text{ as } z \to 1 \ ,$$

which means that

$$G_0(z) = f(z) \cdot z^A \qquad \text{and} \qquad G_1(z) = g(1-z) \cdot (1-z)^B \ ,$$

where $f(z)$ *and* $g(z)$ *are analytic functions in a neighborhood of* $0 \in \mathbb{C}$ *with values in* $\mathbb{C}\langle\langle A, B\rangle\rangle$ *such that* $f(0) = g(0) = 1$*, and the (multivalued) exponential functions are understood as formal power series, i.e.,* $z^A = \exp(A \log z) = \sum_{k\geqslant 0}(A\log z)^k/k!$

Remark. It is often said that the element Φ_{KZ} represents the monodromy of the KZ equation over the horizontal interval from 0 to 1. This phrase has the following meaning. In general, the monodromy along a path γ connecting two points p and q, is the value at q of the solution, analytical over γ and taking value 1 at p. If f_p and f_q are two solutions analytical over γ with initial values $f_p(p) = f_q(q) = 1$, then the monodromy is the element $f_q^{-1}f_p$. The reduced KZ equation has no analytic solutions at the points $p = 0$ and $q = 1$, and the general definition of the monodromy cannot be applied directly in this case. What we do is we choose some natural basic solutions with reasonable asymptotics at these points and define the monodromy as their ratio in the appropriate order.

Proof. Plugging the expression $G_0(z) = f(z) \cdot z^A$ into Equation (10.1.6.1) we get

$$f'(z) \cdot z^A + f \cdot \frac{A}{z} \cdot z^A = \left(\frac{A}{z} + \frac{B}{z-1}\right) \cdot f \cdot z^A \ ,$$

hence $f(z)$ satisfies the differential equation

$$f' - \frac{1}{z}[A, f] = \frac{-B}{1-z} \cdot f \ .$$

Let us look for a formal power series solution $f = 1 + \sum_{k=1}^{\infty} f_k z^k$ with coefficients $f_k \in \mathbb{C}\langle\langle A, B\rangle\rangle$. We have the following recurrence equation for the coefficient of z^{k-1}:

$$kf_k - [A, f_k] = (k - \mathrm{ad}_A)(f_k) = -B(1 + f_1 + f_2 + \cdots + f_{k-1}) \ ,$$

where ad_A denotes the operator $x \mapsto [A, x]$. The operator $k - \mathrm{ad}_A$ is invertible:

$$(k - \mathrm{ad}_A)^{-1} = \sum_{s=0}^{\infty} \frac{\mathrm{ad}_A^s}{k^{s+1}}$$

(the sum is well-defined because the operator ad_A increases the grading), so the recurrence can be solved

$$f_k = \sum_{s=0}^{\infty} \frac{\mathrm{ad}_A^s}{k^{s+1}}\bigl(-B(1 + f_1 + f_2 + \cdots + f_{k-1})\bigr) \ .$$

Therefore the desired solution does exist among formal power series. Since the point 0 is a regular singular point of our equation (10.1.6.1), it follows (see, e.g. [**Wal**]) that this power series converges for $|z| < 1$. We thus get an analytic solution $f(z)$.

To prove the existence of the second solution, $G_1(z)$, it is best to make the change of undependent variable $z \mapsto 1 - z$ which transforms Equation (10.1.6.1) into a similar equation with A and B swapped. $\square$

Remark. If the variables A and B were commutative, then the function explicitly given as the product $z^A(1-z)^B$ would be a solution of Equation 10.1.6.1 satisfying both asymptotic conditions of Lemma 10.1.10 at once. Therefore, the image of Φ_{KZ} under the abelianization map $\mathbb{C}\langle\langle A, B\rangle\rangle \to \mathbb{C}[[A, B]]$ is equal to 1.

The next lemma gives another expression for the associator in terms of the solutions of equation (10.1.6.1).

10.1.11. Lemma ([**LM2**]). *Suppose that $\varepsilon \in \mathbb{C}$, $|\varepsilon| < 1$, $|\varepsilon - 1| < 1$. Let $G_\varepsilon(z)$ be the unique solution of equation (10.1.6.1) satisfying the initial condition $G_\varepsilon(\varepsilon) = 1$. Then*

$$\Phi_{KZ} = \lim_{\varepsilon \to 0} \varepsilon^{-B} \cdot G_\varepsilon(1-\varepsilon) \cdot \varepsilon^A \ .$$

Proof. We rely on, and use the notation of, Lemma 10.1.10. The solution G_ε is proportional to the distinguished solution G_0:

$$G_\varepsilon(z) = G_0(z)G_0(\varepsilon)^{-1} = G_0(z) \cdot \varepsilon^{-A} f(\varepsilon)^{-1} = G_1(z) \cdot \Phi_{\mathrm{KZ}} \cdot \varepsilon^{-A} f(\varepsilon)^{-1}$$

(the function f, as well as g mentioned below, was defined in Lemma 10.1.10). In particular,

$$G_\varepsilon(1-\varepsilon) = G_1(1-\varepsilon) \cdot \Phi_{\mathrm{KZ}} \cdot \varepsilon^{-A} f(\varepsilon)^{-1} = g(\varepsilon)\varepsilon^B \cdot \Phi_{\mathrm{KZ}} \cdot \varepsilon^{-A} f(\varepsilon)^{-1} \ .$$

We must compute the limit

$$\lim_{\varepsilon \to 0} \varepsilon^{-B} g(\varepsilon)\varepsilon^B \cdot \Phi_{\mathrm{KZ}} \cdot \varepsilon^{-A} f(\varepsilon)^{-1} \varepsilon^A \ ,$$

which obviously equals Φ_{KZ} because $f(0) = g(0) = 1$ and $f(z)$ and $g(z)$ are analytic functions in a neighborhood of zero. The lemma is proved. $\square$

10.1.12. The Drinfeld associator and the Kontsevich integral. Putting

$$\omega(z) = A\frac{dz}{z} + B\frac{d(1-z)}{1-z},$$

we can rewrite equation (10.1.6.1) as $dG = \omega \cdot G$. This is a particular case of equation (10.1.3.2) in Section 10.1.1, therefore, its solution $G_\varepsilon(t)$ can be written as a series of iterated integrals

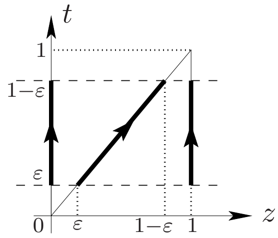
$$G_\varepsilon(t) = 1 + \sum_{m=1}^{\infty} \int\limits_{\varepsilon<t_m<\dots<t_2<t_1<t} \omega(t_1)\wedge\omega(t_2)\wedge\dots\wedge\omega(t_m)\ .$$

The lower limit in the integrals is ε because of the initial condition $G_\varepsilon(\varepsilon) = 1$.

We are interested in the value of this solution at $t = 1-\varepsilon$:

$$G_\varepsilon(1-\varepsilon) = 1 + \sum_{m=1}^{\infty} \int\limits_{\varepsilon<t_m<\dots<t_2<t_1<1-\varepsilon} \omega(t_1)\wedge\omega(t_2)\wedge\dots\omega(t_m)\ .$$

We claim that this series literally coincides with the Kontsevich integral of the following tangle

$Q^\varepsilon =$ 

under the identification $A = \frac{1}{2\pi i}$, $B = \frac{1}{2\pi i}$. Indeed, on every level t_j the differential form $\omega(t_j)$ consists of two summands. The first summand $A\frac{dt_j}{t_j}$ corresponds to the choice of a pair $P = (0, t_j)$ on the first and the second strings and is related to the chord diagram $A =$. The second summand $B\frac{d(1-t_j)}{1-t_j}$ corresponds to the choice of a pair $P = (t_j, 1)$ on the second and third strings and is related to the chord diagram $B =$. The pairing of the first and the third strings does not contribute to the Kontsevich integral, because these strings are parallel and the correspoding differential vanishes. We have thus proved the following proposition.

Proposition. *The value of the solution G_ε at $1-\varepsilon$ is equal to the Kontsevich integral $G_\varepsilon(1-\varepsilon) = Z(Q^\varepsilon)$. Consequently, the KZ associator coincides with the regularization of the Kontsevich integral of the tangle Q^ε:*

$$\Phi_{KZ} = \lim_{\varepsilon\to 0}\ \varepsilon^{-B}\cdot Z(Q^\varepsilon)\cdot\varepsilon^{A},$$

where $A = \frac{1}{2\pi i}$ and $B = \frac{1}{2\pi i}$.

10.2. Combinatorial construction of the Kontsevich integral

In this section we fulfil the promise of Section 8.9 and describe in detail a combinatorial construction for the Kontsevich integral of knots and links. The associator Φ_{KZ} is an essential part of this construction. In Section 10.3

we shall give some formulae for Φ_{KZ}; using these expressions one can perform explicit calculations, at least in low degrees.

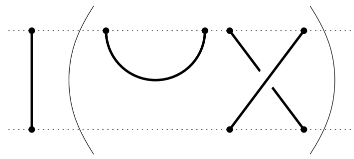
10.2.1. Non-associative monomials. A *non-associative monomial* in one variable is simply a choice of an order (that is, a choice of parentheses) of multiplying n factors; the number n is referred to as the *degree* of the non-associative monomials. The only such monomial in x of degree 1 is x itself. In degree 2 there is also only one monomial, namely xx, in degree 3 there are two monomials $(xx)x$ and $x(xx)$, in degree 4 we have $((xx)x)x$, $(x(xx))x$, $(xx)(xx)$, $x((xx)x)$ and $x(x(xx))$, et cetera.

For each pair u, v of non-associative monomials of the same degree n one can define $\Phi(u,v) \in \widehat{\mathcal{A}}^h(n)$ as follows. If $n < 3$ we set $\Phi(u,v) = 1$, the unit in $\widehat{\mathcal{A}}^h(n)$. Assume $n \geqslant 3$. Then $\Phi(u,v)$ is determined by the following properties:

(1) If $u = w_1(w_2w_3)$ and $v = (w_1w_2)w_3$ where w_1, w_2, w_3 are monomials of degrees n_1, n_2 and n_3 respectively, then
$$\Phi(u,v) = \Delta_{n_1,n_2,n_3}\Phi_{\mathrm{KZ}}.$$
(2) If w is monomial of degree m,
$$\Phi(wu,wv) = \mathbf{1}_m \otimes \Phi(u,v)$$
and
$$\Phi(uw,vw) = \Phi(u,v) \otimes \mathbf{1}_m;$$
(3) If u, v, w are monomials of the same degree, then
$$\Phi(u,v) = \Phi(u,w)\Phi(w,v).$$

These properties are sufficient to determine $\Phi(u,v)$ since each non-associative monomial in one variable can be obtained from any other such monomial of the same degree by moving the parentheses in triple products. It is not immediate that $\Phi(u,v)$ is well-defined, however. Indeed, according to (3), we can define $\Phi(u,v)$ by choosing a sequence of moves that shift one pair of parentheses at a time, and have the effect of changing u into v. A potential problem is that there may be more than one such sequence; however, let us postpone this matter for the moment and work under the assumption that $\Phi(u,v)$ may be multivalued (which it is not, see Section 10.2.7).

Recall from Section 1.7 the notion of an elementary tangle: basically, these are maxima, minima, crossings and vertical segments. Take a tensor product of several elementary tangles and choose the brackets in it, enclosing each elementary tangle other than a vertical segment in its own pair of parentheses. This choice of parentheses produces two non-associative monomials: one formed by the top boundary points of the tangle, and the other, formed by the bottom boundary points. For example, consider the following tensor product, parenthesized as indicated, of three elementary tangles:



The top part of the boundary gives the monomial $x((xx)(xx))$ and the bottom part - $x(xx)$.

Note that here it is important that the factors in the product are not arbitrary, but elementary tangles, since each elementary tangle has at most two upper and at most two lower boundary points.

10.2.2. The construction. Represent a given knot K as a product of tangles

$$K = T_1 T_2 \dots T_n$$

so that each T_i as a tensor product of elementary tangles:

$$T_i = T_{i,1} \otimes \dots \otimes T_{i,k_i}.$$

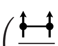
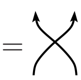
For each T_i choose the parentheses in this tensor product and denote by w^i and w_i the corresponding non-associative monomials coming from the top and bottom parts of the boundary of T_i, respectively. Finally, let α_i be the set of boundary points on the top of T_{i+1} (or on the bottom of T_i, which is the same) where the corresponding strands are oriented downwards. Then the *combinatorial Kontsevich integral* $Z_{comb}(K)$ is defined as

$$Z_{comb}(K) = Z_1 \cdot S_{\alpha_1}(\Phi(w_1, w^2)) \cdot Z_2 \cdot \dots \cdot Z_{n-1} \cdot S_{\alpha_{n-1}}(\Phi(w_{n-1}, w^n)) \cdot Z_n,$$

where Z_i is the tensor product of the Kontsevich integrals of the elementary tangles $T_{i,j}$:

$$Z_i = Z(T_{i,1}) \otimes \dots \otimes Z(T_{i,k_i}).$$

Note that the only elementary tangles for which the Kontsevich integral is non-trivial are the crossings, and for them

$Z(X_+) =$  $\cdot \exp\Big($  $/2\Big)$, $\qquad Z(X_-) =$  $\cdot \exp\Big(-$  $/2\Big)$.

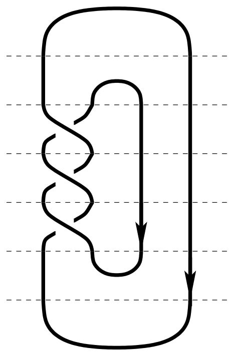
For all other elementary tangles the Kontsevich integral consists of a diagram with no chords (and with the skeleton corresponding to that of the tangle, of course).

We also remind that Z_i in general does not coincide with $Z(T_i)$.

10.2.3. Example of computation. Let us see how the combinatorial Kontsevich integral can be computed, up to order 2, on the example of the left trefoil 3_1. Explicit formulae for the associator will be proved in Section 10.3. In particular, we shall see that

$\Phi_{\mathrm{KZ}} = 1 + \frac{1}{24}\Big($  $-$  $\Big) + \dots .$

Decompose the left trefoil into elementary tangles as shown below and choose the parentheses in the tensor product as shown in the second column:



$\overrightarrow{\max}$

$(\mathrm{id} \otimes \overrightarrow{\max}) \otimes \mathrm{id}^*$

$(X_- \otimes \mathrm{id}^*) \otimes \mathrm{id}^*$

$(X_- \otimes \mathrm{id}^*) \otimes \mathrm{id}^*$

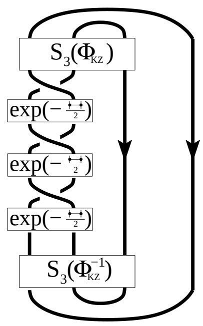
$(X_- \otimes \mathrm{id}^*) \otimes \mathrm{id}^*$

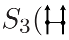
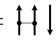
$(\mathrm{id} \otimes \underleftarrow{\min}) \otimes \mathrm{id}^*$

$\underleftarrow{\min}$

The combinatorial Kontsevich integral may then be represented as

$Z_{comb}(3_1) \quad =$

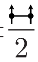
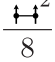


where S_3 is the operation corresponding to the reversal of the 3rd strand, in particular, $S_3($$) =$  and $S_3($$) = -$. The crossings in the above picture are, of course, irrelevant since it shows chord diagrams and not knot diagrams.

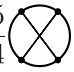
We have that

$S_3(\Phi_{\mathrm{KZ}}^{\pm 1}) = 1 \pm \frac{1}{24}\Big($ $-$ $\Big) + \dots$

and

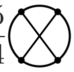
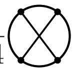
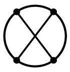
$\exp\Big(\pm$  $/2\Big) = 1 \pm$  $/2 +$  $^2/8 + \dots$

Plugging these expressions into the diagram above we see that, up to degree 2, the Kontsevich integral of the left trefoil is

$Z(3_1) = 1 + \frac{25}{24}$  $+ \dots$

The final Kontsevich integral of the trefoil (in the multiplicative normalization, see page 242) is thus equal to

$I'(3_1) = Z(3_1)/Z(H)$

$= \Big(1 + \frac{25}{24}$  $+ \dots\Big)\Big(1 + \frac{1}{24}$  $+ \dots\Big)^{-1} = 1 +$  $+ \dots$

10.2.4. The main theorem. The main result about the combinatorial Kontsevich integral is the following theorem:

Theorem ([**LM3**]). *The combinatorial Kontsevich integral of a knot is equal to the usual Kontsevich integral:*

$$Z_{comb}(K) = Z(K)\ .$$

The rest of this section is dedicated to the proof of this theorem. We have sketched the idea in Section 8.9 and here we shall make it precise. The most important part of the proof consists of expressing the Kontsevich integral of an associating tangle via Φ_{KZ}. First, let us give a more precise definition of associating tangles.

10.2.5. Boundary configurations and associating tangles. A *boundary configuration* is a finite set of distinct oriented (that is, marked with a sign) points in an interval $[a, b]$. The *cardinality* of a configuration is the number of points in it. To each tangle we can associate two boundary configurations, namely, the top and the bottom of the tangle. The points of the configurations are the boundaries of the strands; the sign of a point is positive if the corresponding strand is oriented upwards and negative otherwise.

Putting two boundary configurations t_1 and t_2 next to each other we obtain the *tensor product* of configurations $t_1 \otimes t_2$. This operation agrees with the tensor product of tangles in the sense that the top (bottom) boundary configuration of a tensor product of two tangles is the tensor product of the corresponding top (bottom) configurations. Note that each boundary configuration of cardinality n can be written as a tensor product of n configurations of cardinality one. This decomposition as a product is not unique since the widths of the factors may vary without affecting the distances between the point of the product.

More generally, we can define the ε-parameterized tensor product of boundary configurations which agrees with that of tangles. As in the case of tangles, the ε-parameterized tensor product is non-associative in general.

A *parenthesized boundary configuration* (t, w) of cardinality n consists of a boundary configuration t of cardinality n together with a non-associative monomial w of degree n in one variable.

10.2.6. Lemma. *Let (t, w) be a parenthesized boundary configuration of cardinality n. For each positive $\varepsilon \leqslant 1$ there is a unique configuration t^w_ε such that $t = t_1 \otimes \ldots \otimes t_n$ with t_i of cardinality one, and t^w_ε is the ε-parameterized tensor product of the t_i with the parentheses coinciding with those of w.*

In particular, each non-associative word of degree n gives a canonical deformation of any boundary configuration of cardinality n.

Definition. Let (t, w) and (t, w') be two parenthesized boundary configurations with the same underlying boundary configuration t. For each positive $\varepsilon \leqslant 1$ the *associating tangle* $AT_\varepsilon(t; w, w')$ is a tangle such that

- its top configuration is t^w_ε and its bottom configuration is $t^{w'}_\varepsilon$;
- the boundary of each component of the tangle has one point on the top and one on the bottom;
- its diagram has no crossings.

Every associating tangle is a product of associating tangles $AT_\varepsilon(t; w, w')$ with one of the monomials w, w' equal to $w_1(w_2w_3)$ and the other - to $w' = (w_1w_2)w_3$ where w_1, w_2 and w_3 are some non-associative monomials. It will be sufficient to calculate the Kontsevich integral for the associating tangles of this form.

The simplest associating tangle is Q^ε from Section 10.1.12. We have seen that its Kontsevich integral, apart from the associator, contains some regularizing factors. For more general tangles the situation is similar, although the regularizing factors become more complicated.

10.2.7. Kontsevich integral for the associating tangles. Let w be non-associative monomial of degree n. The *regularizing factor* $\rho^\varepsilon(w) \in \widehat{\mathcal{A}}^h(n)$ is defined as follows.

First, we define for each integer $i \geqslant 0$ and each non-associative monomial w (in one variable x) the element $c_i(w) \in \widehat{\mathcal{A}}^h(n)$, where n is the degree of w, by setting

- $c_i(x) = 0$ for all i;
- $c_0(w_1w_2) = \Delta_{n_1,n_2}\big(\frac{\;}{2\pi i}\big)$ if $w_1, w_2 \neq 1$, where n_1 and n_2 are the degrees of w_1 and w_2 respectively;

- $c_i(w_1w_2) = c_{i-1}(w_1) \otimes \mathbf{1}_{n_2} + \mathbf{1}_{n_1} \otimes c_{i-1}(w_2)$ if $w_1, w_2 \neq 1$ with $\deg w_1 = n_1$, $\deg w_2 = n_2$ and $i > 0$.

It is easy to see that for each w all the $c_i(w)$ commute with each other and that only a finite number of the c_i is non-zero. Now, we set

$$\rho^\varepsilon(w) = \prod_{k=1}^{\infty} \varepsilon^{kc_k(w)}.$$

This product is, of course, finite since almost all terms in it are equal to the unit in $\widehat{\mathcal{A}}^h(n)$.

Let us introduce, for this chapter only, the following notation. If x and y are two elements of $\widehat{\mathcal{A}}^h(n)$ that depend on a parameter ε, by saying that $x \sim y$ as $\varepsilon \to 0$ we shall mean that in some fixed basis of $\widehat{\mathcal{A}}^h(n)$ (and, hence, in any basis of this algebra) the coefficient of each diagram in $x - y$ is of the same or smaller order of magnitude than $\varepsilon \ln^N \varepsilon$ for some non-negative integer N that may depend on the diagram. Note that for any non-negative N the limit of $\varepsilon \ln^N \varepsilon$ as $\varepsilon \to 0$ is equal to 0.

Proposition. *If all points of the configuration t are positive,*

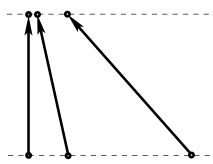
$$(\rho^\varepsilon(w))^{-1} \cdot Z(AT_\varepsilon(t; w, w')) \cdot \rho^\varepsilon(w') \sim \Phi(w, w') \qquad (10.2.7.1)$$

as $\varepsilon \to 0$.

An important corollary of the above formula is that $\Phi(w, w')$ is well-defined, since the left-hand side is. Note that this formula does not depend on the configuration t at all. Here, of course, we have assumed that all points of t are positive only for simplicity. Changing the sign of a point in t results in applying the operation S to the Kontsevich integral of $AT_\varepsilon(t; w, w')$, see ???.

Proof. Let w be a non-associative word and t - a positive boundary configuration. We denote by εt a configuration of the same cardinality and in the same interval as t but whose distances between points are equal to the corresponding distances in t, multiplied by ε. (There are many such configuration, of course, but this is of no importance in what follows.)

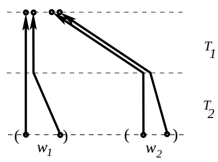
Write $N_\varepsilon(w)$ for a tangle with no crossings which has $\varepsilon t^w_\varepsilon$ and t^w_ε as its top and bottom configurations respectively, and all of whose strands have one boundary point on the top and one on the bottom:



As ε tends to 0, the Kontsevich integral of $N_\varepsilon(w)$ diverges. We have the following asymptotic formula:

$$Z(N_\varepsilon(w)) \sim \prod_{k=0}^{\infty} \varepsilon^{c_k(w)}. \tag{10.2.7.2}$$

If t is a two-point configuration this formula is exact, and amounts to a straightforward computation (see Exercise 4 to Chapter 8). In general, if $w = w_1 w_2$ we can write $N_\varepsilon(w)$ as a product in the following way:



As ε tends to 0, we have

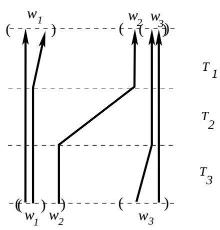
$$Z(T_1) \sim \Delta_{n_1,n_2} \varepsilon^{\;/2\pi i} = \varepsilon^{c_0(w_1 w_2)}$$

and

$$Z(T_2) \sim Z(N_\varepsilon(w_1)) \otimes Z(N_\varepsilon(w_2)).$$

Using induction and the definition of the c_i we arrive to the formula (10.2.7.2).

Now, notice that it is sufficient to prove (10.2.7.1) in the case when $w = w_1(w_2 w_3)$ and $w' = (w_1 w_2) w_3$. Let us draw $AT_\varepsilon(t; w, w')$ as a product $T_1 \cdot T_2 \cdot T_3$ as in the picture:



As $\varepsilon \to 0$ we have:

- $Z(T_1) \sim Z(N_\varepsilon(w_1))^{-1} \otimes \mathbf{1}_{n_2+n_3}$;
- $Z(T_2) \sim (\mathbf{1}_{n_1} \otimes c_0(w_2 w_3)) \cdot \Delta_{n_1,n_2,n_3} \Phi_{\mathrm{KZ}} \cdot (c_0(w_1 w_2) \otimes \mathbf{1}_{n_3})^{-1}$;
- $Z(T_3) \sim \mathbf{1}_{n_1+n_2} \otimes Z(N_\varepsilon(w_3))$.

Notice that these asymptotic expressions for $Z(T_1)$, $Z(T_2)$ and $Z(T_3)$ all commute with each other. Now (10.2.7.1) follows from (10.2.7.2) and the definition of $\rho^\varepsilon(w)$. $\square$

10.2.8. Proof of the main theorem. We have seen in Section 8.9 that given a knot K written as a product $T_1 \cdot \ldots \cdot T_n$ where the T_i are tensor products of elementary tangles, there is a family

$$K^\varepsilon = T_1^\varepsilon \cdot Q_1^\varepsilon \cdot T_2^\varepsilon \cdot \ldots \cdot Q_{n-1}^\varepsilon \cdot T_n^\varepsilon,$$

where the T_i^ε are ε-parameterized tensor products of elementary tangles and the Q_i^ε are associating tangles.

As $\varepsilon \to 0$, the Kontsevich integral of T_i^ε tends to the tensor product of the corresponding elementary tangles, and the integral of Q_i^ε is given by the formula (10.2.7.1) with the appropriate w and w'. Comparing this with the combinatorial Kontsevich integral we see that the only thing to prove is that the regularizing factors in the expressions for the Q_i^ε can be omitted.

...

10.3. Calculation of the KZ Drinfeld associator

In this section, following [**LM2**], we deduce an explicit formula for the Drinfeld associator Φ_{KZ}. It turns out that all coefficients in the corresponding expansion over the monomials in A and B are values of multiple zeta functions (see Section 10.3.11) divided by powers of $2\pi i$.

10.3.1. Put $\omega_0(z) = \frac{dz}{z}$ and $\omega_1(z) = \frac{d(1-z)}{1-z}$. Then the 1-form Ω studied in 10.1.12 is the linear combination $\omega(z) = A\omega_0(z) + B\omega_1(z)$, where $A =$  $/2\pi i$ and $B =$  $/2\pi i$. By definition the terms of the Kontsevich integral $Z(AT_\varepsilon)$ represent the monomials corresponding to all choices of one of the two summands of $\omega(t_j)$ for every level t_j. The coefficients of these monomials are integrals over the simplex $\varepsilon < t_m < \cdots < t_2 < t_1 < 1-\varepsilon$ of all possible products of the forms ω_0 and ω_1. The coefficient of the monomial $B^{i_1}A^{j_1}\ldots B^{i_l}A^{j_l}$ ($i_1 \geqslant 0$, $j_1 > 0$, ..., $i_l > 0$, $j_l \geqslant 0$) is

$$\int\limits_{\varepsilon<t_m<\cdots<t_2<t_1<1-\varepsilon} \underbrace{\omega_1(t_1)\wedge\cdots\wedge\omega_1(t_{i_1})}_{i_1}\wedge\underbrace{\omega_0(t_{i_1+1})\wedge\cdots\wedge\omega_0(t_{i_1+j_1})}_{j_1}\wedge\ldots$$
$$\wedge\underbrace{\omega_0(t_{i_1+\cdots+i_l+1})\wedge\cdots\wedge\omega_0(t_{i_1+\cdots+j_l})}_{j_l},$$

where $m = i_1+j_1+\cdots+i_l+j_l$. For example, the coefficient of AB^2A equals

$$\int\limits_{\varepsilon<t_4<t_3<t_2<t_1<1-\varepsilon} \omega_0(t_1)\wedge\omega_1(t_2)\wedge\omega_1(t_3)\wedge\omega_0(t_4)\,.$$

We are going to divide the sum of all monomials into two parts, convergent Z^{conv} and divergent Z^{div}, depending on the behavior of the coefficients as $\varepsilon \to 0$. We will have $Z(AT_\varepsilon) = Z^{conv} + Z^{div}$ and

$$\Phi = \lim_{\varepsilon\to 0} \varepsilon^{-B} \cdot Z^{conv} \cdot \varepsilon^{-A} + \lim_{\varepsilon\to 0} \varepsilon^{-B} \cdot Z^{div} \cdot \varepsilon^{-A} \ . \tag{10.3.1.1}$$

Then we will prove that the second limit equals zero and find an explicit expression for the first one in terms of multiple zeta values. We will see that although the sum Z^{conv} does not contain any divergent monomials, the first limit in (10.3.1.1) does.

We pass to exact definitions.

10.3.2. Definition. A non-unit monomial in letters A and B with positive powers is said to be *convergent* if it starts with an A and ends with a B. Otherwise the monomial is said to be *divergent* . We regard the unit monomial 1 as convergent.

10.3.3. Example. The integral

$$\int\limits_{a<t_p<\cdots<t_2<t_1<b} \omega_1(t_1) \wedge \cdots \wedge \omega_1(t_p) = \frac{1}{p!} \log^p\Big(\frac{1-b}{1-a}\Big)$$

diverges as $b \to 1$. It is the coefficient of the monomial B^q in $G_\varepsilon(1-\varepsilon)$ when $a = \varepsilon$, $b = 1-\varepsilon$, and this is the reason to call monomials that start with a B divergent.

Similarly, the integral

$$\int\limits_{a<t_q<\cdots<t_2<t_1<b} \omega_0(t_1) \wedge \cdots \wedge \omega_0(t_q) = \frac{1}{q!} \log^q\Big(\frac{b}{a}\Big)$$

diverges as $a \to 0$. It is the coefficient of the monomial A^p in $G_\varepsilon(1-\varepsilon)$ when $a = \varepsilon$, $b = 1-\varepsilon$, and this is the reason to call monomials that end with an A divergent.

Now consider the general case: integral of a product that contains both ω_0 and ω_1. For $\delta_j = 0$ or 1 and $0 < a < b < 1$, introduce the notation

$$I^{a,b}_{\delta_1\ldots\delta_m} = \int\limits_{a<t_m<\cdots<t_2<t_1<b} \omega_{\delta_1}(t_1) \wedge \cdots \wedge \omega_{\delta_m}(t_m) \ .$$

10.3.4. Lemma.

(i) *If* $\delta_1 = 0$, *then the integral* $I^{a,b}_{\delta_1\ldots\delta_m}$ *converges to a non-zero constant as* $b \to 1$, *and it grows as a power of* $\log(1-b)$ *if* $\delta_1 = 1$.

(ii) *If* $\delta_m = 1$, *then the integral* $I^{a,b}_{\delta_1\ldots\delta_m}$ *converges to a non-zero constant as* $a \to 0$, *and it grows as a power of* $\log a$ *if* $\delta_m = 0$.

Proof. Induction on the number of chords m. If $m = 1$ then the integral can be calculated explicitly like in the previous example, and the lemma follows from the result. Now suppose that the lemma is proved for $m-1$ chords. By the Fubini theorem the integral can be represented as

$$I^{a,b}_{1\delta_2\dots\delta_m} = \int\limits_{a<t<b} I^{a,t}_{\delta_2\dots\delta_m}\cdot\frac{dt}{t-1}\ , \qquad I^{a,b}_{0\delta_2\dots\delta_m} = \int\limits_{a<t<b} I^{a,t}_{\delta_2\dots\delta_m}\cdot\frac{dt}{t}\ ,$$

for the cases $\delta_1 = 1$ and $\delta_1 = 0$ respectively. By the induction assumption $0 < c < \left|I^{a,t}_{\delta_2\dots\delta_m}\right| < \left|\log^k(1-t)\right|$ for some constants c and k. The comparison test implies that the integral $I^{a,b}_{0\delta_2\dots\delta_m}$ converges as $b \to 1$ because $I^{a,t}_{\delta_2\dots\delta_m}$ grows slower than any power of $(1-t)$. Moreover, $\left|I^{a,b}_{0\delta_2\dots\delta_m}\right| > c\int_a^1 \frac{dt}{t} = -c\log(a) > 0$ because $0 < a < b < 1$.

In the case $\delta_1 = 1$ we have
$c\log(1-b) = c\int_0^b \frac{dt}{t-1} < \left|I^{a,b}_{1\delta_2\dots\delta_m}\right| < \left|\int_0^b \log^k(1-t)d(\log(1-t))\right| = \left|\frac{\log^{k+1}(1-b)}{k+1}\right|$, which proves assertion (i). The proof of assertion (ii) is similar. □

10.3.5. Here is a plan of our subsequent actions.

Let $\widehat{\mathcal{A}}^{conv}$ (resp. $\widehat{\mathcal{A}}^{div}$) be the subspace of $\widehat{\mathcal{A}} = \mathbb{C}\langle\langle A, B\rangle\rangle$ spanned by all convergent (resp. divergent) monomials. We are going to define a special linear map $f : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$ which kills divergent monomials and preserves the associator Φ. Applying f to both parts of equation (10.3.1.1) we will have

$$\text{(10.3.5.1)}\qquad \Phi = f(\Phi) = f\left(\lim_{\varepsilon\to 0}\ \varepsilon^{-B}\cdot Z^{conv}\cdot\varepsilon^{A}\right) = f\left(\lim_{\varepsilon\to 0} Z^{conv}\right)\ .$$

The last equality here follows from the fact that only the unit terms of ε^{-B} and ε^{A} are convergent and therefore survive under the action of f.

The convergent improper integral

$$\text{(10.3.5.2)}\quad \lim_{\varepsilon\to 0} Z^{conv} = 1 + \sum_{m=2}^{\infty}\ \sum_{\delta_2,\dots,\delta_{m-1}=0,1} I^{0,1}_{0\delta_2\dots\delta_{m-1}1}\cdot AC_{\delta_2}\dots C_{\delta_{m-1}}B$$

can be computed explicitly (here $C_j = A$ if $\delta_j = 0$ and $C_j = B$ if $\delta_j = 1$). Combining equations (10.3.5.1) and (10.3.5.2) we get

$$\text{(10.3.5.3)}\qquad \Phi = 1 + \sum_{m=2}^{\infty}\ \sum_{\delta_2,\dots,\delta_{m-1}=0,1} I^{0,1}_{1\delta_2\dots\delta_{m-1}0}\cdot f(AC_{\delta_2}\dots C_{\delta_{m-1}}B)$$

The knowledge of how f acts on the monomials from $\widehat{\mathcal{A}}$ leads to the desired formula for the associator.

10.3.6. Definition of the linear map $f : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$. Consider the algebra $\widehat{\mathcal{A}}[\alpha, \beta]$ of polynomials in two commuting variables α and β with coefficients in $\widehat{\mathcal{A}}$. Every monomial in $\widehat{\mathcal{A}}[\alpha, \beta]$ can be written uniquely as $\beta^p M \alpha^q$, where M is a monomial in $\widehat{\mathcal{A}}$. Define a $\mathbb{C}$-linear map $j : \widehat{\mathcal{A}}[\alpha, \beta] \to \widehat{\mathcal{A}}$ by $j(\beta^p M \alpha^q) = B^p M A^q$. Now for any element $\Gamma(A, B) \in \widehat{\mathcal{A}}$ let

$$f(\Gamma(A, B)) = j(\Gamma(A - \alpha, B - \beta)) \ .$$

10.3.7. Lemma. *If M is a divergent monomial in $\widehat{\mathcal{A}}$, then $f(M) = 0$.*

Proof. Consider the case where M starts with B, say $M = BC_2 \ldots C_m$, where each C_j is either A or B. Then

$$f(M) = j((B - \beta)M_2) = j(BM_2) - j(\beta M_2) \ ,$$

where $M_2 = (C_2 - \gamma_2) \ldots (C_m - \gamma_m)$ with $\gamma_j = \alpha$ or $\gamma_j = \beta$ depending on C_j. But $j(BM_2)$ equals $j(\beta M_2)$ by the definition of j above. The case where M ends with an A can be done similarly. □

10.3.8. One may notice that for any monomial $M \in \widehat{\mathcal{A}}$ we have $f(M) = M +$ (sum of divergent monomials). Therefore, by the lemma, f is an idempotent map, $f^2 = f$, i.e. f is a projection along $\widehat{\mathcal{A}}^{div}$ (but not onto $\widehat{\mathcal{A}}^{conv}$).

10.3.9. Proposition. $f(\Phi) = \Phi$.

Proof. We use the definition of the associator Φ as the KZ Drinfeld associator from Sec. 10.1.7 (see Proposition in Section 10.1.12).

It is the ratio $\Phi(A, B) = G_1^{-1} \cdot G_0$ of two solutions of the differential equation (10.1.6.1) from Sec. 10.1.7

$$G' = \left(\frac{A}{z} + \frac{B}{z-1}\right) \cdot G$$

with the asymptotics

$$G_0(z) \sim z^A \text{ as } z \to 0 \qquad \text{and} \qquad G_1(z) \sim (1-z)^B \text{ as } z \to 1 \ .$$

Consider the differential equation

$$H' = \left(\frac{A-\alpha}{z} + \frac{B-\beta}{z-1}\right) \cdot H \ .$$

A direct substitution shows that the functions

$$H_0(z) = z^{-\alpha}(1-z)^{-\beta} \cdot G_0(z) \qquad \text{and} \qquad H_1(z) = z^{-\alpha}(1-z)^{-\beta} \cdot G_1(z)$$

are its solutions with the asymptotics

$$H_0(z) \sim z^{A-\alpha} \text{ as } z \to 0 \qquad \text{and} \qquad H_1(z) \sim (1-z)^{B-\beta} \text{ as } z \to 1 \ .$$

Hence we have

$$\Phi(A - \alpha, B - \beta) = H_1^{-1} \cdot H_0 = G_1^{-1} \cdot G_0 = \Phi(A, B) \ .$$

Therefore

$$f(\Phi(A,B)) = j(\Phi(A-\alpha, B-\beta)) = j(\Phi(A,B)) = \Phi(A,B)$$

because j acts as the identity map on the subspace $\widehat{\mathcal{A}} \subset \widehat{\mathcal{A}}[\alpha,\beta]$. The proposition is proved. $\square$

10.3.10. To compute Φ according to formula (10.3.5.3) we must find the integrals $I^{0,1}_{0\delta_2\ldots\delta_{m-1}1}$ and the action of f on the monomials. Let us compute $f(AC_{\delta_2}\ldots C_{\delta_{m-1}}B)$ first.

Represent the monomial $M = AC_{\delta_2}\ldots C_{\delta_{m-1}}B$ in the form

$$M = A^{p_1}B^{q_1}\ldots A^{p_l}B^{q_l}$$

for some positive integers $p_1, q_1, \ldots, p_l, q_l$. Then

$$f(M) = j((A-\alpha)^{p_1}(B-\beta)^{q_1}\ldots(A-\alpha)^{p_l}(B-\beta)^{q_l})\ .$$

We are going to expand the product, collect all β's on the left and all α's on the right, and then replace β by B and α by A. To this end let us introduce the following multi-index notations:

$$\mathbf{r} = (r_1,\ldots,r_l);\quad \mathbf{i} = (i_1,\ldots,i_l);\qquad \mathbf{s} = (s_1,\ldots,s_l);\quad \mathbf{j} = (j_1,\ldots,j_l);$$

$$\mathbf{p} = \mathbf{r}+\mathbf{i} = (r_1+i_1,\ldots,r_l+i_l);\qquad \mathbf{q} = \mathbf{s}+\mathbf{j} = (s_1+j_1,\ldots,s_l+j_l);$$

$$|\mathbf{r}| = r_1+\cdots+r_l;\qquad |\mathbf{s}| = s_1+\cdots+s_l;$$

$$\binom{\mathbf{p}}{\mathbf{r}} = \binom{p_1}{r_1}\binom{p_2}{r_2}\cdots\binom{p_l}{r_l};\qquad \binom{\mathbf{q}}{\mathbf{s}} = \binom{q_1}{s_1}\binom{q_2}{s_2}\cdots\binom{q_l}{s_l};$$

$$(A,B)^{(\mathbf{i},\mathbf{j})} = A^{i_1}\cdot B^{j_1}\cdot\cdots\cdot A^{i_l}\cdot B^{j_l}$$

We have

$$(A-\alpha)^{p_1}(B-\beta)^{q_1}\ldots(A-\alpha)^{p_l}(B-\beta)^{q_l} = \sum_{\substack{0\leqslant\mathbf{r}\leqslant\mathbf{p}\\ 0\leqslant\mathbf{s}\leqslant\mathbf{q}}} (-1)^{|\mathbf{r}|+|\mathbf{s}|}\binom{\mathbf{p}}{\mathbf{r}}\binom{\mathbf{q}}{\mathbf{s}}\cdot\beta^{|\mathbf{s}|}(A,B)^{(\mathbf{i},\mathbf{j})}\alpha^{|\mathbf{r}|}\ ,$$

where the inequalities $0\leqslant\mathbf{r}\leqslant\mathbf{p}$ and $0\leqslant\mathbf{s}\leqslant\mathbf{q}$ mean $0\leqslant r_1\leqslant p_1$, …, $0\leqslant r_l\leqslant p_l$, and $0\leqslant s_1\leqslant q_1$,…, $0\leqslant s_l\leqslant q_l$. Therefore

$$f(M) = \sum_{\substack{0\leqslant\mathbf{r}\leqslant\mathbf{p}\\ 0\leqslant\mathbf{s}\leqslant\mathbf{q}}} (-1)^{|\mathbf{r}|+|\mathbf{s}|}\binom{\mathbf{p}}{\mathbf{r}}\binom{\mathbf{q}}{\mathbf{s}}\cdot B^{|\mathbf{s}|}(A,B)^{(\mathbf{i},\mathbf{j})}A^{|\mathbf{r}|}\ . \tag{10.3.10.1}$$

10.3.11. To complete the formula for the associator we need to compute the coefficient $I^{0,1}_{1\delta_2\ldots\delta_{m-1}0}$ of $f(M)$. It turns out that, up to a sign, they are equal to some values of the multivariate ζ-function

$$\zeta(a_1,\ldots,a_n) \;=\; \sum_{0<k_1<k_2<\cdots<k_n} k_1^{-a_1}\ldots k_n^{-a_n}$$

where a_1, ..., a_n are positive integers (see [**LM1**]). Namely, the coefficients in question are equal, up to a sign, to the values of ζ at integer points $(a_1,\ldots,a_n)\in\mathbb{Z}^n$, which are called (*multiple zeta values*, or MZV for short). Multiple zeta values for $n=2$ were first studied by L. Euler in 1775. His paper [**Eu**] contains several dozens interesting relations between MZVs and values of the univariate Riemann's zeta function. Later, this subject was almost forgotten for more than 200 years until D. Zagier and M. Hoffman revived a general interest to MZVs by their papers [**Zag3**], [**Hoff**].

Exercise. The sum converges if and only if $a_n \geqslant 2$.

10.3.12. Remark. Different conventions about the order of arguments in ζ: schools of Hoffman and Zagier!!

10.3.13. Proposition. *For* $\mathbf{p}>0$ *and* $\mathbf{q}>0$ *let*

(10.3.13.1)
$$\eta(\mathbf{p},\mathbf{q}) := \zeta(\underbrace{1,\ldots,1}_{q_l-1},\ p_l+1,\ \underbrace{1,\ldots,1}_{q_{l-1}-1},\ p_{l-1}+1,\ \ldots\ \underbrace{1,\ldots,1}_{q_1-1},\ p_1+1)\ .$$

Then

(10.3.13.2)
$$I^{0,1}_{\underbrace{0\ldots0}_{p_1}\underbrace{1\ldots1}_{q_1}\ldots\ldots\underbrace{0\ldots0}_{p_l}\underbrace{1\ldots1}_{q_l}} = (-1)^{|\mathbf{q}|}\eta(\mathbf{p},\mathbf{q})\ .$$

The calculations needed to prove the proposition, are best organised in terms of the (univariate) *polylogarithm*[1] function defined by the series

(10.3.13.3)
$$\mathrm{Li}_{a_1,\ldots,a_n}(z) = \sum_{0<k_1<k_2<\cdots<k_n} \frac{z^{k_n}}{k_1^{a_1}\ldots k_n^{a_n}}\ ,$$

which obviously converges for $|z|<1$.

---

[1]It is a generalization of Euler's dilogarithm $\mathrm{Li}_2(z)$ we used on p.247, and a specialization of the multivariate polylogarithm

$$\mathrm{Li}_{a_1,\ldots,a_n}(z_1,\ldots,z_n) = \sum_{0<k_1<k_2<\cdots<k_n} \frac{z_1^{k_1}\ldots z_n^{k_n}}{k_1^{a_1}\ldots k_n^{a_n}}$$

introduced by A. Goncharov in [**Gon**].

10.3.14. Lemma. *For* $|z| < 1$

$$\mathrm{Li}_{a_1,\dots,a_n}(z) = \begin{cases} \displaystyle\int_0^z \mathrm{Li}_{a_1,\dots,a_n-1}(t)\frac{dt}{t}\ , & \text{if } a_n > 1 \\ \displaystyle -\int_0^z \mathrm{Li}_{a_1,\dots,a_{n-1}}(t)\frac{d(1-t)}{1-t}\ , & \text{if } a_n = 1\ . \end{cases}$$

Proof. The lemma follows from the identities below, whose proofs we leave to the reader as an exercise.

$$\tfrac{d}{dz}\mathrm{Li}_{a_1,\dots,a_n}(z) = \begin{cases} \tfrac{1}{z}\cdot \mathrm{Li}_{a_1,\dots,a_n-1}(z)\ , & \text{if } a_n > 1 \\ \tfrac{1}{1-z}\cdot \mathrm{Li}_{a_1,\dots,a_{n-1}}(z)\ , & \text{if } a_n = 1\ ; \end{cases}$$

$$\tfrac{d}{dz}\mathrm{Li}_1(z) = \tfrac{1}{1-z}\ .$$

□

10.3.15. Proof of proposition 10.3.13. From the previous lemma we have

$$\mathrm{Li}_{\underbrace{1,1,\dots,1}_{q_l-1},\ p_l+1,\ \underbrace{1,1,\dots,1}_{q_{l-1}-1},\ p_{l-1}+1,\ \dots,\ \underbrace{1,1,\dots,1}_{q_1-1},\ p_1+1}(z) =$$

$$= (-1)^{q_1+\dots+q_l} \int\limits_{0<t_m<\dots<t_2<t_1<z} \underbrace{\omega_0(t_1)\wedge\dots\wedge\omega_0(t_{p_1})}_{p_1}\wedge$$

$$\wedge\underbrace{\omega_1(t_{p_1+1})\wedge\dots\wedge\omega_1(t_{p_1+q_1})}_{q_1}\wedge\dots\wedge\underbrace{\omega_1(t_{p_1+\dots+p_l+1})\wedge\dots\wedge\omega_1(t_{p_1+\dots+q_l})}_{q_l} =$$

$$= (-1)^{|\mathbf{q}|} I^{0,z}_{\underbrace{0\dots0}_{p_1}\underbrace{1\dots1}_{q_1}\dots\dots\underbrace{0\dots0}_{p_l}\underbrace{1\dots1}_{q_l}}\ .$$

Note that the multiple polylogarithm series (10.3.13.3) converges for $z = 1$ in the case $a_n > 1$. This implies that if $p_l \geqslant 1$ (which holds for a convergent monomial), then we have

$$\begin{aligned} \eta(\mathbf{p},\mathbf{q}) &= \zeta(\underbrace{1,\dots,1}_{q_l-1},\ p_l+1,\ \underbrace{1,\dots,1}_{q_{l-1}-1},\ p_{l-1}+1,\ \dots\ \underbrace{1,\dots,1}_{q_1-1},\ p_1+1) \\ &= \mathrm{Li}_{\underbrace{1,1,\dots,1}_{q_l-1},\ p_l+1,\ \underbrace{1,1,\dots,1}_{q_{l-1}-1},\ p_{l-1}+1,\ \dots,\ \underbrace{1,1,\dots,1}_{q_1-1},\ p_1+1}(1) \\ &= (-1)^{|\mathbf{q}|} I^{0,1}_{\underbrace{0\dots0}_{p_1}\underbrace{1\dots1}_{q_1}\dots\dots\underbrace{0\dots0}_{p_l}\underbrace{1\dots1}_{q_l}}\ . \end{aligned}$$

The Proposition is proved.

10.3.16. Explicit formula for the associator. Combining equations (10.3.5.3), (10.3.10.1), and (10.3.13.2) we get the following formula for the associator.

$$\boxed{\Phi = 1 + \sum_{m=2}^{\infty} \sum_{\substack{0<\mathbf{p},0<\mathbf{q} \\ |\mathbf{p}|+|\mathbf{q}|=m}} \eta(\mathbf{p},\mathbf{q}) \cdot \sum_{\substack{0\leqslant\mathbf{r}\leqslant\mathbf{p} \\ 0\leqslant\mathbf{s}\leqslant\mathbf{q}}} (-1)^{|\mathbf{r}|+|\mathbf{j}|} \binom{\mathbf{p}}{\mathbf{r}} \binom{\mathbf{q}}{\mathbf{s}} \cdot B^{|\mathbf{s}|}(A,B)^{(\mathbf{i},\mathbf{j})} A^{|\mathbf{r}|}}$$

where $\mathbf{i}$ and $\mathbf{j}$ are multi-indices of the same length, $\mathbf{p} = \mathbf{r} + \mathbf{i}$, $\mathbf{q} = \mathbf{s} + \mathbf{j}$, and $\eta(\mathbf{p},\mathbf{q})$ is the multiple zeta value given by (10.3.13.1).

This formula was obtained by Le and Murakami in [**LM1, LM2, LM4**].

10.3.17. Example. Degree 2 terms of the associator. There is only one possibility to represent $m = 2$ as the sum of two positive integers: $2 = 1 + 1$. So we have only one possibility for $\mathbf{p}$ and $\mathbf{q}$: $\mathbf{p} = (1)$, $\mathbf{q} = (1)$. In this case $\eta(\mathbf{p},\mathbf{q}) = \zeta(2) = \pi^2/6$ according to (10.3.13.1). The multi-indices $\mathbf{r}$ and $\mathbf{s}$ have length 1 and thus consist of a single number $\mathbf{r} = (r_1)$ and $\mathbf{s} = (s_1)$. There are two possibilities for each of them: $r_1 = 0$ or $r_1 = 1$, and $s_1 = 0$ or $s_1 = 1$. In all these cases the binomial coefficients $\binom{\mathbf{p}}{\mathbf{r}}$ and $\binom{\mathbf{q}}{\mathbf{s}}$ are equal to 1. We arrange all the possibilities in the following table.

r_1	s_1	i_1	j_1	$(-1)^{\lvert\mathbf{r}\rvert+\lvert\mathbf{j}\rvert} \cdot B^{\lvert\mathbf{s}\rvert}(A,B)^{(\mathbf{i},\mathbf{j})}A^{\lvert\mathbf{r}\rvert}$
0	0	1	1	$-A\ B$
0	1	1	0	$B\ A$
1	0	0	1	$B\ A$
1	1	0	0	$-B\ A$

Hence, for the degree 2 terms of the associator we get the formula:

$$-\zeta(2)[A,B] = -\frac{\zeta(2)}{(2\pi i)^2}[a,b] = \frac{1}{24}[a,b]\ ,$$

where $a = (2\pi i)A =$ , and $b = (2\pi i)B =$ .

10.3.18. Example. Degree 3 terms of the associator. There are two ways to represent $m = 3$ as the sum of two positive integers: $3 = 2 + 1$ and $3 = 1 + 2$. In each case either $\mathbf{p} = (1)$ or $\mathbf{q} = (1)$. Hence $l = 1$ and both multi-indices consist of just one number $\mathbf{p} = (p_1)$, $\mathbf{q} = (q_1)$. Therefore all other multi-indices $\mathbf{r}$, $\mathbf{s}$, $\mathbf{i}$, $\mathbf{j}$ also consist of one number.

Here is the corresponding table.

p_1	q_1	$\eta(\mathbf{p},\mathbf{q})$	r_1	s_1	i_1	j_1	$(-1)^{\lvert\mathbf{r}\rvert+\lvert\mathbf{j}\rvert}\binom{\mathbf{p}}{\mathbf{r}}\binom{\mathbf{q}}{\mathbf{s}}\cdot B^{\lvert\mathbf{s}\rvert}(A,B)^{(\mathbf{i},\mathbf{j})}A^{\lvert\mathbf{r}\rvert}$
			0	0	2	1	$-A\ A\ B$
			0	1	2	0	$B\ A\ A$
2	1	$\zeta(3)$	1	0	1	1	$2A\ B\ A$
			1	1	1	0	$-2B\ A\ A$
			2	0	0	1	$-B\ A\ A$
			2	1	0	0	$B\ A\ A$
			0	0	1	2	$A\ B\ B$
			1	0	0	2	$-B\ B\ A$
1	2	$\zeta(1,2)$	0	1	1	1	$-2B\ A\ B$
			1	1	0	1	$2B\ B\ A$
			0	2	1	0	$B\ B\ A$
			1	2	0	0	$-B\ B\ A$

Using the Euler identity $\zeta(1,2)=\zeta(3)$ (see section 10.3.20) we can sum up the degree 3 part of Φ into the formula

$$\zeta(3)\Big(-A\ A\ B+2A\ B\ A-B\ A\ A+A\ B\ B-2B\ A\ B+B\ B\ A\Big)$$
$$=\zeta(3)\left(-\Big[A,\big[A,B\big]\Big]-\Big[B,\big[A,B\big]\Big]\right)=-\frac{\zeta(3)}{(2\pi i)^3}\big[a+b,[a,b]\big]\ .$$

10.3.19. Example. Degree 4 terms of the associator. Proceeding in the same way and using the the identities from Sec.10.3.20:

$$\zeta(1,1,2)=\zeta(4)=\pi^4/90,\qquad \zeta(1,3)=\pi^4/360,\qquad \zeta(2,2)=\pi^4/120\ ,$$

we can write out the associator Φ up to degree 4:

$$\boxed{\begin{aligned}\Phi_{\mathrm{KZ}}\ =\ 1\ &+\ \frac{1}{24}[a,b]\ -\ \frac{\zeta(3)}{(2\pi i)^3}\big[a+b,[a,b]\big]\ -\ \frac{1}{1440}\Big[a,\big[a,[a,b]\big]\Big]\\ &-\ \frac{1}{5760}\Big[a,\big[b,[a,b]\big]\Big]\ -\ \frac{1}{1440}\Big[b,\big[b,[a,b]\big]\Big]\ +\ \frac{1}{1152}[a,b]^2\\ &+\ (\text{terms of order}>4)\ .\end{aligned}}$$

10.3.20. Multiple zeta values. There is a lot of relations between MZV's and powersof π. Some of them, like $\zeta(2)=\frac{\pi^2}{6}$ or $\zeta(1,2)=\zeta(3)$, were already known to Euler. The last one can be obtained in the following way.

According to (10.3.13.1) and (10.3.13.2) we have

$$\begin{aligned}\zeta(1,2) &= \eta((1),(2)) = I^{0,1}_{011} = \int\limits_{0<t_3<t_2<t_1<1} \omega_0(t_1)\wedge\omega_1(t_2)\wedge\omega_1(t_3)\\ &= \int\limits_{0<t_3<t_2<t_1<1} \frac{dt_1}{t_1}\wedge\frac{d(1-t_2)}{1-t_2}\wedge\frac{d(1-t_3)}{1-t_3}\ .\end{aligned}$$

The change of variables $(t_1,t_2,t_3)\mapsto(1-t_3,1-t_2,1-t_1)$ transforms the last integral to

$$\begin{aligned}&\int\limits_{0<t_3<t_2<t_1<1} \frac{d(1-t_3)}{1-t_3}\wedge\frac{dt_2}{t_2}\wedge\frac{dt_1}{t_1}\\ &\quad = -\int\limits_{0<t_3<t_2<t_1<1} \omega_0(t_1)\wedge\omega_0(t_2)\wedge\omega_1(t_3) = -I^{0,1}_{001} = \eta((2),(1)) = \zeta(3)\ .\end{aligned}$$

In the general case a similar change of variables

$$(t_1,t_2,\dots,t_m)\mapsto(1-t_m,\dots,1-t_2,1-t_1)$$

gives the identity

$$I^{0,1}_{\underbrace{0\dots0}_{p_1}\underbrace{1\dots1}_{q_1}\dots\dots\underbrace{0\dots0}_{p_l}\underbrace{1\dots1}_{q_l}} = (-1)^m I^{0,1}_{\underbrace{0\dots0}_{q_l}\underbrace{1\dots1}_{p_l}\dots\dots\underbrace{0\dots0}_{q_l}\underbrace{1\dots1}_{p_l}}\ .$$

By (10.3.13.2), we have

$$\begin{aligned}I^{0,1}_{\underbrace{0\dots0}_{p_1}\underbrace{1\dots1}_{q_1}\dots\dots\underbrace{0\dots0}_{p_l}\underbrace{1\dots1}_{q_l}} &= (-1)^{|q|}\eta(\mathbf{p},\mathbf{q}),\\ I^{0,1}_{\underbrace{0\dots0}_{q_l}\underbrace{1\dots1}_{p_l}\dots\dots\underbrace{0\dots0}_{q_l}\underbrace{1\dots1}_{p_l}} &= (-1)^{|p|}\eta(\overline{\mathbf{q}},\overline{\mathbf{p}}),\end{aligned}$$

where the bar denotes the inversion of a sequence: $\overline{\mathbf{p}} = (p_l,p_{l-1},\dots,p_1)$, $\overline{\mathbf{q}} = (q_l,q_{l-1},\dots,q_1)$.

Since $|p|+|q| = m$, we deduce that

$$\eta(\mathbf{p},\mathbf{q}) = \eta(\overline{\mathbf{q}},\overline{\mathbf{p}}),$$

This relation is called the *duality relation* between MZV's. After the conversion from η to ζ according to equation (10.3.13.1), the duality relations become picturesque and unexpected.

—!! write about the rotation of the picture by 180° and its relation to duality—

As an example, we give a table of all nontrivial duality relations of weight $m \leqslant 5$:

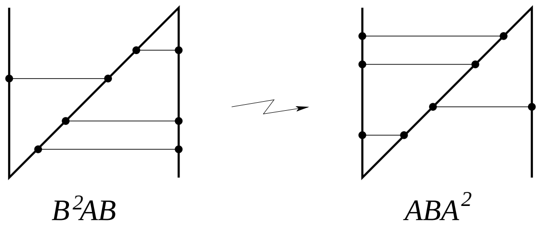


Figure 10.3.20.1. Rotation of a tangle through 180°

$\mathbf{p}$	$\mathbf{q}$	$\bar{\mathbf{q}}$	$\bar{\mathbf{p}}$	relation
(1)	(2)	(2)	(1)	$\zeta(1,2)=\zeta(3)$
(1)	(3)	(3)	(1)	$\zeta(1,1,2)=\zeta(4)$
(1)	(4)	(4)	(1)	$\zeta(1,1,1,2)=\zeta(5)$
(2)	(3)	(3)	(2)	$\zeta(1,1,3)=\zeta(1,4)$
$(1,1)$	$(1,2)$	$(2,1)$	$(1,1)$	$\zeta(1,2,2)=\zeta(2,3)$
$(1,1)$	$(2,1)$	$(1,2)$	$(1,1)$	$\zeta(2,1,2)=\zeta(3,2)$

The reader may want to check this table by way of exercise.

There are other relations between the multiple zeta values that do not follow from the duality law. Let us quote just a few.

1. Euler's relations:

$$\zeta(1,n-1)+\zeta(2,n-2)+\cdots+\zeta(n-2,2)=\zeta(n), \tag{10.3.20.1}$$

$$\zeta(m)\cdot\zeta(n)=\zeta(m,n)+\zeta(n,m)+\zeta(m+n)\ . \tag{10.3.20.2}$$

2. Relations obtained by Le and Murakami [**LM1**] computing the Kontsevich integral of the unknot by the combinatorial procedure explained below in Section 10.2 (the first one was earlier proved by M. Hoffman [**Hoff**]):

$$\zeta(\underbrace{2,2,\dots,2}_{m})=\frac{\pi^{2m}}{(2m+1)!} \tag{10.3.20.3}$$

$$\begin{aligned}\Big(\frac{1}{2^{2n-2}}-1\Big)\zeta(2n)-\zeta(1,2n-1)+\zeta(1,1,2n-1)-\dots\\ +\zeta(\underbrace{1,1,\dots,1}_{2n-2},2)=0\ .\end{aligned} \tag{10.3.20.4}$$

These relations are sufficient to express all multiple zeta values with the sum of arguments equal to 4, through powers of π. Indeed, we have:

$$\zeta(1,3)+\zeta(2,2)=\zeta(4),$$
$$\zeta(2)^2=2\zeta(2,2)+\zeta(4),$$
$$\zeta(2,2)=\frac{\pi^4}{120},$$
$$-\frac{3}{4}\zeta(4)-\zeta(1,3)+\zeta(1,1,2)=0.$$

Solving these equations one by one and using the identity $\zeta(2)=\pi^2/6$, we find the values of all MZVs of weight 4: $\zeta(2,2)=\pi^4/120$, $\zeta(1,3)=\pi^4/360$, $\zeta(1,1,2)=\zeta(4)=\pi^4/90$.

There exists an extensive literature about the relations between MZV's, e.g. [**BBBL, Car2, Hoff, HoOh, OU**], and the interested reader is invited to consult it.

An attempt to overview the whole variety of relations between MZV's was undertaken by D. Zagier [**Zag3**]. Call the *weight* of a multiple zeta value $\zeta(n_1,\dots,n_k)$ the sum of all its arguments $w=n_1+\dots+n_k$. Let $\mathcal{Z}_w$ be the vector subspace of the reals $\mathbb{R}$ over the rationals $\mathbb{Q}$ spanned by all MZV's of a fixed weight w. For completeness we put $\mathcal{Z}_0=\mathbb{Q}$ and $\mathcal{Z}_1=0$. Denote the formal direct sum of all $\mathcal{Z}_w$ by $\mathcal{Z}_\bullet := \bigoplus\limits_{w\geqslant 0}\mathcal{Z}_w$.

Proposition. *The vector space $\mathcal{Z}_\bullet$ forms a graded algebra over $\mathbb{Q}$, i.e. $\mathcal{Z}_u\cdot\mathcal{Z}_v\subseteq\mathcal{Z}_{u+v}$.*

Euler's product formula (10.3.20.2) illustrates this statement. A proof can be found in [**Gon**]. D. Zagier made a conjecture about the Poincaré series of this algebra.

Conjecture ([**Zag3**]).

$$\sum_{w=0}^{\infty}\dim_{\mathbb{Q}}(\mathcal{Z}_w)\cdot t^w=\frac{1}{1-t^2-t^3}\ .$$

This series turns out to be related to the dimensions of various subspaces in the primitive space of the chord diagram algebra $\mathcal{A}$ (see [**Br, Kre**]).

10.3.21. Logarithm of the KZ associator modulo the second commutant. The associator Φ_{KZ} is group-like (see exercise 3 at the end of the chapter). Therefore its logarithm can be expressed as a Lie series in variables A and B. Let L be the completion of a free Lie algebra generated by A and B and let $L'':=[[L,L],[L,L]]$ be its second commutant. We can consider L as a subspace of $\mathbb{C}\langle\langle A, B\rangle\rangle$. V. Drinfeld [**Dr2**] found the following formula

$$\ln \Phi_{\mathrm{KZ}} = \sum_{k,l\geqslant 0} c_{kl} \operatorname{ad}_B^l \operatorname{ad}_A^k [A, B] \qquad (\mathrm{mod}\ L'') ,$$

where the coefficients c_{kl} are defined by the generating function

$$1 + \sum_{k,l\geqslant 0} c_{kl} u^{k+1} v^{l+1} = \exp\left(\sum_{n=2}^{\infty} \frac{\zeta(n)}{(2\pi i)^n n} \big(u^n + v^n - (u+v)^n\big)\right)$$

expressed in terms of the univariate zeta function $\zeta(n) := \sum_{k=1}^{\infty} k^{-n}$. In particular, $c_{kl} = c_{lk}$ and $c_{k0} = c_{0k} = -\frac{\zeta(k+2)}{(2\pi i)^{k+2}}$.

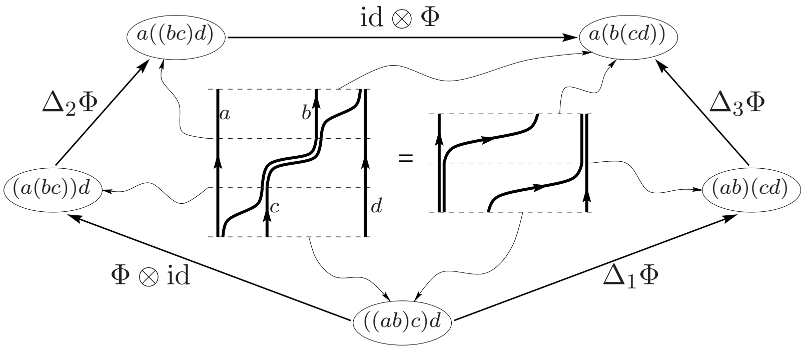
10.4. General associators

Let $\mathcal{A}(n)$ be the algebra of tangled chord diagrams on n strings (we will always depict them as vertical and oriented upwards). (In distinction with the previously considered algebra $\mathcal{A}^h(n)$, the chords need not be horizontal.) The space $\mathcal{A}(n)$ is graded by the number of chords in the diagram. We denote the graded completion of this graded space by $\widehat{\mathcal{A}}(n)$.

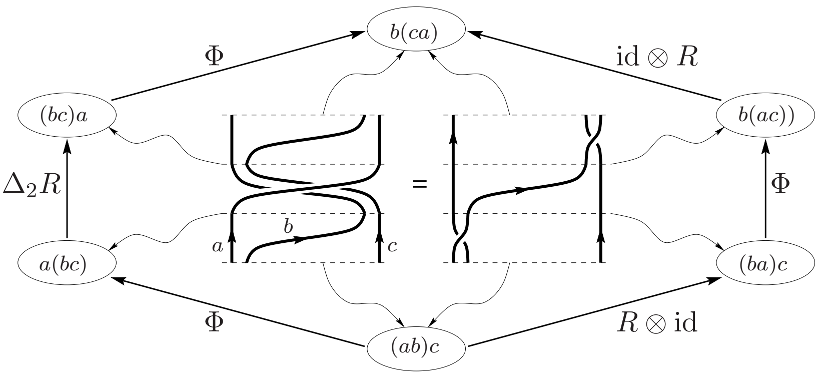
10.4.1. Definition. An *associator* Φ is an element of the algebra $\widehat{\mathcal{A}}(3)$ satisfying the following axioms:

- (*strong invertibility*) $\varepsilon_1(\Phi) = \varepsilon_2(\Phi) = \varepsilon_3(\Phi) = 1$ (operations ε_i are defined on page **??**; in particular, this property means that the series Φ starts with 1 and thus represents an invertible element of the algebra $\widehat{\mathcal{A}}(3)$).
- (*skew symmetry*) $\Phi^{-1} = \Phi^{321}$, where Φ^{321} is obtained from Φ by changing the first and the third strings (i.e. by adding a permutation (321) both to the top and the bottom of each diagram appearing in the series Φ).
- (*pentagon relation*) $\quad (\mathrm{id}\otimes\Phi)\cdot(\Delta_2\Phi)\cdot(\Phi\otimes\mathrm{id}) = (\Delta_3\Phi)\cdot(\Delta_1\Phi).$

Diagrammatically this relation can be depicted as follows.



- (*hexagon relation*) $\quad \Phi\cdot(\Delta_2 R)\cdot\Phi = (\mathrm{id}\otimes R)\cdot\Phi\cdot(R\otimes\mathrm{id}).$



This relation also contains the element R defined as follows

$R =$  $\cdot \exp\Big($  $/\,2\Big)$.

A version of the last two relations appears in abstract category theory where they make part of the definition of a monoidal category (see [**ML**, Sec.XI.1]).

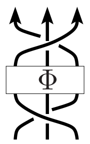
10.4.2. Theorem. *The Knizhnik–Zamolodchikov Drinfeld associator* Φ_{KZ} *satisfies the axioms above.*

Proof. The main observation is that the pentagon and the hexagon relations hold because Φ_{KZ} can be expressed through the Kontsevich integral and therefore possesses the property of horizontal invariance. The details of the proof are as follows.

Property 1 immediately follows from the explicit formula 10.3.16 for the associator Φ_{KZ}, which shows that the series starts with 1 and every term appearing with non-zero coefficient has endpoints of chords on each of the three strings.

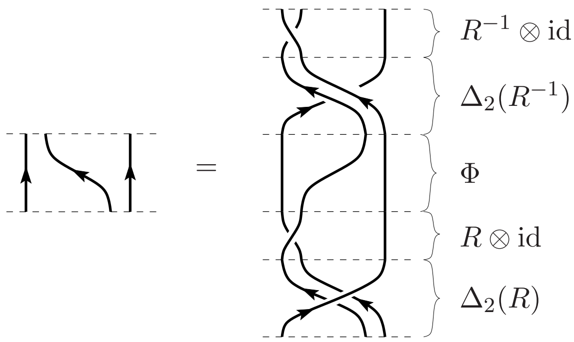
Property 2 We must prove that

10.4.3. Lemma.



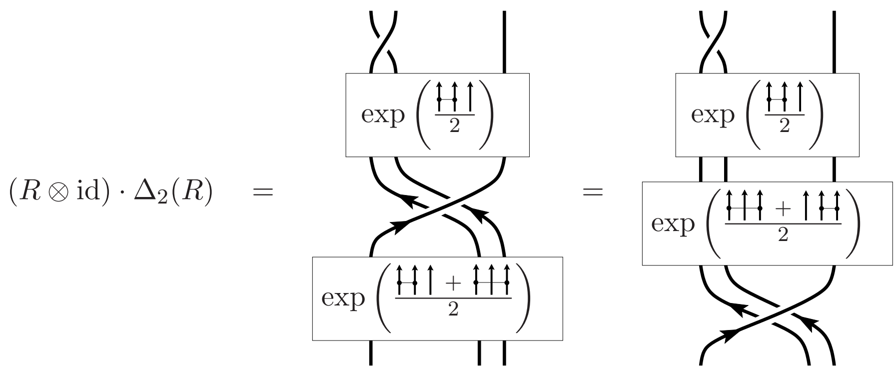
$$= \quad \Phi^{-1} \ .$$

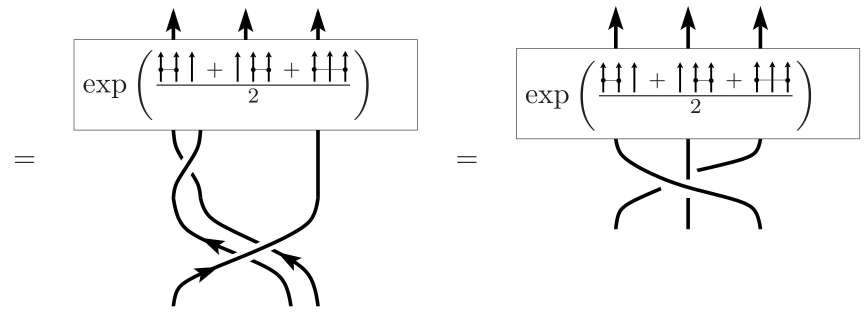
Indeed, the inverse associator Φ^{-1} is equal to the combinatorial Kontsevich integral of the following parenthesized tangle which can be deformed to the picture on the right.



Therefore, $\Phi^{-1} = (R^{-1} \otimes \mathrm{id}) \cdot \Delta_2(R^{-1}) \cdot \Phi \cdot (R \otimes \mathrm{id}) \cdot \Delta_2(R)$.

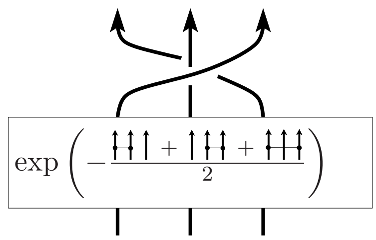
Now we have

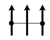




Similarly we can get

$$\Delta_2(R^{-1})\cdot(R^{-1}\otimes \mathrm{id}) \quad =$$



Since the element  +  +  belongs to the center of the algebra of horizontal chord diagrams it commutes with any other element of this algebra. In particular, the exponent of this elements commutes with Φ.

Property 3. Now let us prove the pentagon relation. We will follow a procedure similar to that in Example **??**. Assume that the distances between the strings of the tangles are as indicated in Figure 10.4.3.1.

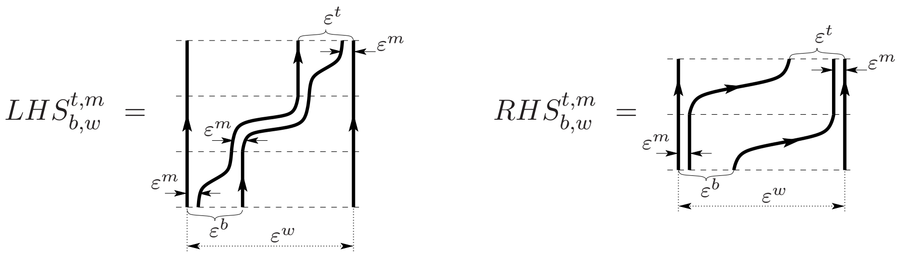


Figure 10.4.3.1. The two sides of the pentagon relation

Since the tangles are isotopic, we have $Z(LHS^{t,m}_{b,w}) = Z(RHS^{t,m}_{b,w})$. Hence

$$\varepsilon^{-\frac{t-w}{2\pi i}\cdot\Delta_3(\;)}\cdot\varepsilon^{-\frac{m-w}{2\pi i}\cdot\;}\cdot Z(LHS^{t,m}_{b,w})\cdot\varepsilon^{\frac{m-w}{2\pi i}\cdot\;}\cdot\varepsilon^{\frac{b-w}{2\pi i}\cdot\Delta_1(\;)}$$
$$=\varepsilon^{-\frac{t-w}{2\pi i}\cdot\Delta_3(\;)}\cdot\varepsilon^{-\frac{m-w}{2\pi i}\cdot\;}\cdot Z(RHS^{t,m}_{b,w})\varepsilon^{\frac{m-w}{2\pi i}\cdot\;}\cdot\varepsilon^{\frac{b-w}{2\pi i}\cdot\Delta_1(\;)}\;.$$

Now, following **??**, we are going to insert the commuting and mutually canceling factors into the left and right hand sides and then pass to the limit as $\varepsilon\to 0$. A possible choice of insertions for the left hand side is shown in Figure 10.4.3.2 and for the right hand side, in Figure 10.4.3.3. Values of the limits at the right columns are justified in problems 15 and 16 at the end of Chapter 8.

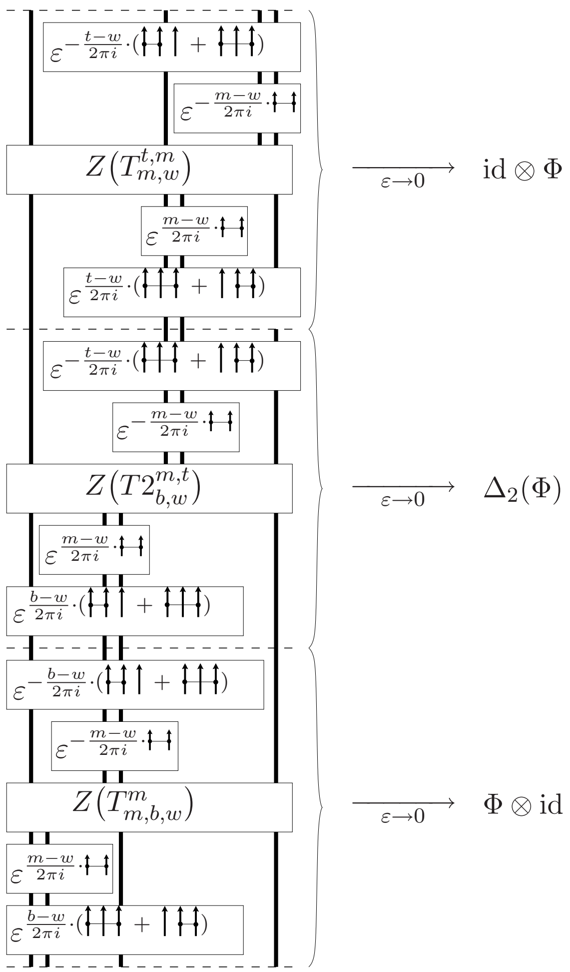


Figure 10.4.3.2. Inserted factors for the LHS of the pentagon relation

Property 4, the hexagon relation, can be proven in the same spirit as the pentagon relation. We leave details to the reader as an exercise (problem 8 at the end of this chapter). $\square$

10.4.4. Rational associators. Drinfeld proved that there exists an associator with rational coefficients. In [**BN2**] D. Bar-Natan, following [**Dr2**], gives a construction of an associator by induction on the degree. He implemented the inductive procedure in `Mathematica` ([**BN5**]) and computed the logarithm of the associator up to degree 7. With the notation $a =$,

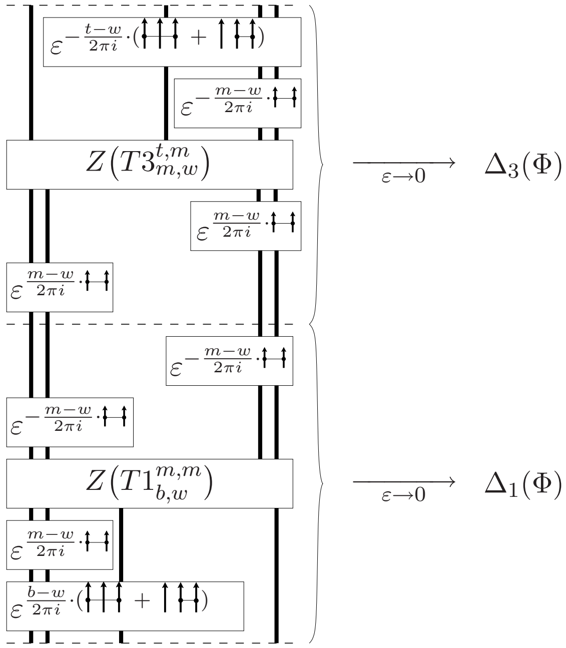


Figure 10.4.3.3. Inserted factors for the RHS of the pentagon relation

$b =$ ⟂⟂⟂ his answer looks as follows.

$$\begin{aligned}\ln\Phi \;=\;& \tfrac{1}{48}[ab] - \tfrac{1}{1440}[a[a[ab]]] - \tfrac{1}{11520}[a[b[ab]]]\\ &+\tfrac{1}{60480}[a[a[a[a[ab]]]]] + \tfrac{1}{1451520}[a[a[a[b[ab]]]]]\\ &+\tfrac{13}{1161216}[a[a[b[b[ab]]]]] + \tfrac{17}{1451520}[a[b[a[a[ab]]]]]\\ &+\tfrac{1}{1451520}[a[b[a[b[ab]]]]]\\ &-(\text{the similar terms with interchanged } a \text{ and } b) + \dots\end{aligned}$$

10.4.5. Remark. This expression is obtained from Φ_{KZ} expanded to degree 7, by substitutions $\zeta(3)\to 0$, $\zeta(5)\to 0$, $\zeta(7)\to 0$.

10.4.6. Remark. V. Kurlin [**Kur**] described all group-like associators modulo the second commutant. The Drinfeld associator Φ_{KZ} is one of them (see Sec. 10.3.21).

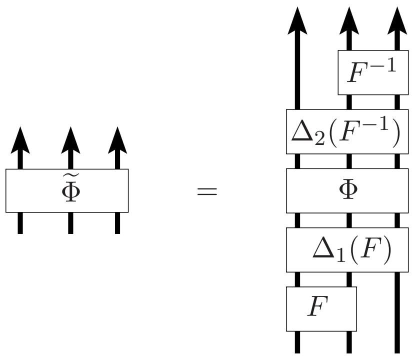
10.4.7. The axioms do not define the associator uniquely. The following theorem describes the totality of all associators.

Theorem. *(*[**Dr1, LM2**]*). Let Φ and $\widetilde{\Phi}$ be two associators. Then there is a symmetric invertible element $F\in\widehat{\mathcal{A}}(2)$ such that*

$$\widetilde{\Phi} = (\mathrm{id}\otimes F^{-1})\cdot\Delta_2(F^{-1})\cdot\Phi\cdot\Delta_1(F)\cdot(F\otimes\mathrm{id})\ .$$

Conversely, for any associator Φ *and an invertible element* $F \in \widehat{\mathcal{A}}(2)$ *this equation produces an associator.*

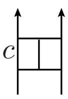
The operation $\Phi \mapsto \widetilde{\Phi}$ is called *twisting* by F. Diagrammatically, it looks as follows:



10.4.8. Exercise. Prove that the twist by an element $F = \exp(\alpha$ $)$ is identical on any associator.

10.4.9. Exercise. Prove that the elements $f \otimes \mathrm{id}$ and $\Delta_1(F)$ commute.

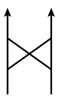
10.4.10. Example. Take $\Phi = \Phi_{\mathrm{BN}}$ (the associator of the previous remark) and $\widetilde{\Phi} = \Phi_{\mathrm{KZ}}$. It is remarkable that these associators are both horizontal, i.e. belong to the subalgebra $\mathcal{A}^h(3)$ of horizontal diagrams, but one can be converted into another only by a non-horizontal twist. For example, twisting Φ_{BN} by the element

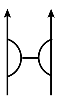
$$F = 1 + c$$ 

with an appropriate constant c ensures the coincidence with Φ_{KZ} up to degree 4.

— the coresponding beautiful calculation will be added by S.D.!! —

10.4.11. Exercise. Prove that

(a) twisting by $1 +$  adds $2([A, B + AC - BC)$ to the degree 2 term of an associator.

(b) twisting by $1 +$  does not change the degree 3 term of an associator.

Twisting and the previous theorem were discovered by V. Drinfeld [**Dr1**] in the context of quasi-triangular quasi-Hopf algebras. Then it was adapted for algebras of tangled chord diagrams in [**LM2**]

Proof. ......Sketch of the proof to be written by S.D.......... □

10.4.12. We can use an arbitrary associator Φ to define a combinatorial Kontsevich integral $Z_\Phi(K)$ corresponding to Φ replacing Φ_{KZ} by Φ in the constructions of section **??**.

Theorem ([**LM2**]). *For any two associators Φ and $\widetilde{\Phi}$ the corresponding combinatorial integrals are equal: $Z_\Phi(K) = Z_{\widetilde{\Phi}}(K)$.*

The proof is based on the fact that for any parenthesized tangle T the integrals $Z_\Phi(T)$ and $Z_{\widetilde{\Phi}}(T)$ are conjugate in the sense that $Z_{\widetilde{\Phi}}(T) = \mathcal{F}_b \cdot Z_\Phi(T) \cdot \mathcal{F}_t^{-1}$, where the elements $\mathcal{F}_b$ and $\mathcal{F}_t$ depend only on the parenthesis structures on the bottom and top of the tangle T respectively. Since for a knot K, considered as a tangle, the bottom and top are empty, the integrals are equal.

10.4.13. Corollary ([LM2]). *For any knot K the coefficients of the Kontsevich integral $Z(K)$, expanded over an arbitrary basis consisting of chord diagrams, are rational.*

Indeed, V. Drinfeld [**Dr2**] (see also [**BN6**]) showed that there exists an associator $\Phi_{\mathbb{Q}}$ with rational coefficients. According to Theorems 10.2.4 and 10.4.12 $Z(K) = Z_{comb}(K) = Z_{\Phi_{\mathbb{Q}}}(K)$. The last combinatorial integral has rational coefficients.

Exercises

(1) Find the monodromy of the reduced KZ equation (p. 281) around the points 0, 1 and ∞.

(2) Using the action of the permutation group S_n on the configuration space $X = \mathbb{C}^n \setminus \mathcal{H}$ determine the algebra of values and the KZ equation on the quotient space X/S_n in such a way that the monodromy gives the Kontsevich integral of braids (not necessarily pure). Compute the result for $n = 2$ and compare it with Exercise 8.4.3.

(3) Prove that the associator Φ_{KZ} is group-like.

(4) Find $Z_{comb}(3_1)$ up to degree 4 using the paranthesized presentation of the trefoil knot given in Figure **??** (page **??**).

(5) Draw a diagrammatic expression for the combinatorial Kontsevich integral of the knot 4_1 corresponding to the parenthesized presentation from **??**. Prove Theorem 10.2.4 for the parenthesized presentation of the knot 4_1 like it was done for the trefoil knot in Example **??**.

(6) Compute the Kontsevich integral of the knot 4_1 up to degree 4.

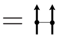
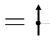
(7) Prove that the condition $\varepsilon_2(\Phi) = 1$ and the pentagon relation imply the other two equalities for strong invertibility: $\varepsilon_1(\Phi) = 1$ and $\varepsilon_3(\Phi) = 1$.

(8) Prove that Φ_{KZ} satisfies the hexagon relation.

(9) Prove the second hexagon relation

$$\Phi^{-1} \cdot (\Delta_1 R) \cdot \Phi^{-1} = (R \otimes \mathrm{id}) \cdot \Phi^{-1} \cdot (\mathrm{id} \otimes R) \ .$$

for an arbitrary associator Φ.

(10) Any associator Φ in the algebra of horizontal diagrams $\mathcal{A}^h(3)$ can be written as a power series in non commuting variables $a =$ , $b =$ , $c =$ : $\Phi = \Phi(a, b, c)$.

a). Check that Lemma 10.4.3 is equivalent to the identity $\Phi^{-1}(a, b, c) = \Phi(b, a, c)$. In particular, for an associator $\Phi(A, B)$ with values in $\mathbb{C}\langle\langle A, B\rangle\rangle$ (like Φ_{BN}, or Φ_{KZ}), we have $\Phi^{-1}(A, B) = \Phi(B, A)$.

b). Prove that the hexagon relation from page 305 can be written in the form $\Phi(a, b, c) \exp\left(\frac{b+c}{2}\right) \Phi(c, a, b) = \exp\left(\frac{b}{2}\right) \Phi(c, b, a) \exp\left(\frac{c}{2}\right)$.

c). (*V. Kurlin* [**Kur**]) Prove that for a horizontal associator the hexagon relation is equivalent to the relation

$$\Phi(a, b, c) e^{\frac{-a}{2}} \Phi(c, a, b) e^{\frac{-c}{2}} \Phi(b, c, a) e^{\frac{-b}{2}} = e^{\frac{-a-b-c}{2}} \ .$$

d). Show that for a horizontal associator Φ,

$$\Phi\Delta_2(R) \cdot \Phi\Delta_2(R) \cdot \Phi\Delta_2(R) = \exp(a + b + c) \ .$$

(11) Express $Z(H)$ through Φ_{KZ}. Is it true that the resulting power series contains only even degree terms?

Part 4

Other Topics

Chapter 11

The Kontsevich integral: advanced features

11.1. Relation with quantum invariants

The quantum knot invariant corresponding to the Lie algebra $\mathfrak{g}$ and its finite-dimensional representation ρ can be obtained from the Kontsevich integral as follows. Let $W_{\mathfrak{g}}^V(K)$ be this invariant evaluated on a knot K. It is a function of the parameter q. Let $\varphi_{\mathfrak{g}}^V$ be the weight system defined by the pair $(\mathfrak{g}, V)$ as in 6.1.4. Take the value of $\varphi_{\mathfrak{g}}^V$ on the n-th homogeneous term of the Kontsevich invariant $I(K)$, multiply by h_n and sum for n from 0 to ∞. The resulting function coincides with $W_{\mathfrak{g}}^V(K)$, where q is replaced by e^h. In other words the quantum invariant $W_{\mathfrak{g}}^V(K)$ is the *canonical* invariant in the sense of the next section.

11.2. Canonical Vassiliev invariants

The Fundamental Theorem 4.2.1 (more precisely, Theorem 8.8.1) and its framed version 9.6.2 provide a means to recover a Vassiliev invariant of order $\leqslant n$ from its symbol up to invariants of smaller order. It is natural to consider those remarkable Vassiliev invariants whose recovery gives precisely the original invariant.

11.2.1. Definition. ([**BNG**]) A (framed) Vassiliev invariant v of order $\leqslant n$ is called *canonical* if for every (framed) knot K,

$$v(K) = \mathrm{symb}(v)(I(K))\,.$$

In the case of framed invariants one should write $I^{fr}(K)$ instead of $I(K)$. (Recall that I denotes the final Kontsevich integral).

Suppose we have a Vassilev invariant f_n of order $\leqslant n$ for every n. Then we can construct a formal power series invariant $f = \sum_{n=0}^{\infty} f_n h^n$ in a formal parameter h. Let $w_n = \mathrm{symb}(f_n)$ be their symbols. Then the series $w = \sum_{n=0}^{\infty} w_n \in \overline{\mathcal{W}}$ (see page 111) can be understood as the symbol of the whole series $\mathrm{symb}(f)$.

A series f is called *canonical* if

$$f(K) = \sum_{n=0}^{\infty} \big(w_n(I(K))\big) h^n\,,$$

for every knot K. And again in the framed case one should use $I^{fr}(K)$.

Canonical invariants define a grading in the filtered space of Vassiliev invariants which is consistent with the filtration.

Example. The trivial invariant of order 0 which is identically equal to 1 on all knots is a canonical invariant. Its weight system is equal to $\mathbf{I}_0$ in the notation of Section 4.5.

Example. The Casson invariant c_2 is canonical. This follows from the explicit formula 3.6.7 that defines it in terms of the knot diagram.

11.2.2. Exercise. Prove that the invariant j_3 (see 3.6.1) is canonical.

Surprisingly many of the classical knot invariants discussed in Chapters 2 and 3 turn out to be canonical.

The notion of a canonical invariant allows one to reduce various relations between Vassiliev knot invariants to some combinatorial relations between their symbols, which gives a powerful tool to investigate knot invariants. This approach will be used in section 14.1 to prove the Melvin–Morton conjecture. Now we will give examples of canonical invariants following [**BNG**].

11.2.3. Quantum invariants. Building on the work of Drinfeld [**Dr1, Dr2**] and Kohno [**Koh2**], T. Le and J. Murakami [**LM3**, Th 10], and C. Kassel [**Kas**, Th XX.8.3] (see also [**Oht1**, Th 6.14]) proved that the quantum knot invariants $\theta^{fr}(K)$ and $\theta(K)$ introduced in Section 2.6 become canonical series after substitution $q = e^h$ and expansion into a power series in h.

The initial data for these invariants is a semi-simple Lie algebra $\mathfrak{g}$ and its finite dimensional irreducible representation V_λ, where λ is its highest weight. To emphasize this data, we shall write $\theta_{\mathfrak{g}}^{V_\lambda}(K)$ for $\theta(K)$ and $\theta_{\mathfrak{g}}^{fr,V_\lambda}(K)$ for $\theta^{fr}(K)$.

The quadratic Casimir element c (see Section 6.1) acts on V_λ as multiplication by a constant, call it c_λ. The relation between the framed and unframed quantum invariants is

$$\theta_{\mathfrak{g}}^{fr,V_\lambda}(K) = q^{\frac{c_\lambda \cdot w(K)}{2}} \theta_{\mathfrak{g}}^{V_\lambda}(K)\,,$$

where $w(K)$ is the writhe of K.

Set $q = e^h$. Write $\theta_{\mathfrak{g}}^{fr,V_\lambda}$ and $\theta_{\mathfrak{g}}^{V_\lambda}$ as power series in h:

$$\theta_{\mathfrak{g}}^{fr,V_\lambda} = \sum_{n=0}^{\infty} \theta_{\mathfrak{g},n}^{fr,\lambda} h^n \qquad \theta_{\mathfrak{g}}^{V_\lambda} = \sum_{n=0}^{\infty} \theta_{\mathfrak{g},n}^{\lambda} h^n.$$

According to the Birman–Lin theorem (3.6.6), the coefficients $\theta_{\mathfrak{g},n}^{fr,\lambda}$ and $\theta_{\mathfrak{g},n}^{\lambda}$ are Vassiliev invariants of order n. The Le–Murakami—Kassel Theorem states that they both are canonical series.

It is important that the symbol of $\theta_{\mathfrak{g}}^{fr,V_\lambda}$ is precisely the weight system $\varphi_{\mathfrak{g}}^{V_\lambda}$ described in Chapter 6. The symbol of $\theta_{\mathfrak{g}}^{V_\lambda}$ equals $\varphi_{\mathfrak{g}}^{\prime V_\lambda}$. In other words, it is obtained from the previous symbol $\varphi_{\mathfrak{g}}^{V_\lambda}$ by the standard deframing procedure of Sec. 4.5.6. Hence the knowledge the Kontsevich integral and these weight systems allows us to restore the quantum invariants $\theta_{\mathfrak{g}}^{fr,V_\lambda}$ and $\theta_{\mathfrak{g}}^{V_\lambda}$ without the quantum procedure of Sec. 2.6.

11.2.4. Colored Jones polynomial. The *colored Jones polynomials* $J^k := \theta_{\mathfrak{sl}_2}^{V_\lambda}$ and $J^{fr,k} := \theta_{\mathfrak{sl}_2}^{fr,V_\lambda}$ are particular cases of quantum invariants for $\mathfrak{g} = \mathfrak{sl}_2$. For this Lie algebra, the highest weight is an integer $\lambda = k-1$, where k is the dimension of the representation, so in our notation we may use k instead of λ. The quadratic Casimir number in this case is $c_\lambda = \frac{k^2-1}{2}$, and the relation between the framed and unframed colored Jones polynomials is

$$J^{fr,k}(K) = q^{\frac{k^2-1}{4} \cdot w(K)} J^k(K)\,.$$

The ordinary Jones polynomial of Section 2.4 corresponds to the case $k = 2$, i.e., to the standard 2-dimensional representation of the Lie algebra $\mathfrak{sl}_2$.

Set $q = e^h$. Write $J^{fr,k}$ and J^k as power series in h:

$$J^{fr,k} = \sum_{n=0}^{\infty} J_n^{fr,k} h^n \qquad J^k = \sum_{n=0}^{\infty} J_n^k h^n.$$

Both series are canonical with the symbols

$$\mathrm{symb}(J^{fr,k}) = \varphi_{\mathfrak{sl}_2}^{V_k}\,, \qquad \mathrm{symb}(J^k) = \varphi_{\mathfrak{sl}_2}^{\prime V_k}$$

defined in Sections 6.1.3 and 6.2.3.

11.2.5. Alexander–Conway polynomial. Consider the unframed quantum invariant $\theta^{St}_{\mathfrak{sl}_N}$ as a function of the parameter N. Let us think about N not as a discrete parameter but rather as a continuous variable, where for non integer N the invariant $\theta^{St}_{\mathfrak{sl}_N}$ is defined by the skein and initial relations above. Its symbol $\varphi'^{St}_{\mathfrak{sl}_N} = \varphi'^{St}_{\mathfrak{gl}_N}$ (see problem (14) of the Chapter 6) also makes sense for all real values of N, because for every chord diagram D, $\varphi'^{St}_{\mathfrak{gl}_N}(D)$ is a polynomial of N. Even more, since this polynomial is divisible by N, we may consider the limit

$$\lim_{N\to 0} \frac{\varphi'^{St}_{\mathfrak{sl}_N}}{N}\,.$$

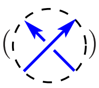
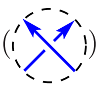
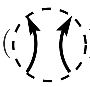
Exercise. Prove that the weight system defined by this limit coincides with the symbol of the Conway polynomial, $\mathrm{symb}(C) = \sum\limits_{n=0}^{\infty} \mathrm{symb}(c_n)$.

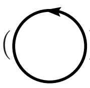
Hint. Use exercise (16) from Chapter 3.

Make the substitution $\theta^{St}_{\mathfrak{sl}_N}\big|_{q=e^h}$. The skein and initial relations for $\theta^{St}_{\mathfrak{sl}_N}$ allow us to show (see exercise (3)) that the limit

$$A := \lim_{N\to 0} \frac{\theta^{St}_{\mathfrak{sl}_N}\big|_{q=e^h}}{N}$$

does exist and satisfies the relations

(11.2.5.1) $A($$) - A($$) = (e^{h/2} - e^{-h/2})A($$)$;

(11.2.5.2) $A($$) = \dfrac{h}{e^{h/2} - e^{-h/2}}\,.$

A comparison of these relations with the defining relation for the Conway polynomial 2.3.1 shows that

$$A = \frac{h}{e^{h/2} - e^{-h/2}} C\big|_{t=e^{h/2}-e^{-h/2}}\,.$$

Despite of the fact that the Conway polynomial C itself is not a canonical series it becomes canonical after the substitution $t = e^{h/2} - e^{-h/2}$ and multiplication by $\frac{h}{e^{h/2}-e^{-h/2}}$. The weight system of this canonical series is the same as for the Conway polynomial. Or, in other words,

$$\frac{h}{e^{h/2} - e^{-h/2}} C\big|_{t=e^{h/2}-e^{-h/2}}(K) = \sum_{n=0}^{\infty} \big(\mathrm{symb}(c_n) \circ I(K)\big) h^n\,.$$

Warning. We cannot do the same for framed invariants because none of the limits

$$\lim_{N\to 0} \frac{\theta^{fr,St}_{\mathfrak{sl}_N}\big|_{q=e^h}}{N}, \qquad \lim_{N\to 0} \frac{\varphi^{St}_{\mathfrak{sl}_N}}{N}$$

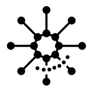
exists.

11.3. Wheeling

We mentioned in Section 5.8 of Chapter 5 that the relation between the algebras $\mathcal{B}$ and $\mathcal{C}$ is similar to the relation between the invariants in the symmetric algebra of a Lie algebra and the centre of its universal enveloping algebra. One may then expect that there exists an algebra isomorphism between $\mathcal{B}$ and $\mathcal{C}$ similar to the *Duflo-Kirillov isomorphism* for Lie algebras (see ???).

This isomorphism indeed exists. It is called *wheeling* and we describe it in this section. It will be used in the next section to deduce an explicit *wheels* formula for the Kontsevich integral of the unknot.

11.3.1. The Wheeling map. The *wheel* w_n in the algebra $\mathcal{B}$ is the diagram

$w_n =$ 

n spokes

The wheels w_n with n odd are equal to zero; this follows directly from Lemma 5.6.3.

Define the *wheels element* Ω as the formal power series

$$\Omega = \exp \sum_{n=1}^{\infty} b_{2n} w_{2n}$$

where b_{2n} are the *modified Bernoulli numbers*, and the products are understood to be in the algebra $\mathcal{B}$.

The modified Bernoulli numbers b_{2n} are the coefficients at x^{2n} in the Taylor expansion of the function

$$\frac{1}{2} \ln \frac{\sinh x/2}{x/2}.$$

We have $b_2 = 1/48$, $b_4 = -1/5760$ and $b_6 = 1/362880$. In general,

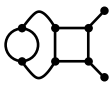
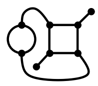
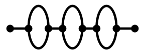
$$b_{2n} = \frac{B_{2n}}{4n \cdot (2n)!},$$

where B_{2n} are the usual Bernoulli numbers.

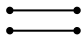
For an open diagram C with n legs, let us now define the *diagrammatic differential operator*

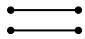
$$\partial_C : \mathcal{B} \to \mathcal{B}.$$

Take an open diagram D. If D has at most n legs, set $\partial_C D = 0$. If D has more than n legs, we define $\partial_C(D) \in \mathcal{B}$ as the sum of all those ways of glueing all the legs of C to some legs of D that produce diagrams having at least one leg on each connected component. For example,

$\partial_{w_2}(w_4) = 8$  $+\ 4$  $=\ 10$  .

Also,

$\partial_{w_2}($  $)\ =\ 8$  ,

since the other four ways of glueing w_2 into  produce diagrams one of whose components has no legs (see page 325).

Extending the definition by linearity, we can replace the diagram C in the definition of ∂_C by any linear combination of diagrams.

The operators ∂_C can be expressed via the disconnected cabling operation and the bilinear pairing introduced in 5.10.2. If $D \in \mathcal{B}(\boldsymbol{y})$, the labels $\boldsymbol{y}_1, \boldsymbol{y}_2$ are obtained by doubling $\boldsymbol{y}$ and the diagram C is considered as an element of $\mathcal{B}(\boldsymbol{y}_1)$, we have

$$\partial_C(D) = \langle C, \Delta^{(2)}(D)\rangle_{\boldsymbol{y}_1}.$$

Given a diagram $D \in \mathcal{B}$, the element $\partial_{w_{2n}} D$, if non-zero, has the same degree as D. Since for any D there is only a finite number of operators $\partial_{w_{2n}} D$ that do not annihilate D, the operator

$$\partial_\Omega = \exp \sum_{n=1}^{\infty} b_{2n} \partial_{w_{2n}}$$

gives a well-defined linear map $\mathcal{B} \to \mathcal{B}$, called the *wheeling* map. The wheeling map is, clearly, an isomorphism since $\partial_{\Omega^{-1}}$ is an inverse for it.

11.3.2. Theorem (Wheeling Theorem)**.** *The map $\chi \circ \partial_\Omega : \mathcal{B} \to \mathcal{C}$ is an algebra isomorphism.*

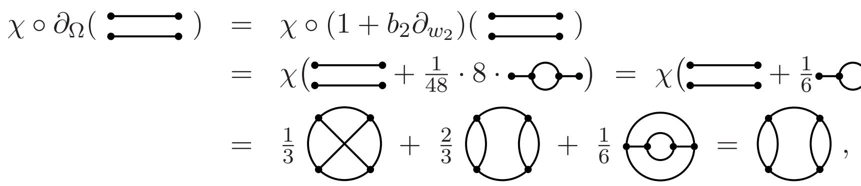
The map $\chi \circ \partial_\Omega$ plays the role of the Duflo-Kirillov isomorphism for the algebras of diagrams. In particular, for any metrized Lie algebra $\mathfrak{g}$ the diagram

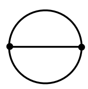
$$\begin{tikzcd} \mathcal{B} \arrow[r, "\psi_{\mathfrak{g}}"] \arrow[d, "\partial_\Omega"'] & S(\mathfrak{g}) \arrow[d, "\partial_{j^{\frac{1}{2}}}"] \\ \mathcal{B} \arrow[r, "\psi_{\mathfrak{g}}"] & S(\mathfrak{g}) \end{tikzcd}$$

commutes. We shall see this after calculating the Kontsevich integral of the unknot in the next section.

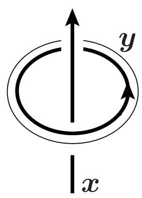
There are several approaches to the proof of the above theorem. It has been noted by Kontsevich [**Kon3**] that the Duflo-Kirillov isomorphism holds for a Lie algebra in any rigid tensor category; Hinich and Vaintrob showed in [**HV**] that the wheeling map can be interpreted as a particular case of such a situation. Here, we shall follow the proof of Bar-Natan, Le and Thurston [**BLT**]. In the next two sections we shall often write # for the product in $\mathcal{C}$ and $\cup$ for the product in $\mathcal{B}$.

11.3.3. Example. At the beginning of section 5.8 (page 151) we saw that χ is not compatible with the multiplication. Let us check the multiplicativity of $\chi \circ \partial_\Omega$ on the same example:



which is the square of the element $\chi \circ \partial_\Omega(\bullet\!\!-\!\!\!-\!\!\bullet) =$  in the algebra $\mathcal{C}$.

11.3.4. The Hopf link and the map Φ_0. Consider the framed Hopf link ⏀ with one interval component labelled $\boldsymbol{x}$, one closed component labelled $\boldsymbol{y}$, zero framing, and orientations as indicated:



The framed Kontsevich integral I^{fr}(⏀) lives in $\mathcal{C}(\boldsymbol{x}, \boldsymbol{y})$ or, via the isomorphism

$$\chi_{\boldsymbol{y}}^{-1} : \mathcal{C}(\boldsymbol{x}, \boldsymbol{y}) \to \mathcal{C}(\boldsymbol{x} \,|\, \boldsymbol{y}),$$

in $\mathcal{C}(\boldsymbol{x} \,|\, \boldsymbol{y})$.

Let us write $\mathcal{Z}$(⏀) for $\chi_{\boldsymbol{y}}^{-1} I^{fr}$(⏀). For a diagram $D \in \mathcal{B}(\boldsymbol{y})$, the pairing $\langle \mathcal{Z}$(⏀)$, D\rangle_{\boldsymbol{y}}$ lives in $\mathcal{C}(\boldsymbol{x})$. Identifying $\mathcal{B}(\boldsymbol{y})$ with $\mathcal{B}$ and $\mathcal{C}(\boldsymbol{x})$ with $\mathcal{C}$, we have a map

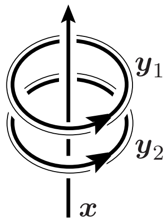
$$\Phi : \mathcal{B} \to \mathcal{C}$$

defined by

$$D \to \langle \mathcal{Z}(⏀), D\rangle_{\boldsymbol{y}}.$$

Lemma. *The map $\Phi : \mathcal{B} \to \mathcal{C}$ is a homomorphism of algebras.*

Proof. Taking the disconnected cabling of the Hopf link ϕ along the component $\boldsymbol{y}$, we obtain a link $\phi\!\!\phi$ with one interval component labelled $\boldsymbol{x}$ and two closed parallel components labelled $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$:



In the same spirit as Φ, we define the map

$$\Phi_2 : \mathcal{B} \otimes \mathcal{B} \to \mathcal{C}$$

using the link $\phi\!\!\phi$ instead of ϕ. Namely, given two diagrams, $D_1 \in \mathcal{B}(\boldsymbol{y}_1)$ and $D_2 \in \mathcal{B}(\boldsymbol{y}_2)$ we have $D_1 \otimes D_2 \in \mathcal{B}(\boldsymbol{y}_1, \boldsymbol{y}_2)$. The Kontsevich integral $I^{fr}(\phi\!\!\phi)$ via the map $\chi^{-1}_{\boldsymbol{y}_1,\boldsymbol{y}_2}$ can be pulled to $\mathcal{C}(\boldsymbol{x} \,|\, \boldsymbol{y}_1, \boldsymbol{y}_2)$; we shall write $\mathcal{Z}(\phi\!\!\phi)$ for $\chi^{-1}_{\boldsymbol{y}_1,\boldsymbol{y}_2}(I^{fr}(\phi\!\!\phi))$. Identify $\mathcal{B}(\boldsymbol{y}_1, \boldsymbol{y}_2)$ with $\mathcal{B} \otimes \mathcal{B}$ and $\mathcal{C}(\boldsymbol{x})$ with $\mathcal{C}$, and define

$$\Phi_2 : \mathcal{B} \otimes \mathcal{B} \to \mathcal{C}$$

as

$$D_1 \otimes D_2 \to \langle \mathcal{Z}(\phi\!\!\phi), D_1 \otimes D_2 \rangle_{\boldsymbol{y}_1,\boldsymbol{y}_2}.$$

The map Φ_2 glues the legs of the diagram D_1 to the $\boldsymbol{y}_1$ legs of $\mathcal{Z}(\phi\!\!\phi)$, and the legs of D_2 — to the $\boldsymbol{y}_2$ legs of $\mathcal{Z}(\phi\!\!\phi)$.

There are two ways of expressing $\Phi_2(D_1 \otimes D_2)$ in terms of $\Phi(D_i)$. First, we can use the fact that $\phi\!\!\phi$ is a product (as tangles) of two copies of the Hopf link ϕ. Since the legs of D_1 and D_2 are glued independently to the legs corresponding to $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, it follows that

$$\Phi_2(D_1 \otimes D_2) = \Phi(D_1) \# \Phi(D_2).$$

On the other hand, we can apply the formula (???) that relates the disjoint union multiplication with disconnected cabling. We have

$$\begin{aligned}
\Phi_2(D_1 \otimes D_2) &= \langle \Delta^{(2)}_{\boldsymbol{y}}(\mathcal{Z}(\phi)), D_1 \otimes D_2 \rangle_{\boldsymbol{y}_1,\boldsymbol{y}_2} \\
&= \langle \mathcal{Z}(\phi), D_1 \cup D_2 \rangle_{\boldsymbol{y}} \\
&= \Phi(D_1 \cup D_2),
\end{aligned}$$

and, therefore,

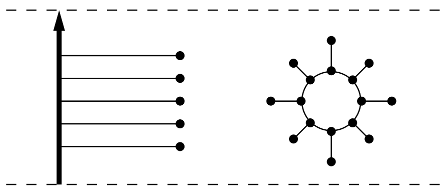
$$\Phi(D_1) \# \Phi(D_2) = \Phi(D_1 \cup D_2).$$

□

Given a diagram $D \in \mathcal{C}(\boldsymbol{x} \,|\, \boldsymbol{y})$, the map $\mathcal{B} \to \mathcal{C}$ given by sending $C \in \mathcal{B}(\boldsymbol{y})$ to $\langle D, C\rangle_{\boldsymbol{y}} \in \mathcal{C}(\boldsymbol{x})$ shifts the degree of C by the amount equal to the degree of D minus the number of $\boldsymbol{y}$ legs of D. If D appears in $\mathcal{Z}(\phi)$ with a non-zero coefficient, this difference is non-negative. Indeed, in $\mathcal{Z}(\phi)$ there are no intervals both of whose ends are labelled with $\boldsymbol{y}$, since the $\boldsymbol{y}$ component comes with zero framing. Also, if two $\boldsymbol{y}$ legs are attached to the same internal vertex, the diagram is zero, because of the antisymmetry relation, and therefore, the number of inner vertices of D is at least as big as the number of $\boldsymbol{y}$ legs.

It follows that the Kontsevich integral $\mathcal{Z}(\phi)$ can be written as $\mathcal{Z}_0(\phi) + \mathcal{Z}_1(\phi) + \ldots$, where $\mathcal{Z}_i(\phi)$ is the part consisting of diagrams whose degree exceeds the number of $\boldsymbol{y}$ legs by i. We shall be interested in the term $\mathcal{Z}_0(\phi)$ of this sum.

Each diagram that appears in this term is a union of a *"comb"* with some wheels:



Indeed, each vertex of such diagram is either a $\boldsymbol{y}$ leg, or is adjacent to exactly one $\boldsymbol{y}$ leg.

Denote a "comb" with n teeth by u^n. Strictly speaking, u^n is not really a product of n copies of u since $\mathcal{C}(\boldsymbol{x} \,|\, \boldsymbol{y})$ is not an algebra. However, we can introduce a Hopf algebra structure in the space of all diagrams in $\mathcal{C}(\boldsymbol{x} \,|\, \boldsymbol{y})$ that consist of combs and wheels. The product of two diagrams is the disjoint union of all components followed by the concatenation of the combs; in particular $u^k u^m = u^{k+m}$. The coproduct is the same as in $\mathcal{C}(\boldsymbol{x} \,|\, \boldsymbol{y})$. This Hopf algebra is nothing else but the free commutative Hopf algebra on a countable number of generators.

The Kontsevich integral is group-like, and this implies that

$$\delta(\mathcal{Z}_0(\phi)) = \mathcal{Z}_0(\phi) \otimes \mathcal{Z}_0(\phi).$$

Group-like elements in the completion of the free commutative Hopf algebra are the exponentials of linear combinations of generators and, therefore

$$\mathcal{Z}_0(\phi) = \exp(cu \cup \sum_n a_{2n} w_{2n}),$$

where c and a_{2n} are some constants.

In fact, the constant c is precisely the linking number of the components $\boldsymbol{x}$ and $\boldsymbol{y}$, and, hence is equal to 1. We can write

$$\mathcal{Z}_0(\phi) = \sum_n \frac{u^n}{n!} \cup \Omega',$$

where Ω' the part of $\mathcal{Z}_0(\phi)$ containing wheels:

$$\Omega' = \exp \sum_n a_{2n} w_{2n}.$$

Define the map $\Phi_0 : \mathcal{B} \to \mathcal{C}$ by taking the pairing of a diagram in $\mathcal{B}(\boldsymbol{y})$ with $\mathcal{Z}_0(\phi)$:

$$D \to \langle \mathcal{Z}_0(\phi), D \rangle_{\boldsymbol{y}}.$$

The map Φ_0 can be thought of as the part of Φ that shifts the degree by zero. The map Φ_0, like Φ, is multiplicative. In fact, we shall see later that $\Phi_0 = \Phi$.

11.3.5. Lemma. $\Phi_0 = \chi \circ \partial_{\Omega'}$.

Proof. Let us notice first that if $C \in \mathcal{C}(\boldsymbol{x} \,|\, \boldsymbol{y})$ and $D \in \mathcal{B}(\boldsymbol{y})$, we have

$$\langle C \cup w_{2n}, D \rangle_{\boldsymbol{y}} = \langle C, \partial_{w_{2n}}(D) \rangle_{\boldsymbol{y}}.$$

Also, for any $D \in \mathcal{B}$ the expression

$$\langle \sum_n \frac{u^n}{n!}, D \rangle_{\boldsymbol{y}}$$

is precisely the average of all possible ways of attaching the legs of D to the line $\boldsymbol{x}$.

Therefore, for $D \in \mathcal{B}(\boldsymbol{y})$

$$\Phi_0(D) = \langle \sum_n \frac{u^n}{n!} \cup \Omega',\, D \,\rangle_{\boldsymbol{y}} = \langle \sum_n \frac{u^n}{n!},\, \partial_{\Omega'} D \,\rangle_{\boldsymbol{y}} = \chi \circ \partial_{\Omega'} D.$$

□

11.3.6. The algebra $\mathcal{B}^\circ$. For the calculations that follow it will be convenient to enlarge the algebra $\mathcal{B}$.

A diagram in the enlarged algebra $\mathcal{B}^\circ$ is a union of a unitrivalent graph with a finite number of circles with no vertices on them; a cyclic order of half-edges at every trivalent vertex is given. The algebra $\mathcal{B}^\circ$ is spanned by all such diagrams modulo IHX and antisymmetry relations. The multiplication in $\mathcal{B}^\circ$ is the disjoint union. The algebra $\mathcal{B}$ is the subalgebra of $\mathcal{B}^\circ$ spanned by graphs which have at least one univalent vertex in each connected component. Killing all diagrams which have components with no legs, we get a homomorphism $\mathcal{B}^\circ \to \mathcal{B}$, which restricts to the identity map on $\mathcal{B} \subset \mathcal{B}^\circ$.

The algebra of 3-graphs Γ from Chapter 7 is also a subspace of $\mathcal{B}^\circ$. In fact, the algebra $\mathcal{B}^\circ$ is the tensor product of $\mathcal{B}$, the symmetric algebra $\mathrm{Sym}(\Gamma)$ of the vector space Γ and the polynomial algebra in one variable (which counts the circles with no vertices on them).

The advantage of considering $\mathcal{B}^\circ$ instead of $\mathcal{B}$ is the existence of a bilinear symmetric pairing $\mathcal{B}^\circ \otimes \mathcal{B}^\circ \to \mathcal{B}^\circ$. For two diagrams $C, D \in \mathcal{B}^\circ$ with the same number of legs we define $\mathcal{B}^\circ$ to be the sum of all ways of glueing all legs of C to those of D. If the numbers of legs of C and D do not coincide we set $\langle C, D\rangle = 0$.

This definition is very similar to the definition of the pairing

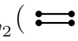
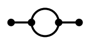
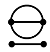
$$\mathcal{C}(\boldsymbol{x} \,|\, \boldsymbol{y}) \otimes \mathcal{B}(\boldsymbol{y}) \to \mathcal{C}(\boldsymbol{x}).$$

The essential difference between the two lies in the fact that diagrams in $\mathcal{B}^\circ$ can have components with no legs; in fact, the whole image of the pairing on $\mathcal{B}^\circ$ lies in the subspace spanned by legless diagrams.

For an open diagram C the *diagrammatic differential operator*

$$\partial^\circ_C : \mathcal{B}^\circ \to \mathcal{B}^\circ$$

is defined as follows. If $D \in \mathcal{B}^\circ$ has less legs than C, we set $\partial^\circ_C(D) = 0$. If D has at least as many legs as C, we define $\partial^\circ_C(D)$ as the sum of all ways of glueing the legs of C to those of D. For example,

$\partial^\circ_{w_2}($  $) \;=\; 8$  $+\; 4$  .

This definition of diagrammatic operators is consistent with the definition of $\partial_{w_{2n}} : \mathcal{B} \to \mathcal{B}$. Namely, we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\partial_{w_{2n}}} & \mathcal{B} \\ \uparrow & & \uparrow \\ \mathcal{B}^\circ & \xrightarrow{\partial^\circ_{w_{2n}}} & \mathcal{B}^\circ \end{array}$$

11.3.7. The coefficients of the wheels in Φ_0. First, let us make the following observation:

Lemma. *Let $\boldsymbol{y}$ be a circle and denote by $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ the components of the disconnected cabling $\Delta^{(2)}(\boldsymbol{y})$. Identify $\mathcal{B}$ with $\mathcal{B}(\boldsymbol{y})$. Then*

$$\Delta^{(2)}_{\boldsymbol{y}} \Omega' = \Omega' \otimes \Omega' \in \mathcal{B} \otimes \mathcal{B} = \mathcal{B}(\boldsymbol{y}_1, \boldsymbol{y}_2).$$

Proof. We again use the fact that the sum of the Hopf link ϕ with itself coincides with its two-fold disconnected cabling along the closed component $\boldsymbol{y}$. Since the Kontsevich integral is multiplicative, we see that

$$\Delta^{(2)}_{\boldsymbol{y}} \mathcal{Z}_0(\phi) = \mathcal{Z}_0(\phi) \otimes \mathcal{Z}_0(\phi),$$

where the right-hand side lives in the tensor product of graded completions of $\mathcal{C}(\boldsymbol{x}\,|\,\boldsymbol{y}_1)$ and $\mathcal{C}(\boldsymbol{x}\,|\,\boldsymbol{y}_2)$ respectively. Now, if we factor out on both sides the diagrams that have at least one vertex on the $\boldsymbol{x}$ component, we obtain the statement of the lemma. $\square$

Lemma. *For any* $D \in \mathcal{B}$

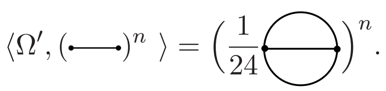
$$\partial^\circ_D(\Omega') = \langle D, \Omega' \rangle \Omega'.$$

Proof. By identifying $\mathcal{B}$ with $\mathcal{B}(\boldsymbol{y}_i)$ we obtain formal power series $\Omega_{\boldsymbol{y}_i}$ and diagrams $D_{\boldsymbol{y}_i}$ for $i = 1, 2$. It is then clear from the definitions and from the preceding lemma that

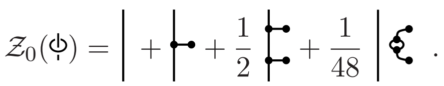
$$\partial^\circ_D(\Omega') = \langle D_{\boldsymbol{y}_1}, \Delta^{(2)}_{\boldsymbol{y}} \Omega' \rangle = \langle D_{\boldsymbol{y}_1}, \Omega'_{\boldsymbol{y}_1} \Omega'_{\boldsymbol{y}_2} \rangle = \langle D, \Omega' \rangle \Omega'.$$

$\square$

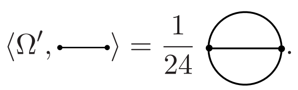
Lemma. *The following holds in* $\mathcal{B}^\circ$*:*



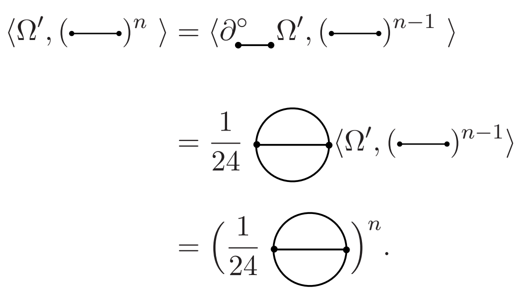
Proof. According to the problem 14 at the end of this chapter (page 337), the Kontsevich integral of the Hopf link up to degree two is equal to



It follows that the coefficient a_2 in Ω' is equal to $1/48$ and that



This establishes the lemma for $n = 1$. Now, use induction:



The first equality follows from the obvious identity valid for arbitrary $A, B, C \in \mathcal{B}^\circ$:

$$\langle C, A \cup B \rangle = \langle \partial^\circ_B(C), A \rangle.$$

The second and the third equalities follow from the lemmas above. $\square$

In order to establish that $\Omega' = \Omega$ we have to show that the coefficients a_{2n} in the expression $\Omega' = \exp \sum_n a_{2n} w_{2n}$ are equal to the modified Bernoulli numbers b_{2n}. In other words, we have to prove that

(11.3.7.1)
$$\sum_n a_{2n} x^{2n} = \frac{1}{2} \ln \frac{\sinh x/2}{x/2}, \quad \text{or} \quad \exp\Big(2\sum_n a_{2n} x^{2n}\Big) = \frac{\sinh x/2}{x/2}.$$

To do this we compute the value of the $\mathfrak{sl}_2$-weight system $\eta_{\mathfrak{sl}_2}(\cdot)$ from section 7.6 (p.212) on the 3-graph $\langle \Omega', (\bullet\!\!-\!\!\bullet)^n \rangle \in \Gamma$ in two ways.

Using the last lemma and Theorem 6.2.3 on page 181 we have

$$\eta_{\mathfrak{sl}_2}\Big(\langle \Omega', (\bullet\!\!-\!\!\bullet)^n \rangle\Big) = \eta_{\mathfrak{sl}_2}\left(\Big(\frac{1}{24}\ominus\Big)^n\right) = \frac{1}{2^n} \ .$$

From the other hand, according to the exercise 25 from Chapter 6, page 193, the $\mathfrak{sl}_2$ tensor corresponding to a wheel w_{2n} is equal to $2^{n+1}\ (\bullet\!\!-\!\!\bullet)^n$. Therefore,

$$\eta_{\mathfrak{sl}_2}\Big(\langle \Omega', (\bullet\!\!-\!\!\bullet)^n \rangle\Big) = \eta_{\mathfrak{sl}_2}\Big(\langle \exp \sum_m a_{2m} 2^{m+1}\ (\bullet\!\!-\!\!\bullet)^m, (\bullet\!\!-\!\!\bullet)^n \rangle\Big) \ .$$

Denote by f_n the coefficient at z^n of the power series expansion of the function $\exp\big(2\sum_n a_{2n} z^n\big) = \sum_n f_n z^n$. We get

$$\eta_{\mathfrak{sl}_2}\Big(\langle \Omega', (\bullet\!\!-\!\!\bullet)^n \rangle\Big) = 2^n f_n\ \eta_{\mathfrak{sl}_2}\Big(\langle (\bullet\!\!-\!\!\bullet)^n, (\bullet\!\!-\!\!\bullet)^n \rangle\Big)$$

Now using the exercise 17 on page 337 and the fact that the value $\eta_{\mathfrak{sl}_2}(\bigcirc)$ of the $\mathfrak{sl}_2$-weight system $\eta_{\mathfrak{sl}_2}(\cdot)$ on a circle without vertices is equal to 3 (page 212) we obtain

$$\begin{aligned} \eta_{\mathfrak{sl}_2}\Big(\langle (\bullet\!\!-\!\!\bullet)^n, (\bullet\!\!-\!\!\bullet)^n \rangle\Big) &= (2n+1)(2n)\ \eta_{\mathfrak{sl}_2}\Big(\langle (\bullet\!\!-\!\!\bullet)^{n-1}, (\bullet\!\!-\!\!\bullet)^{n=1} \rangle\Big) = \dots \\ &= (2n+1)! \ . \end{aligned}$$

Comparing these two calculations we find that $f_n = \frac{1}{4^n(2n+1)!}$, which is the coefficient at z^n of the power series expansion of $\frac{\sinh \sqrt{z}/2}{\sqrt{z}/2}$. Hence

$$\exp\Big(2\sum_n a_{2n} z^n\Big) = \frac{\sinh \sqrt{z}/2}{\sqrt{z}/2} \ .$$

Substituting $z = x^2$ we get the equality (11.3.7.1) which establishes that $\Omega' = \Omega$ and completes the proof of the Wheeling Theorem 11.3.2.

11.3.8. Wheeling for tangle diagrams. A version of the Wheeling theorem exists for more general spaces of tangle diagrams. For our purposes it is sufficient to consider the spaces of diagrams for links with two closed components $\boldsymbol{x}$ and $\boldsymbol{y}$.

For $D \in \mathcal{B}$ define the operator

$$(\partial_D)_{\boldsymbol{x}} : \mathcal{B}(\boldsymbol{x}, \boldsymbol{y}) \to \mathcal{B}(\boldsymbol{x}, \boldsymbol{y})$$

as the sum of all possible ways of glueing all the legs of D to some of the $\boldsymbol{x}$-legs of a diagram in $\mathcal{B}(\boldsymbol{x}, \boldsymbol{y})$ that do not produce components without legs.

Exercise. Show that $(\partial_D)_{\boldsymbol{x}}$ respects the link relations, and, therefore, is well-defined.

Define the wheeling map $\Phi_{\boldsymbol{x}}$ as $\chi_{\boldsymbol{x}} \circ (\partial_\Omega)_{\boldsymbol{x}}$. (Strictly speaking, we should have called it $(\Phi_0)_{\boldsymbol{x}}$ since we have not yet proved that $\Phi = \Phi_0$.) The Wheeling theorem can now be generalized as follows:

Theorem. *The map*

$$\Phi_{\boldsymbol{x}} : \mathcal{B}(\boldsymbol{x}, \boldsymbol{y}) \to \mathcal{C}(\boldsymbol{x} \,|\, \boldsymbol{y})$$

identifies the $\mathcal{B}(\boldsymbol{x})$*-module* $\mathcal{B}(\boldsymbol{x}, \boldsymbol{y})$ *with the* $\mathcal{C}(\boldsymbol{x})$*-module* $\mathcal{C}(\boldsymbol{x} \,|\, \boldsymbol{y})$.

The proof is, essentially, the same as the proof of the Wheeling theorem, and we leave it to the reader.

11.4. The Kontsevich integral of the unknot and of the Hopf link

The arguments similar to those used in the proof of the wheeling theorem allow us to write down an explicit expression for the framed Kontsevich integral of the zero-framed unknot O. Let us denote by $\mathcal{Z}(O)$ the image $\chi^{-1} I^{fr}(O)$ of the Konstevich integral of O in the graded completion of $\mathcal{B}$. (Note that we use the notation $\mathcal{Z}(\phi)$ in a similar, but not exactly the same context.)

11.4.1. Theorem.

$$\mathcal{Z}(O) = \Omega = \exp \sum_{n=1}^{\infty} b_{2n} w_{2n}. \tag{11.4.1.1}$$

A very similar formula holds for the Kontsevich integral of the Hopf link ϕ:

11.4.2. Theorem.

$$\mathcal{Z}(\phi) = \sum_n \frac{u^n}{n!} \cup \Omega.$$

This formula implies that the maps Φ and Φ_0 of the previous section, in fact, coincide.

We start the proof with a lemma.

11.4.3. Lemma. *If $C_1, \dots, C_n$ are non-trivial elements of $\mathcal{C}$, then $\chi^{-1}(C_1 \# \dots \# C_n)$ is a combination of diagrams in $\mathcal{B}$ with at least n legs.*

Proof. Let us first introduce some notation. If $D \in \mathcal{B}$ is a diagram, we denote by $D_{\boldsymbol{y}}$ the result of decorating all the legs of D with the label $\boldsymbol{y}$. In other words, if $\mathcal{B}$ is identified with $\mathcal{B}(\boldsymbol{y})$, D is identified with $D_{\boldsymbol{y}}$.

Now recall from the proof of the wheeling theorem that $\Phi_0 = \chi \circ \partial_\Omega$. Let $D_i = \Phi_0^{-1}(C_i) \in \mathcal{B}$.

We have that

$$\partial_\Omega(D_1 \cup \ldots \cup D_n) = \langle \Omega_{\boldsymbol{y}_1}, \Delta^{(2)}_{\boldsymbol{y}}(D_1 \cup \ldots \cup D_n) \rangle.$$

Decompose $\Delta^{(2)}_{\boldsymbol{y}}(D_i)$ as a sum $A_i' + A_i''$ where A_i' consists of diagrams without legs labelled by $\boldsymbol{y}_1$ and A_i'' contains only diagrams with at least one leg labelled by $\boldsymbol{y}_1$.

Recall that in the completion of the algebra $\mathcal{B}^\circ$ we have $\partial^\circ_D(\Omega) = \langle D, \Omega \rangle \Omega$. By projecting this equality to $\mathcal{B}$ we see that $\partial_D(\Omega)$ vanishes unless D is empty. Hence,

$$\langle \Omega_{\boldsymbol{y}_1}, (D_1')_{\boldsymbol{y}_1} \Delta^{(2)}_{\boldsymbol{y}}(D_2 \cup \ldots \cup D_n) \rangle = \langle (\partial_{D_1}\Omega)_{\boldsymbol{y}_1}, \Delta^{(2)}_{\boldsymbol{y}}(D_2 \cup \ldots \cup D_n) \rangle = 0.$$

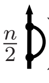
As a result we have

$$\begin{aligned}
\partial_\Omega(D_1 \cup \ldots \cup D_n) &= \langle \Omega_{\boldsymbol{y}_1}, A_1' \Delta^{(2)}_{\boldsymbol{y}}(D_2 \cup \ldots \cup D_n) \rangle \\
&\qquad + \langle \Omega_{\boldsymbol{y}_1}, A_1'' \Delta^{(2)}_{\boldsymbol{y}}(D_2 \cup \ldots \cup D_n) \rangle \\
&= \langle \Omega_{\boldsymbol{y}_1}, A_1'' \Delta^{(2)}_{\boldsymbol{y}}(D_2 \cup \ldots \cup D_n) \rangle \\
&= \langle \Omega_{\boldsymbol{y}_1}, A_1'' \ldots A_n'' \rangle.
\end{aligned}$$

Each of the A_i'' has at least one leg labelled $\boldsymbol{y}_2$, and these legs are preserved by taking the pairing with respect to the label $\boldsymbol{y}_1$. $\square$

11.4.4. The Kontsevich integral of the unknot. The calculation of the Kontsevich integral for the unknot is based on the following geometric fact: the nth (connected) cabling of the unknot is again an unknot.

The cabling formula (9.8) in this case reads

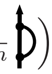
(11.4.4.1) $\psi^{(n)}\Big((I^{fr}(O) \# \exp\big(\tfrac{1}{2n}$  $\big)\Big) = I^{fr}(O) \# \exp\big(\tfrac{n}{2}$  $\big).$

In each degree, the right-hand side of this formula depends on n polynomially. The term of degree 0 in n is precisely the Kontsevich integral of the unknot $I^{fr}(O)$.

As a consequence, the left-hand side also contains only non-negative powers of n. We shall be specifically interested in the terms that are of degree 0 in n.

The operator $\psi^{(n)}$ has a particularly simple form in the algebra $\mathcal{B}$ (see Corollary 9.7.6 on page 265): it multiplies a diagram with k legs by n^k. Let us expand the argument of $\psi^{(n)}$ into a power series and convert it to $\mathcal{B}$ term by term.

It follows from Lemma 11.4.3 that if a diagram D is contained in

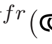
$\chi^{-1}\Big(I^{fr}(O)\#\Big(\frac{1}{2n}$  $\Big)^k\Big),$

then it has at least $k' \geqslant k$ legs. Applying $\psi^{(n)}$, we multiply D by $n^{k'}$, hence the coefficient of D depends on n as $n^{k'-k}$. We see that if the coefficient of D is of degree 0 in n, then the number of legs of D must be equal to the degree of D.

Thus we have proved that $\mathcal{Z}(O)$ contains only diagrams whose number of legs is equal to their degree. We have seen in 11.3.4 that the part of the Kontsevich integral of the Hopf link that consists of such diagrams has the form $\sum \frac{u^n}{n!} \cup \Omega$. Deleting from this expression the diagrams with legs attached to the interval component, we obtain Ω. On the other hand, this is the Kontsevich integral of the unknot $\mathcal{Z}(O)$.

11.4.5. The Kontsevich integral of the Hopf link. The Kontsevich integral of the Hopf link both of whose components are closed with zero framing is computed in [**BLT**]. Such Hopf link can be obtained from the zero-framed unknot in three steps: first, change the framing of the unknot from 0 to +1, then take the disconnected twofold cabling, and, finally, change the framings of the resulting components from +1 to 0. We know how the Kontsevich integral behaves under all these operations and this gives us the following theorem (see 11.3.8 for notation):

Theorem. *Let*  *be the Hopf link both of whose components are closed with zero framing and oriented counterclockwise. Then*

$I^{fr}($  $) = (\Phi_{\boldsymbol{x}} \circ \Phi_{\boldsymbol{y}})(\exp |^{\boldsymbol{x}}_{\boldsymbol{y}}),$

where $|^{\boldsymbol{x}}_{\boldsymbol{y}} \in \mathcal{B}(\boldsymbol{x}, \boldsymbol{y})$ *is an interval with one* $\boldsymbol{x}$ *leg and one* $\boldsymbol{y}$ *leg.*

We shall obtain Theorem 11.4.2 from the above statement.

Proof. Let O^{+1} be the unknot with $+1$ framing. Its Kontsevich integral is related to that of the zero-framed unknot by (???):

$$I^{fr}(O^{+1}) = I^{fr}(O) \# \exp\left(\tfrac{1}{2}\right).$$

Using the Wheeling theorem and the expression for $I^{fr}(O)$ we can re-write this expression as

$$\mathcal{Z}(O^{+1}) = \partial_\Omega\big(\partial_\Omega^{-1}(\Omega) \cup \exp \partial_\Omega^{-1}(\bullet\!\!-\!\!\!-\!\!\bullet)\big),$$

since $\partial_\Omega^{-1}(\bullet\!\!-\!\!\!-\!\!\bullet) = \bullet\!\!-\!\!\!-\!\!\bullet$.

Recall that in the proof of Lemma 11.4.3 we have seen that $\partial_D(\Omega) = 0$ unless D is empty. In particular, $\partial_\Omega^{-1}(\Omega) = \Omega$. We see that

$$\mathcal{Z}(O^{+1}) = \partial_\Omega\big(\Omega \cdot \exp\big(\tfrac{1}{2}\,\bullet\!\!-\!\!\!-\!\!\bullet\big)\big). \tag{11.4.5.1}$$

Our next goal is the following formula:

$$\partial_\Omega^{-2}(\mathcal{Z}(O^{+1})) = \exp\big(\tfrac{1}{2}\,\bullet\!\!-\!\!\!-\!\!\bullet\big)\ . \tag{11.4.5.2}$$

Applying ∂_Ω to both sides of this equation and using (11.4.5.1), we obtain an equivalent form of (11.4.5.2):

$$\partial_\Omega\big(\exp\big(\tfrac{1}{2}\,\bullet\!\!-\!\!\!-\!\!\bullet\big)\big) = \Omega \cdot \exp\big(\tfrac{1}{2}\,\bullet\!\!-\!\!\!-\!\!\bullet\big)\ .$$

To prove it, we observe that

$$\begin{aligned}\partial_\Omega\big(\exp\big(\tfrac{1}{2}\,\bullet\!\!-\!\!\!-\!\!\bullet\big)\big) &= \langle \Omega_{\boldsymbol{y}_1}, \Delta^{(2)}_{\boldsymbol{y}} \exp\big(\tfrac{1}{2}\,\bullet\!\!-\!\!\!-\!\!\bullet\big)\rangle_{\boldsymbol{y}_1} \\ &= \langle \Omega_{\boldsymbol{y}_1}, \exp\big(\tfrac{1}{2}\,\big|^{\boldsymbol{y}_1}_{\boldsymbol{y}_1}\big) \exp\big(\big|^{\boldsymbol{y}_2}_{\boldsymbol{y}_1}\big) \exp\big(\tfrac{1}{2}\,\big|^{\boldsymbol{y}_2}_{\boldsymbol{y}_2}\big)\rangle_{\boldsymbol{y}_1}.\end{aligned}$$

The pairing $\mathcal{B}(\boldsymbol{y}_1, \boldsymbol{y}_2) \otimes \mathcal{B}(\boldsymbol{y}_1) \to \mathcal{B}(\boldsymbol{y}_2)$ satisfies

$$\langle C, A \cup B\rangle_{\boldsymbol{y}_1} = \langle \partial_B(C), A\rangle_{\boldsymbol{y}_1}.$$

for all $A, B \in \mathcal{B}(\boldsymbol{y}_1)$, $C \in \mathcal{B}(\boldsymbol{y}_1, \boldsymbol{y}_2)$. Therefore, the last expression can be re-written as

$$\langle \partial_{\exp\left(\frac{1}{2}\left|^{\boldsymbol{y}_1}_{\boldsymbol{y}_1}\right.\right)} \Omega_{\boldsymbol{y}_1}, \exp\big(\big|^{\boldsymbol{y}_2}_{\boldsymbol{y}_1}\big)\rangle_{\boldsymbol{y}_1} \cdot \exp\big(\tfrac{1}{2}\,\big|^{\boldsymbol{y}_2}_{\boldsymbol{y}_2}\big)\ .$$

Taking into the account the fact that $\partial_D(\Omega) = 0$ unless D is empty, we see that this is the same thing as

$$\langle \Omega_{\boldsymbol{y}_1}, \exp\big(\big|^{\boldsymbol{y}_2}_{\boldsymbol{y}_1}\big)\rangle_{\boldsymbol{y}_1} \cdot \exp\big(\tfrac{1}{2}\,\big|^{\boldsymbol{y}_2}_{\boldsymbol{y}_2}\big) = \Omega \cdot \exp\big(\tfrac{1}{2}\,\bullet\!\!-\!\!\!-\!\!\bullet\big),$$

and this proves (11.4.5.2).

To proceed, we need the following simple observation:

Lemma.

$$\Delta^{(2)}_{\boldsymbol{y}}\partial_C(D) = (\partial_C)_{\boldsymbol{y}_1}(\Delta^{(2)}_{\boldsymbol{y}}(D)) = (\partial_C)_{\boldsymbol{y}_2}(\Delta^{(2)}_{\boldsymbol{y}}(D)).$$

Now, let $\bigcirc\!\!\!\bigcirc^{+1}$ be the Hopf link both of whose components are closed with $+1$ framing. The above lemma together with the disconnected cabling formula (???) implies that

$$\Delta^{(2)}\partial_\Omega^{-2}(\mathcal{Z}(O^{+1})) = (\partial_\Omega)^{-1}_{\boldsymbol{y}_1}(\partial_\Omega)^{-1}_{\boldsymbol{y}_2}\big(\chi^{-1}_{\boldsymbol{y}_1,\boldsymbol{y}_2}I^{fr}(\bigcirc\!\!\!\bigcirc^{+1})\big).$$

On the other hand, this, by (11.4.5.2) is equal to

$$\Delta^{(2)}\exp\big(\tfrac{1}{2}\bullet\!\!-\!\!\!-\!\!\bullet\big) = \exp(\big|^{\boldsymbol{y}_2}_{\boldsymbol{y}_1}) \cdot \exp\big(\tfrac{1}{2}\big|^{\boldsymbol{y}_1}_{\boldsymbol{y}_1}\big) \cdot \exp\big(\tfrac{1}{2}\big|^{\boldsymbol{y}_2}_{\boldsymbol{y}_2}\big)\ .$$

Applying $\Phi_{\boldsymbol{y}_1}\circ\Phi_{\boldsymbol{y}_2}$ to the first expression, we get exactly $I^{fr}(\bigcirc\!\!\!\bigcirc^{+1})$. On the second expression, this evaluates to

$$\Phi_{\boldsymbol{y}_1}(\Phi_{\boldsymbol{y}_2}(\exp(\big|^{\boldsymbol{y}_2}_{\boldsymbol{y}_1}))\#\exp_\#\Big(\tfrac{1}{2}\ {}_{\boldsymbol{y}_1}\Big)\#\exp_\#\Big(\tfrac{1}{2}\ {}_{\boldsymbol{y}_2}\Big).$$

Changing the framing, we see that

$$I^{fr}(\bigcirc\!\!\!\bigcirc) = (\Phi_{\boldsymbol{y}_1}\circ\Phi_{\boldsymbol{y}_2})(\exp\big|^{\boldsymbol{y}_1}_{\boldsymbol{y}_2}).$$

The statement of the theorem follows by a simple change of notation. $\square$

Proof of Theorem 11.4.2. First, let us observe that for any diagram $D \in \mathcal{B}$ we have

$$(\partial_D)_{\boldsymbol{x}}\exp\big|^{\boldsymbol{y}}_{\boldsymbol{x}} = D_{\boldsymbol{y}}\cup\exp\big|^{\boldsymbol{y}}_{\boldsymbol{x}}.$$

Now, we have

$$\begin{aligned} I^{fr}(\phi)\#\chi_{\boldsymbol{x}}(\Omega_{\boldsymbol{x}}) &= I^{fr}(\bigcirc\!\!\!\bigcirc) \\ &= \Phi_{\boldsymbol{x}}(\Phi_{\boldsymbol{y}}(\exp\big|^{\boldsymbol{y}}_{\boldsymbol{x}})) \\ &= \Phi_{\boldsymbol{x}}(\exp\big|^{\boldsymbol{y}}_{\boldsymbol{x}}\cup\Omega_{\boldsymbol{x}}) \qquad \text{by the observation above.} \end{aligned}$$

Since $\partial_\Omega = \Omega$, it follows that $\Phi_0(\Omega) = \chi(\Omega)$ and

$$\begin{aligned} I^{fr}(\phi) &= \Phi_{\boldsymbol{x}}(\Omega^{-1}_{\boldsymbol{x}})\#\Phi_{\boldsymbol{x}}(\exp\big|^{\boldsymbol{y}}_{\boldsymbol{x}}\cup\Omega_{\boldsymbol{x}}) \\ &= \Phi_{\boldsymbol{x}}(\exp\big|^{\boldsymbol{y}}_{\boldsymbol{x}}\cup\Omega_{\boldsymbol{x}}\cup\Omega^{-1}_{\boldsymbol{x}}) \qquad \text{by the Wheeling theorem of 11.3.8} \\ &= \Phi_{\boldsymbol{x}}(\exp\big|^{\boldsymbol{y}}_{\boldsymbol{x}}) \\ &= \chi_{\boldsymbol{x}}(\Omega\cup\exp\big|^{\boldsymbol{y}}_{\boldsymbol{x}}). \end{aligned}$$

$\square$

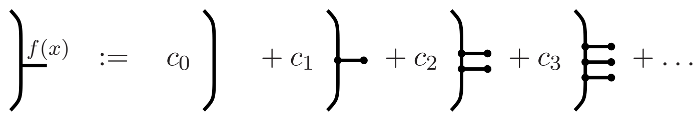
11.4.6. The wheeling map and the Duflo-Kirillov isomorphism $\boxed{\textbf{TBW}}$**.**

11.5. Rozansky's rationality conjecture

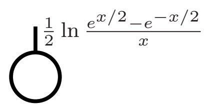
This section concerns a generalization of the wheels formula for the Kontsevich integral of the unknot to arbitrary knots. The generalization is, however, not complete – the Rozansky–Kricker theorem does not give an explicit formula, it only suggests that $I^{fr}(K)$ can be written in a certain *form*.

It turns out that the terms of the Kontsevich integral $I^{fr}(K)$ with values in $\mathcal{B}$ can be rearranged into lines corresponding to the number of loops in open diagrams from $\mathcal{B}$. Namely, for any term of $I^{fr}(K)$, shaving off all legs of the corresponding diagram $G \in \mathcal{B}$, we get a 3-graph $\gamma \in \Gamma$. Infinitly many terms of $I^{fr}(K)$ have the same γ. It turns out that these terms behave in a regular fashion, so that it is possible to recover all of them from γ and some finite information.

To make this statement precise we introduce *marked open diagrams* which are represented by a 3-graph whose edges are marked by power series (it does matter on which side of the edge the mark is located, and we will indicate the side in question by a small leg near the mark). We use such marked open diagrams to represent infinite series of open diagrams which differ by the number of legs. More specifically, an edge marked by a power series $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$ means the following series of open diagrams:

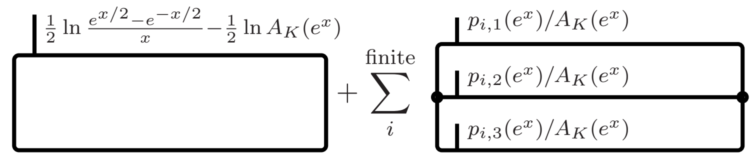


In this notation the wheels formula (Theorem 11.4.1) can be written as

$\ln I^{fr}(O) =$ 

Now we can state the

Rozansky's rationality conjecture. [**Roz2**]

$\ln I^{fr}(K) \;=$ 

$+$ (terms with $\geqslant 3$ loops) ,

where $A_K(t)$ is the Alexander polynomial of K normalized so that $A_K(t) = A_K(t^{-1})$ and $A_K(1) = 1$, $p_{i,j}(t)$ are polynomials, and the higher loop terms *mean the sum over marked 3-graphs (with finitely many copies of each graph) whose edges are marked by a polynomial in e^x divided by $A_K(t)$.*

The word 'rationality' refers to the fact that the labels on all 3-graphs of degree $\geqslant 1$ are rational functions of e^x. The conjecture was proved by A. Kricker in [**Kri2**]. Due to AS and IHX relations the specified presentation of the Kontsevich integral is not unique. Hence the polynomials $p_{i,j}(t)$ themselves cannot be knot invariants. However, there are certain combinations of these polynomials that are genuine knot invariants. For example, consider the polynomial

$$\Theta'_K(t_1,t_2,t_3) = \sum_i p_{i,1}(t_1)p_{i,2}(t_2)p_{i,3}(t_3)\ .$$

Its symmetrization,

$$\Theta_K(t_1,t_2,t_3) = \sum_{\substack{\varepsilon=\pm1\\ \{i,j,k\}=\{1,2,3\}}} \Theta'_K(t_i^\varepsilon,t_j^\varepsilon,t_k^\varepsilon) \in \mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1}]/(t_1t_2t_3=1)\ ,$$

over the order 12 group of symmetries of the theta graph, is a knot invariant. It is called the *2-loop polynomial* of K. Its values on knots with few crossings are tabulated in [**Roz2**]. T. Ohtsuki [**Oht2**] found a cabling formula for the 2-loop polynomial and its values on torus knots $T(p,q)$.

Exercises

(1) Find a basis in the space of canonical invariants of degree 4.
Answer: $j_4, c_4 + c_2/6, c2^2$.

(2) Show that the self-linking number defined in Section 2.2.3 is a canonical framed Vassiliev invariant of order 1.

(3) Show the existence of the limit from Sec.11.2.5

$$A = \lim_{N\to 0} \frac{\theta_{\mathfrak{sl}_N,V}\big|_{q=e^h}}{N}\ .$$

Hint. Choose a complexity function on link diagrams in such a way that two of the diagrams participating in the skein relation for $\theta_{\mathfrak{sl}_N,V}$ are strictly simpler then the third one. Then use induction on complexity.

(4) Let $f(h) = \sum_{n=0}^{\infty} f_n h^n$ and $g(h) = \sum_{n=0}^{\infty} g_n h^n$ be two power series Vassiliev invariants, i.e., for every n both f_n and g_n are Vassiliev invariants of order $\leqslant n$.

a). Prove that their product $f(h) \cdot g(h)$ as formal power series in h is a Vassiliev power series invariant, and

$$\mathrm{symb}(f \cdot g) = \mathrm{symb}(f) \cdot \mathrm{symb}(g)\,.$$

b). Suppose that f and g are related to each other via substitution and multiplication:

$$f(h) = \beta(h) \cdot g\big(\alpha(h)\big)\,,$$

where $\alpha(h)$ and $\beta(h)$ are formal power series in h, and

$$\alpha(h) = ah + \big(\text{terms of degree} \geqslant 2\big)\,, \qquad \beta(h) = 1 + \big(\text{terms of degree} \geqslant 1\big)\,.$$

Prove that $\mathrm{symb}(f_n) = a^n \mathrm{symb}(g_n)$.

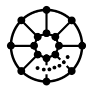
(5) Prove that a canonical Vassiliev invariant is primitive if its symbol is primitive.

(6) Prove that the product of any two canonical Vassiliev power series is a canonical Vassiliev power series.

(7) If v is a canonical Vassiliev invariant of odd order and K an amphicheiral knot, then $v(K) = 0$.

(8) Let $\kappa \in \mathcal{W}_n$ be a weight system of degree n. Construct another weight system $(\psi_j^* \kappa)' \in \mathcal{W}'_n$, where ψ_j^* is the j-th cabling operator from Section 9.7, and $(\cdot)'$ is the deframing operator from Section 4.5.6. Thus we have a function $f_\kappa \colon j \mapsto (\psi_j^* \kappa)'$ with values in $\mathcal{W}'_n$. Prove that

a). $f_\kappa(j)$ is a polynomial in j of degree $\leqslant n$ if n is even, and of degree $\leqslant n-1$ if n is odd.

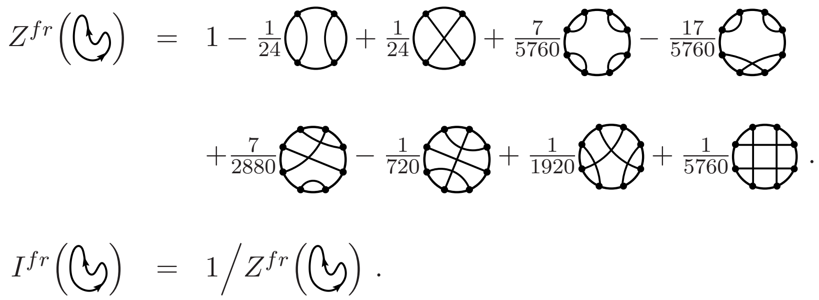
$\overline{w}_n =$ 

n spokes

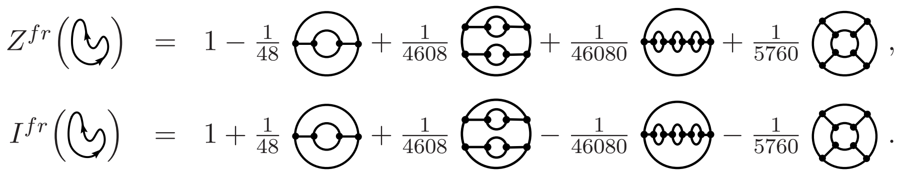
b). The n-th degree term of the polynomial $f_\kappa(j)$ is equal to $-\dfrac{\kappa(\overline{w}_n)}{2}\mathrm{symb}(c_n) j^n$, where $\overline{w}_n$ is the wheel with n spokes, and c_n is the n-th coefficient of the Conway polynomial.

(9) Find the framed Kontsevich integrals $Z^{fr}\Big($  $\Big)$ and $I^{fr}\Big($  $\Big)$ up to order 4 for the indicated unknot with blackboard framing.

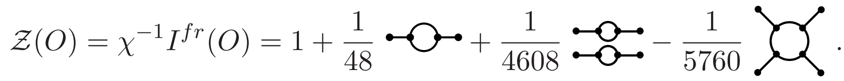
Answer.



(10) Using problem 4 from Chapter 5 (page 162) show that up to degree 4

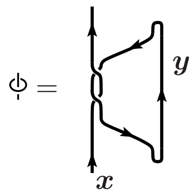


(11) Using the previous problem and problem 23 from Chapter 5 (page 166) prove that up to degree 4

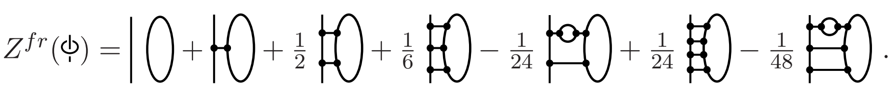


This result confirms Theorem 11.4.1 from page 328 up to degree 4.

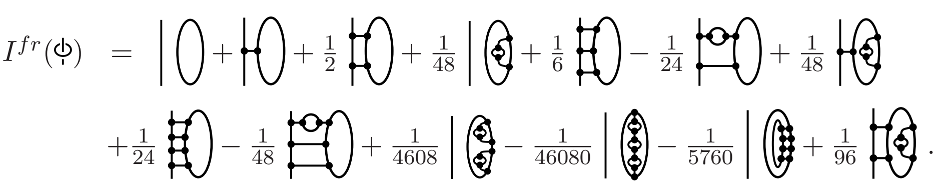
(12) Compute the framed Kontsevich integral $Z^{fr}(\phi)$ up to degree 4 for the Hopf link ϕ with one vertical interval component $\boldsymbol{x}$ and one closed component $\boldsymbol{y}$ represented by a parametrized tangle on the picture. Write the result as an element of $\mathcal{C}(\boldsymbol{x},\boldsymbol{y})$.



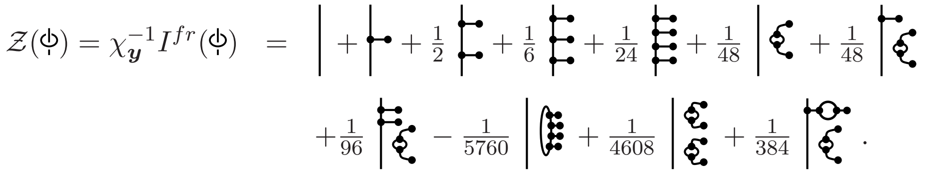
Answer.



(13) Compute the final framed Kontsevich integral $I^{fr}(\phi)$ up to degree 4:



(14) Using the previous problem and problem 23 from Chapter 5 (page 162) prove that up to degree 4



Indicate the parts of this expression forming $\mathcal{Z}_0(\cdot)$, $\mathcal{Z}_1(\cdot)$, $\mathcal{Z}_2(\cdot)$ up to degree 4. This result confirms Theorem 11.4.2 from page 328 up to degree 4.

(15) Prove that $\chi \circ \partial_\Omega : \mathcal{B} \to \mathcal{C}$ is a bialgebra isomorphism.

Hint. In view of Theorem 11.3.2 it remains to check that $\chi \circ \partial_\Omega$ is compatible with the comultiplication. The last follows from the fact that $\chi \circ \partial_\Omega$ transforms primitive elements of $\mathcal{B}$ into primitive elements of $\mathcal{C}$.

(16) Compute $\chi \circ \partial_\Omega(\ )$, $\chi \circ \partial_\Omega(\ )$, $\chi \circ \partial_\Omega(w_6)$.

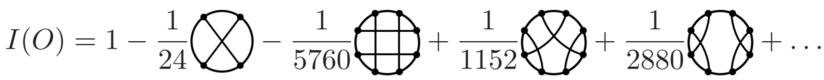
(17) Show that the pairing $\langle (\bullet\!\!-\!\!\bullet)^n, (\bullet\!\!-\!\!\bullet)^n \rangle$ satisfies the recursive relation

$$\langle (\bullet\!\!-\!\!\bullet)^n, (\bullet\!\!-\!\!\bullet)^n \rangle = 2n \cdot \big(\bigcirc + 2n - 2\big) \cdot \langle (\bullet\!\!-\!\!\bullet)^{n-1}, (\bullet\!\!-\!\!\bullet)^{n-1} \rangle \ ,$$

where $\bigcirc$ is a 3-graph in $\Gamma_0 \subset \Gamma \subset \mathcal{B}^\circ$ of degree 0 represented by a circle without vertices and multiplication is understood in algebra $\mathcal{B}^\circ$ (disjoint union).

(18) Prove that, after being carried over from $\mathcal{B}$ to $\mathcal{A}$, the right hand part of Equation 11.4.1.1 (page 328) belongs in fact to the subalgebra $\mathcal{A}' \subset \mathcal{A}$. Find an explicit expression of the series through some basis of $\mathcal{A}'$ up to degree 4.

Answer. The first terms of the infinite series giving the Kontsevich integral of the unknot, are:



Note that this agrees well with the answer to Exercise 9.

Chapter 12

Braids and string links

Essentially, the theory of Vassiliev invariants of braids is a particular case of the Vassiliev theory for tangles, and the main constructions are very similar to the case of knots. There is, however, one big difference: many of the questions that are still open for knots, are rather easy to answer in the case of braids. This, in part, can be explained by the fact that braids form a group, and it turns out that the whole Vassiliev theory for braids can be described in group-theoretic terms. In this chapter we shall see that the Vassiliev filtration on the pure braid groups coincides with the filtrations coming from the nilpotency theory of groups.

The group-theoretic techniques of this chapter can be used to study knots and links. One such application is the theorem of Goussarov that n-equivalence classes of string links on m strands form a group. Another application of the same methods is a proof that Vassiliev invariants of pure braids extend to invariants of string links of the same order. In order to make these connections we shall describe a certian braid closure that produces string links out of pure braids.

The theory of the finite type invariants for braids was first developed by T. Kohno [**Koh1, Koh2**] several years before Vassiliev knot invariants were introduced. The connection between the theory of commutators in braid groups and the Vassiliev knot invariants was first made by T. Stanford [**Sta4**].

12.1. Vassiliev invariants for free groups

We shall start by treating what may seem to be a very particular case: braids on $m+1$ strands whose all strands, apart from the last (the rightmost) one, are straight. Such a braid can be thought of as a graph of a path of a particle in a plane with m punctures. (The punctures correspond to the vertical strands.) The set of equivalence classes of such braids can be identified with the fundamental group of the punctured plane, that is, with the free group F_m on m generators x_i, where $1 \leqslant i \leqslant m$.

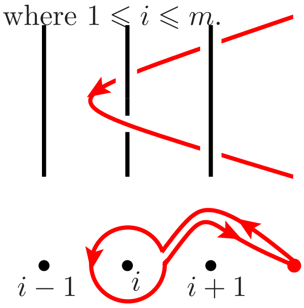


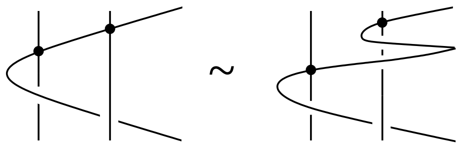
Figure 12.1.0.1. The generator x_i of F_m as a braid and as a path in a plane with m punctures.

A *singular path* in the m-punctured plane is represented by a braid with double points, whose first m strands are vertical. Resolving the double points of a singular path with the help of the Vassiliev skein relation we obtain an element of the space $\mathbb{Z}F_m$ of linear combinations of elements of F_n. Singular paths with k double points span the kth term of a descending filtration on $\mathbb{Z}F_n$ which is analogous to the singular knot filtration on $\mathbb{Z}\mathcal{K}$, defined in Section 3.2.1. A Vassiliev invariant of order k for the free group F_m is, of course, just a linear map from $\mathbb{Z}F_m$ to some abelian group that vanishes on singular paths with k double points.

The radical difference between the singular knots and singular paths (and, in fact, arbitrary singular braids) lies in the following

12.1.1. Lemma. *A singular path in the m-punctured plane with k double points is a product of k singular paths with one double point each.*

Indeed, this is clear from the picture:



Lemma 12.1.1 allows to describe the singular path filtration in purely algebraic terms. Recall that $\mathbb{Z}F_m$ is a ring whose product is the linear extension of the product in F_m. Singular paths then span an ideal JF_m in this ring and singular paths with k double points span the kth power of this ideal.

This exact situation has been studied in great detail in the nilpotency theory of groups. Let us recall some generalities.

12.1.2. The dimension series. Let G be an arbitrary group and $\mathcal{R}$ be a commutative unital ring. The *group algebra* $\mathcal{R}G$ of the group G consists of finite linear combinations $\sum a_i g_i$ with $a_i \in \mathcal{R}$ and $g_i \in G$. The product in $\mathcal{R}G$ is the linear extension of the product in G.

Let $JG \subset \mathcal{R}G$ be the *augmentation ideal*, that is, the kernel of the homomorphism $\mathcal{R}G \to G$ that sends each $g \in G$ to $1 \in \mathcal{R}$. Elements of JG are the linear combinations $\sum a_i g_i$ with $\sum a_i = 0$. The powers $J^n G$ of the augmentation ideal form a descending filtration on $\mathcal{R}G$. We denote by $\mathcal{A}^{\mathcal{R}}(G)$, or simply by $\mathcal{A}(G)$, the graded algebra associated to this filtration:

$$\mathcal{A}(G) = \oplus \mathcal{A}_k(G),$$

where $\mathcal{A}_k(G) = J^k G / J^{k+1} G$.

Let $\mathcal{D}_k G$ (or $\mathcal{D}_k^{\mathcal{R}} G$) be the subset of G consisting of all $g \in G$ such that $g - 1 \in J^k G$. $\mathcal{D}_k G$ is called the kth *dimension subgroup* of G (over $\mathcal{R}$).

Exercise. Show that $\mathcal{D}_k G$ is a normal subgroup of G.

In what follows we shall usually assume that $\mathcal{R} = \mathbb{Z}$, otherwise $\mathcal{R}$ will be stated explicitly.

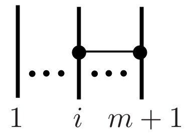
When $G = F_m$, the augmentation ideal is spanned by singular paths. Indeed, each singular path is an alternating sum of non-singular paths, and, hence, it defines an element of the augmentation ideal of F_m. On the other hand, the augmentation ideal of F_m is spanned by differences of the form $g - 1$ where g is some path. By successive crossing changes on its braid diagram, the path g can be made trivial. Let $g_1, \ldots, g_s$ be the sequence of paths obtained in the process of changing the crossings from g to 1. Then

$$g - 1 = (g - g_1) + (g_1 - g_2) + \ldots + (g_s - 1),$$

where the difference enclosed by each pair of brackets is a singular path with one double point.

We see that the Vassiliev invariants are those that vanish on some power of the augmentation ideal of F_m. The dimension subgroups of F_m are the counterpart of the Goussarov filtration: $\mathcal{D}_k F_m$ consists of elements that cannot be distinguished from the unit by Vassiliev invariants of order less than k. We shall refer to these as to being $k-1$-*trivial*.

The algebra $\mathcal{A}(F_m)$ is nothing else but the algebra of chord diagrams for paths: indeed, $\mathcal{A}_k(F_m)$ is the space of paths with k double points modulo those with $k+1$ double points. Graphically, elements of $\mathcal{A}_k(F_m)$ are represented by diagrams with $m+1$ vertical strands and k horizontal chords joining the first m strands to the last strand. For example, the class of the element $x_i - 1$, where x_i is the ith generator of F_m, is represented by the diagram



12.1.3. Commutators and the lower central series. For many groups, the dimension subgroups can be described in terms of *group commutators.* The commutator of two elements $x, y \in G$ can be[1] defined as

$$[x, y] = x^{-1}y^{-1}xy.$$

For H, K normal subgroups of G, denote by $[H, K]$ the subgroup of G generated by all the commutators of the form $[h, k]$ with $h \in H$ and $k \in K$. The *lower central series* subgroups $\gamma_k G$ of a group G are defined inductively by setting $\gamma_1 G = G$ and $\gamma_k G = [\gamma_{k-1}G, G]$.

Exercise. Show that $\gamma_k G$ is a normal subgroup of G.

It is not hard to show by induction that $\gamma_k G$ is always contained in $\mathcal{D}_k G$. If $\gamma_k G$ is actually the same thing as the kth dimension subgroup of G over the integers, it is said that G has the *dimension subgroup property.*

Theorem. *The free group F_m has the dimension subgroup property.*

A proof can be found, for example, in Section 5.7 of [**MKS**].

It was thought for some time that all groups have the dimension subgroup property, until E. Rips found a counterexample in 1972, [**Rips**]. In general, if $x \in \mathcal{D}_k G$, there exists q such that $x^q \in \gamma_k G$, and the group $x \in \mathcal{D}_k^{\mathbb{Q}} G$ consists of all x with this property, [**Jen**].

Exercise. Show that $\gamma_2 G = \mathcal{D}_2 G$.

A group G is called *nilpotent* if $\gamma_n G = \{1\}$ for some n. The maximal n such that $\gamma_n G \neq \{1\}$ is called the *nilpotency class* of G.

12.1.4. The Taylor formula for the free group. Let $x_1, \ldots, x_m$ be a set of free generators of the free group F_m and set $X_i = x_i - 1 \in \mathbb{Z}F_m$. Then, for $k > 0$ each element $w \in F_m$ can be uniquely expressed inside $\mathbb{Z}F_m$ as

$$w = 1 + \sum_{1 \leqslant i \leqslant m} a_i X_i + \ldots + \sum_{1 \leqslant i_1, \ldots, i_k \leqslant m} a_{i_1, \ldots, i_k} X_{i_1} \ldots X_{i_k} + r(w),$$

where $a_{i_1, \ldots, i_j}$ are integers and $r(w) \in J^{k+1} F_m$. To show that such formula exists, it is enough to have it for the generators of F_m and their inverses:

$$x_i = 1 + X_i$$

and

$$x_i^{-1} = 1 - X_i + X_i^2 - \ldots + (-1)^k X_i^k + (-1)^{k+1} X_i^{k+1} x_i^{-1}.$$

[1]there are other, equally good, options, such as $[x, y] = xyx^{-1}y^{-1}$.

The uniqueness will be clear from the construction of the next paragraph. In fact, the coefficients $a_{i_1,\dots,i_j}$ can be interpreted as some kind of derivatives, see [**Fox**].

12.1.5. The Magnus expansion. Having defined the Taylor formula we can go further and define something like the Taylor series.

Let $\mathbb{Z}[[X_1,\dots,X_m]]$ be the algebra of formal power series in m non-commuting variables X_i. Consider the homomorphism of F_m into the group of units of this algebra

$$\mathcal{M}: F_m \to \mathbb{Z}[[X_1,\dots,X_m]],$$

which sends the ith generator x_i of F_m to $1+X_i$. In particular,

$$\mathcal{M}(x_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \dots$$

This homomorphism is called the *Magnus expansion.* It is injective: the Magnus expansion of a reduced word $x_{\alpha_1}^{\varepsilon_1}\dots x_{\alpha_k}^{\varepsilon_k}$ contains the monomial $X_{\alpha_1}\dots X_{\alpha_k}$ with the coefficient $\varepsilon_1\dots\varepsilon_k$.

The Magnus expansion is very useful since it allows to describe the dimension filtration on the free group in very concrete terms.

Lemma. *For $w\in F_m$ the power series $\mathcal{M}(w)-1$ starts with terms of degree k if and only if $w\in\mathcal{D}_kF_m$.*

Proof. Extend the Magnus expansion by linearity to the group algebra $\mathbb{Z}F_m$. The augmentation ideal is sent by $\mathcal{M}$ to the set of power series with no constant term and, hence, the Magnus expansion of anything in $J^{k+1}F_m$ starts with terms of degree at least $k+1$. It follows that the first k terms of the Magnus expansion coincide with the first k terms of the Taylor formula. Notice that this implies the uniqueness of the coefficients in the Taylor formula. Now, the term of lowest non-zero degree on right-hand side of the Taylor formula has degree k if and only if $w-1\in J^kF_m$. □

In fact, the Magnus expansion identifies the algebra $\mathbb{Z}[[X_1,\dots,X_m]]$ with the completion $\overline{\mathcal{A}(F_m)}$ of the algebra of the chord diagrams $\overline{\mathcal{A}(F_m)}$. The following statement is now obvious:

Theorem. *The Magnus expansion is a universal Vassiliev invariant.*

Since the Magnus expansion is injective, we have

Corollary. *The Vassiliev invariants distinguish elements of the free group.*

To put it differently, the dimension subgroups of the free group intersect in the identity.

The Magnus expansion is not the only universal Vassiliev invariant. (See Exercise 2 in the end of the Chapter.) Another important universal invariant is, of course, the Kontsevich integral. In contrast to the Kontsevich integral, the Magnus expansion has integer coefficients. We shall see that it also gives rise to a universal Vassiliev invariant of pure braids with integer coefficients; however, unlike the Kontsevich integral, this invariant fails to be multiplicative.

12.1.6. Observation. If a word $w \in F_m$ contains only positive powers of the generators x_i, the Magnus expansion of w has a transparent combinatorial meaning: $\mathcal{M}(w)$ is simply the sum of all subwords of w, with the letters capitalized. This is also the logic behind the construction of the universal invariant for virtual knots discussed in Chapter 13: it associates to a diagram the sum of all its subdiagrams.

12.2. Vassiliev invariants of pure braids

Here we shall treat the case of the pure braids, that is, the braids whose associated permutation is trivial. The descriptions of invariants for braids with arbitrary permutations will follow from the results of this section.

12.2.1. Pure braids as tangles. Pure braids are a particular case of tangles and thus we have a general recipe for constructing their Vassiliev invariants. The only special feature of braids is the requirement that the tangent vector to a strand is nowhere horizontal. This leads to the fact that the chord diagrams for braids have only horizontal chords on a skeleton consisting of vertical lines; the relations they satisfy are the horizontal 4T-relations.

The Kontsevich integral provides the universal $\mathbb{Q}$-valued Vassiliev invariant for pure braids. Therefore, the study of finite-type invariants for braids reduces to studying the algebra $\mathcal{A}^h(m)$ of horizontal chord diagrams (see page 160). While the multiplicative structure of $\mathcal{A}^h(m)$ is rather complex, an explicit additive basis for this algebra can be easily described. This is due to the very particular structure of the pure braid groups.

12.2.2. Pure braids and free groups. Pure braid groups are, in some sense, very close to being direct products of free groups.

Erasing one (say, the rightmost) strand of a pure braid on m strands produces a pure braid on $m-1$ strands. This procedure respects braid multiplication, so, in fact, it gives a homomorphism $P_m \to P_{m-1}$. Note that this homomorphism has a section $P_{m-1} \to P_m$ defined by adding a vertical non-interacting strand on the right.

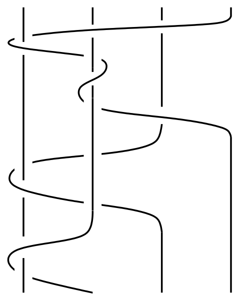


Figure 12.2.2.1. An example of a combed braid.

The kernel of erasing the rightmost strand consists of braids on m strands whose first $m-1$ strands are vertical. Such braids are graphs of paths in a plane with $m-1$ punctures, and they form a group isomorphic to the free group on $m-1$ letters F_{m-1}.

All the above can be re-stated as follows: there is a split extension

$$1 \to F_{m-1} \to P_m \leftrightarrows P_{m-1} \to 1.$$

It follows that P_m is a semi-direct product $F_{m-1} \ltimes P_{m-1}$, and, proceeding inductively, we see that

$$P_m \cong F_{m-1} \ltimes \dots F_2 \ltimes F_1.$$

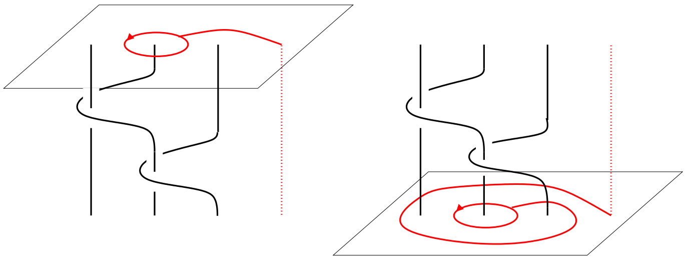
Here F_{k-1} can be identified with the free subgroup of P_m formed by pure braids which can be made to be totally straight apart from the kth strand which is allowed to braid around the strands to the left. As a consequence, every braid in P_n can be written uniquely as a product $\beta_{m-1}\beta_{m-2}\dots\beta_1$, where $\beta_k \in F_k$. This decomposition is called the *combing* of a pure braid.

One can show that the above semi-direct products are not direct (see Exercise 3 at the end of the chapter). However, they are close to direct products in the following sense. Having a semi-direct product $A \ltimes B$ is the same as having an action of B on A by automorphisms. An action of B on A gives rise to an action of B on the *abelianization* (that is, the maximal abelian quotient) of A; we say that a semi-direct product $A \ltimes B$ is *almost direct* if this latter action is trivial.

Lemma. *The semi-direct product $P_m = F_{m-1} \ltimes P_{m-1}$ is almost direct.*

Proof. The abelianization F_{m-1}^{ab} of F_{m-1} is a direct sum of $m-1$ copies of $\mathbb{Z}$. Given a path $x \in F_{m-1}$, its image in F_{m-1}^{ab} is given by the $m-1$ linking numbers with each puncture. `Do we know the definition of a linking number of a curve and a point in the plane?`

The action of a braid $b \in P_{m-1}$ on a generator $x_i \in F_{m-1}$ consists in "pushing" the x_i through the braid:



It is clear the linking numbers of the path $b^{-1}xb$ with the punctures in the plane are the same as those of x_i, therefore the action of P_{m-1} on F_{m-1}^{ab} is trivial. $\square$

12.2.3. Vassiliev invariants and the Magnus expansion. The Vassiliev filtration on the group algebra $\mathbb{Z}P_m$ can be described in the same algebraic terms as in the Section 12.1. Indeed, singular braids can be identified with the augmentation ideal $JP_m \subset \mathbb{Z}P_m$. It is still true that each singular braid with k double points can be written as a product of k singular braids with one double point each; therefore, such singular braids span the kth power of JP_m. The (linear combinations of) chord diagrams with k chords are identified with $J^kP_m/J^{k+1}P_m = \mathcal{A}_k(P_m)$ and the Goussarov filtration on P_m is given by the dimension subgroups $\mathcal{D}_kP_m$.

Now, the augmentation ideals, the dimension series and the associated graded object behave in a predictable way under taking direct products of groups. When $G = G_1 \times G_2$ we have

$$J^kG = \sum_{i+j=k} J^iG_1 \otimes J^jG_2,$$

and this implies that

$$\mathcal{A}_k(G) = \bigoplus_{i+j=k} \mathcal{A}_iG_1 \otimes \mathcal{A}_jG_2,$$

and

$$\mathcal{D}_kG = \mathcal{D}_kG_1 \times \mathcal{D}_kG_2.$$

When G is a semi-direct product of G_1 and G_2 these isomorphisms break down. However, if the semi-direct product is almost direct, the first two of the isomorphisms above still hold additively and the third remains valid with the direct product replaced by the semi-direct product; see [**Pap**], or [**MW**] for the particular case of pure braid groups.

Remark. For an almost direct product $G = G_1 \ltimes G_2$ both $\mathcal{D}_kG = \mathcal{D}_kG_1 \ltimes \mathcal{D}_kG_2$ and $\gamma_kG = \gamma_kG_1 \ltimes \gamma_kG_2$. Since for the free groups the dimension subgroups coincide with the lower central series, we see that the same is true for the pure braid groups.

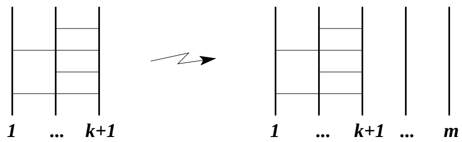
The above algebraic facts can be re-stated in the language of Vassiliev invariants as follows.

Firstly, each singular braid with k double points is a linear combination of *combed singular braids* with the same number of double points. A combed singular braid with k double points is a product $b_{m-1}b_{m-2}\dots b_1$ where b_i is a singular path in $\mathbb{Z}F_i$ with k_i double points, and $k_{m-1}+\dots+k_1=k$.

Secondly, *combed diagrams* form a basis in the space of all horizontal chord diagrams. A combed diagram D is a product $D_{m-1}D_{m-2}\dots D_1$ where D_i is a diagram whose all chords have their rightmost end on the ith strand.

Thirdly, a pure braid is n-trivial if and only if, when combed, it becomes a product of n-trivial elements of free groups. In particular, the only braid that is n-trivial for all n is the trivial braid.

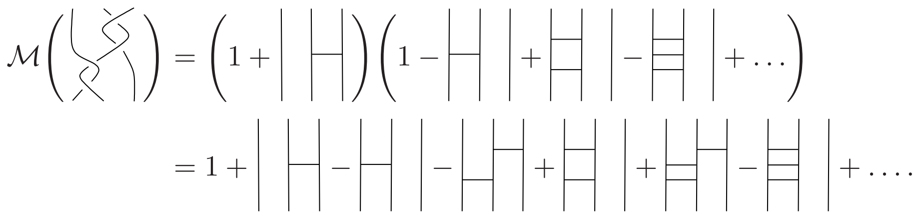
Let $\beta\in P_m$ be a combed braid: $\beta=\beta_{m-1}\beta_{m-2}\dots\beta_1$, where $\beta_k\in F_k$. The Magnus expansions of the elements β_i can be "glued together". Let $i_k:\mathcal{A}(F_k)\hookrightarrow\mathcal{A}^h(m)$ be the map that adds $m-k-1$ vertical strands, with no chords on them, to the right:



The maps i_k extend to the completions of the algebras $\mathcal{A}(F_k)$ and $\mathcal{A}^h(m)$. Define the Magnus expansion

$$\mathcal{M}:P_m\to\widehat{\mathcal{A}}^h(m)$$

as the map sending β to $i_{m-1}\mathcal{M}(\beta_{m-1})\dots i_1\mathcal{M}(\beta_1)$. For example:



Theorem. *The Magnus expansion is a universal Vassiliev invariant of pure braids.*

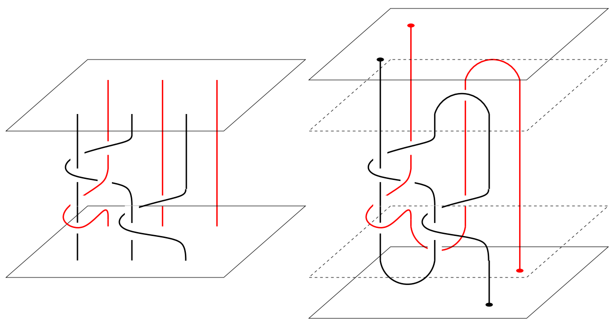
As in the case of free groups, the Magnus expansion is injective, and, therefore, Vassiliev invariants distinguish pure braids.

12.3. String links as closures of pure braids

The above description of the Vassiliev invariants for pure braids can be used to prove some facts about the invariants of knots, and, more generally, string links.

12.3.1. The short-circuit closure. String links can be obtained from pure braids by a procedure called *short-circuit closure.* Essentially, it is a modification of the *plat closure* construction described in [**Bir3**]. Short-circuit closure produces a string link on m strands out of a pure braid on $(2k+1)m$ strands in the following way.

Let us draw a braid in such a way that its top and bottom consist of the integer points of the rectangle $[1,m] \times [0,2k]$ in the plane. A string link on m strands can be obtained from such a braid by joining the points $(i,2j-1)$ and $(i,2j)$ (with $0 < j \leqslant k$) in the top plane and $(i,2j)$ and $(i,2j+1)$ (with $0 \leqslant j < k$) in the bottom plane by little arcs, and extending the strands at the points $(i,0)$ in the top plane and $(i,2k)$ in the bottom plane. Here is an example with $m=2$ and $k=1$:



The short-circuit closure can be thought of as a map $\mathcal{S}_k$ from the pure braid group $P_{(2k+1)m}$ to the monoid $\mathcal{L}_m$ of string links on m strands. This map is compatible with the *stabilization*, which consists of adding $2m$ unbraided strands to the braid on the right, as in Figure 12.3.1.1.

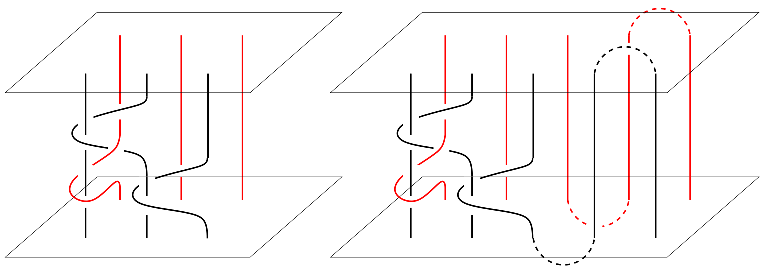


Figure 12.3.1.1. The stabilization map.

Therefore, if P_∞ denotes the union of the groups $P_{(2k+1)m}$ with respect to the inclusions $P_{(2k+1)m} \to P_{(2k+3)m}$, there is a map

$$\mathcal{S} : P_\infty \to \mathcal{L}_m.$$

The map $\mathcal{S}$ is onto, while $\mathcal{S}_k$, for any finite k, is not[2].

One can say when two braids in P_∞ give the same string link after the short-circuit closure:

12.3.2. Theorem. *There exist two subgroups H^T and H^B of P_∞ such that the map $\mathcal{S}_n$ is constant on the double cosets of the form $H^T x H^B$. The preimage of every string link is a coset of this form.*

In other words, $\mathcal{L}_m = H^T \backslash P_\infty / H^B$.

Theorem 12.3.2 generalizes a similar statement for knots (the case $m = 1$), which was proved for the first time by J. Birman in [**Bir3**] in the setting of the plat closure. Below we sketch a proof which closely follows the argument given for knots in [**MSt**].

First, notice that the short-circuit closure of a braid in $P_{(2k+1)m}$ is not just a string link, but a *Morse* string link: the height in the 3-space is a function on the link with a finite number of isolated critical points, none of which is on the boundary. We shall say that two Morse string links are *Morse equivalent* if one of them can be deformed into the other through Morse string links.

Lemma. *Assume that the short-circuit closures of $b_1, b_2 \in P_{(2k+1)m}$ are isotopic. There exist $k' \geqslant k$ such that the short-circuit closures of the images of b_1 and b_2 in $P_{(2k'+1)m}$ under the (iterated) stabilization map are Morse equivalent.*

The proof of this Lemma is not difficult; it is identical to the proof of Lemma 4 in [**MSt**] and we omit it.

Let us now describe the groups H^T and H^B. The group H^T is generated by elements of two kinds. For each pair of strands joined on top by the short-circuit map take (a) the full twist of this pair of strands (b) the braid obtained by taking this pair of strands around some strand, as in Figure 12.3.2.1:

The group H^B is defined similarly, but instead of pairs of strands joined on top we consider those joined at the bottom. Clearly, multiplying a braid x on the left by an element of H^T and on the right by an element of H^B does not change the string link $\mathcal{S}(x)$.

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[2] To show this one has to use the *bridge number* (see page 64) of knots.

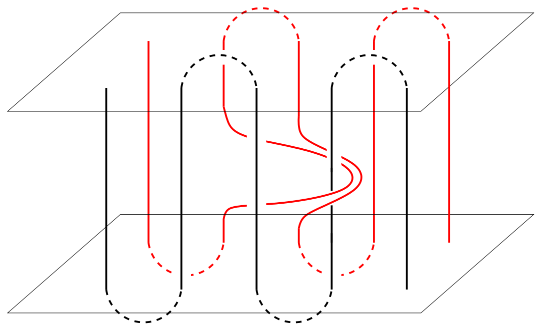


Figure 12.3.2.1. A generator of H^T.

Now, given a Morse string link with the same numbers of maxima of the height function on each component (say, k), we can reconstruct a braid whose short-circuit closure it is, as follows.

Suppose that the string link is situated between the top and the bottom planes of the braid. Without loss of generality we can also assume that the top point of ith strand is the point $(i, 0)$ in the top plane and the bottom point of the same strand is $(i, 2k)$ in the bottom plane. For the jth maximum on the ith strand, choose an ascending curve that joins it with the point $(i, 2j - 1/2)$ in the top plane, and for the jth minimum choose a descending curve joining it to the point $(i, 2k - 3/2)$ in the bottom plane. We choose the curves in such a way that they are all disjoint from each other and only have common points with the string link at the corresponding maxima and minima. On each of these curves choose a framing that is tangent to the knot at one end and is equal to $(1, 0, 0)$ at the other end. Then, doubling each of this curves in the direction of its framing, we obtain a braid as in Figure 12.3.2.2.

Each braid representing a given string link can be obtained in this way. Given two Morse equivalent string links decorated with systems of framed curves, there exists a deformation of one string link into the other through Morse links. It extends to a deformation of the systems of framed curves if we allow a finite number of transversal intersections of curves with each other or with the string link, all at distinct values of the parameter of the deformation, and changes of framing. When a system of framed curves passes such a singularity, the braid that it represents changes. A change of framing on a curve ascending from a maximum produces the multiplication on the left by some power of the twist on the pair of strands corresponding to the curve. An intersection of the curve ascending from a maximum with the link or with another curve gives the multiplication on the left by a braid in H^T obtained by taking the pair of strands corresponding to the curve around

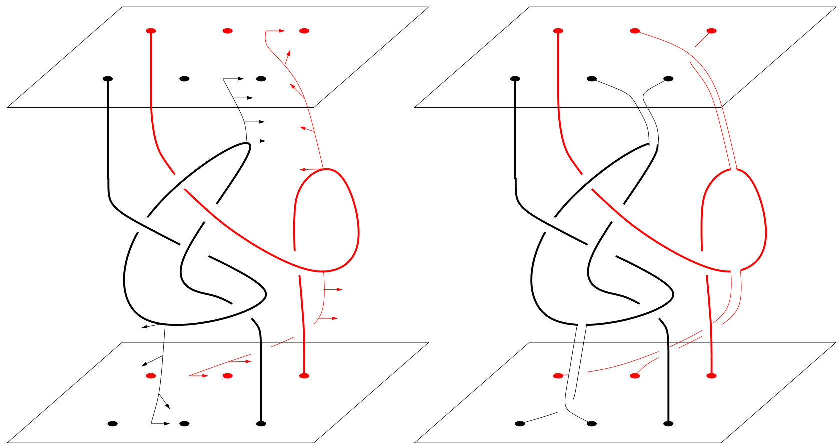


Figure 12.3.2.2. Obtaining a braid from a string link.

some other strands. Similarly, singularities involving a curve descending from a minimum produce multiplications on the right by elements of H^B.

12.3.3. Remark. The subgroups H^T and H^B can be described in the following terms. The short-circuit map $\mathcal{S}$ can be thought of as consisting of two independent steps: joining the top ends of the strands and joining the bottom ends. A braid belongs to H^T if and only if the tangle obtained from it after joining the top strands only is "trivial", that is, equivalent to the tangle obtained in this way from the trivial braid. The subgroup H^B is described in the same way.

12.4. Goussarov groups of string links

Definition. Two string links L_1 and L_2 are *n-equivalent* if there are $x_1, x_2 \in P_\infty$ such that $L_i = \mathcal{S}(x_i)$ and $x_1 x_2^{-1} \in \gamma_{n+1} P_\infty$.

The product of string links descends to their n-equivalence classes.

12.4.1. Theorem. [**G1, Ha2**] *For each m and n, the n-equivalence classes of string links on n strands form a group under the string link product.*

The groups of string links modulo n-equivalence are known as *Goussarov groups*. We shall denote these groups by $\mathcal{L}_m(n)$, or by $\mathcal{L}(n)$, dropping the reference to the number of strands. Let $\mathcal{L}(n)_k$ be the subgroup of $\mathcal{L}(n)$ consisting of the classes of k-trivial links. Note that $\mathcal{L}(n)_k = 1$ for $k \geqslant n$.

12.4.2. Theorem. [**G1, Ha2**] *For all p, q we have*

$$[\mathcal{L}(n)_p, \mathcal{L}(n)_q] \subset \mathcal{L}(n)_{p+q},$$

where by $[\mathcal{L}(n)_p, \mathcal{L}(n)_q]$ *we mean the subgroup of* $\mathcal{L}(n)$ *generated by the commutators of elements of* $\mathcal{L}(n)_p$ *with those of* $\mathcal{L}(n)_q$*. In particular,* $\mathcal{L}(n)$ *is nilpotent of nilpotency class at most* n*.*

Finally,

12.4.3. Theorem. *Two string links cannot be distinguished by* $\mathbb{Q}$*-valued Vassiliev invariants of degree* n *and smaller if and only if the elements they define in* $\mathcal{L}(n)$ *differ by an element of finite order.*

In the case of knots we can say a little bit more.

Theorem. *Two knots cannot be distinguished by Vassiliev invariants of degree* n *and smaller with values in any abelian group if and only if the elements they define in the group of* n*-equivalence classes coincide.*

In particular, the notion of n-equivalence for knots coincides with the definition of Section 3.2.1:

$$\mathcal{L}_1(n) = \Gamma_n \mathcal{K}.$$

12.4.4. The shifting endomorphisms. For $k > 0$, define τ_k to be the endomorphism of P_∞ which triples the kth row of strands. In other words, τ_k replaces each strand with ends at the points $(i, k-1)$ in the top and bottom planes, with $1 \leqslant i \leqslant n$, by three parallel copies of itself as in Figure 12.4.4.1:

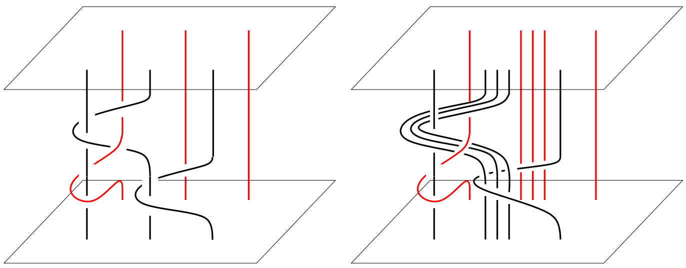


Figure 12.4.4.1

Denote by τ_0 the endomorphism of P_∞ which adds $2m$ non-interacting strands, arranged in 2 rows, to the left of the braid (this is in contrast to the stabilization map, which adds $2m$ strands to the right and is defined only for $P_{(2k+1)m}$ with finite k).

Strand-tripling preserves both H^T and H^B. Also, since τ_k is an endomorphism, it respects the lower central series of P_∞.

Lemma. [**CMSt**] *For any* n *and any* $x \in \gamma_n P_{(2N-1)m}$ *there exist* $t \in H^T \cap \gamma_n P_{(2N+1)m}$ *and* $b \in H^B \cap \gamma_n P_{(2N+1)m}$ *such that* $\tau_0(x) = txb$*.*

Proof. Let $t_{2k-1} = \tau_{2k-1}(x)(\tau_{2k}(x))^{-1}$, and let $b_{2k} = (\tau_{2k+1}(x))^{-1}\tau_{2k}(x)$. Notice that $t_{2k-1}, b_{2k} \in \gamma_n P_\infty$. Moreover, t_{2k-1} looks as in Figure 12.4.4.2 and, by the Remark 12.3.3, lies in H^T. Similarly, $b_{2k} \in H^B$. We have

$$\tau_{2k-1}(x) = t_{2k-1}\tau_{2k}(x),$$

$$\tau_{2k}(x) = \tau_{2k+1}(x)b_{2k}.$$

There exists N such that $\tau_{2N+1}(x) = x$. Thus the following equality holds:

$$\tau_0(x) = t_1 \cdots t_{2N-1}xb_{2N} \cdots b_0,$$

and this completes the proof. □

Figure 12.4.4.2. Braids x and t_{2k-1}.

12.4.5. Existence of inverses. Theorem 12.4.1 is a consequence of the following, stronger, statement:

Proposition. *For any $x \in \gamma_k P_{(2N-1)m}$ and any n there exists $y \in \gamma_k P_\infty$ such that:*

- *y is contained in the image of τ_0^N;*
- *$xy = thb$ with $h \in \gamma_n P_\infty$ and $t, b \in \gamma_k P_\infty$.*

The first condition implies that $\mathcal{S}(xy) = \mathcal{S}(x) \cdot \mathcal{S}(y)$. It follows from the second condition that the class of $\mathcal{S}(y)$ is the inverse for $\mathcal{S}(x)$. The fact that t and b lie in $\gamma_k P_\infty$ is not needed for the proof of Theorem 12.4.1, but will be useful for Theorem 12.4.2.

Proof. Fix n. For $k \geqslant n$ there is nothing to prove.

Assume there exist braids for which the statement of the proposition fails; among such braids choose x with the maximal possible value of k. By Lemma 12.4.4 we have $\tau_0^N(x^{-1}) = t_1x^{-1}b_1$ with $t_1 \in H^T \cap \gamma_k P_{4N-1}$ and $b_1 \in H^B \cap \gamma_k P_{4N-1}$. Then

$$x\tau_0^N(x^{-1}) = xt_1x^{-1}b_1 = t_1 \cdot t_1^{-1}xt_1x^{-1} \cdot b_1.$$

Since $t_1^{-1}xt_1x^{-1} \in \gamma_{k+1}P_{4N-1}$, there exists $y' \in \gamma_{k+1}P_\infty \cap \mathrm{Im}\tau_0^{2N}$ such that $t_1^{-1}xt_1x^{-1} \cdot y' = t_2hb_2$ where $h \in \gamma_n P_\infty$, $t_2 \in H^T \cap \gamma_{k+1}P_\infty$ and $b_2 \in H^B \cap \gamma_{k+1}P_\infty$. Note that y' commutes with b_1, and, hence,

$$x \cdot \tau_0^N(x^{-1})y' = t_1t_2 \cdot h \cdot b_2b_1.$$

Setting $y = \tau_0^N(x^{-1})y'$, $t = t_1t_2$ and $b = b_2b_1$ we see that for x the statement of the proposition is satisfied. We get a contradiction, and the proposition is proved. □

12.4.6. The nilpotency of $\mathcal{L}(n)$. Let $x \in \gamma_p P_\infty$ and $x' \in \gamma_q P_\infty$. Choose the braids y and y' representing the inverses in $\mathcal{L}(n)$ of x and x', respectively, such that the conditions of Proposition 12.4.5 are satisfied, with n replaced by $n+1$: $xy = t_1h_1b_1$ and $x'y' = t_2h_2b_2$ with $h_i \in \gamma_{n+1}P_\infty$, $t_1, b_1 \in \gamma_p P_\infty$ and $t_2, b_2 \in \gamma_q P_\infty$. Replacing the braids by their iterated shifts to the right, if necessary, we can achieve that the braids x, x', y and y' all involve different blocks of strands, and, therefore, commute with each other. Then

$$\begin{aligned}\mathcal{S}(x) \cdot \mathcal{S}(x') \cdot \mathcal{S}(y) \cdot \mathcal{S}(y') = \mathcal{S}(xx'yy') &= \mathcal{S}(xyx'y') \\ &= \mathcal{S}(t_1h_1b_1t_2h_2b_2) = \mathcal{S}(h_1b_1t_2h_2).\end{aligned}$$

The latter link is n-equivalent to $\mathcal{S}(t_2^{-1}b_1t_2b_1^{-1})$ which lives in $\mathcal{L}(n)_{p+q}$.

It follows that each n-fold (that is, involving $n+1$ terms) commutator in $\mathcal{L}(n)$ is trivial, which means that $\mathcal{L}(n)$ is nilpotent of nilpotency class at most n. Theorem 12.4.2 is proved.

12.4.7. Vassiliev invariants and n-equivalence of string links. Let $\{G_i\}$ be a series of subgroups

$$G = G_1 \supseteq G_2 \supseteq \dots$$

of a group G with the property that $[G_p, G_q] \subseteq G_{p+q}$. For $x \in G$ denote by $\mu(x)$ the maximal k such that $x \in G_k$. Recall that $\mathcal{R}$ denotes a commutative unital ring. Let E_nG be the ideal of the group algebra $\mathcal{R}G$ spanned by the products of the form $(x_1 - 1) \cdot \ldots \cdot (x_s - 1)$ with $\sum_{i=1}^s \mu(x_i) \geqslant n$. We have the filtration of $\mathcal{R}G$:

$$\mathcal{R}G \supset J_{\mathcal{R}}G = E_1G \supseteq E_2G \supseteq \dots.$$

This filtration is referred to as the *canonical filtration induced by the series* $\{G_n\}$.

Recall that a string link invariant is called a *Vassiliev invariant of order* n if it vanishes on links with more than n double points. In terms of the short-circuit closure, this means that a Vassiliev invariant of order n is required to vanish on $\mathcal{S}(J^{n+1}P_\infty)$, where JP_∞ is the augmentation ideal of P_∞ with integer coefficients. The value of any order n Vassiliev invariant on a string link depends only on the n-equivalence class of the link. Indeed, $\gamma_{n+1}P_\infty - 1$ is contained in $J^{n+1}P_\infty$.

The following proposition is the key to determining when two different n-equivalence classes of string links cannot be distinguished by Vassiliev invariants of order n:

12.4.8. Proposition. *The filtration by the powers of the augmentation ideal JP_∞ is carried by short-circuit map to the canonical filtration $\{E_i\mathcal{L}(n)\}$ of the group ring $\mathbb{Z}\mathcal{L}(n)$, induced by $\{\mathcal{L}(n)_i\}$.*

Proof. We use induction on the power k of JP_∞. For $k=1$ there is nothing to prove.

Any product of the form

$$(*) \qquad (x_1-1)(x_2-1)\dots(x_s-1)y$$

with $y \in P_\infty$, $x_i \in \gamma_{d_i}P_\infty$ and $\sum d_i = d$ belongs to J^dP_∞ since for any d_i we have $\gamma_{d_i}P_\infty - 1 \subset J^{d_i}P_\infty$. We shall refer to s as the *length* of such product, and to d as its *degree*. The maximal d such that a product of the form $(*)$ is of degree d, will be referred to as the *exact degree* of the product.

The short-circuit closure of a product of length 1 and degree k is in $E_k\mathcal{L}(n)$. Assume there exists a product of the form $(*)$ of degree k whose image R is not in $E_k\mathcal{L}(n)$; among such products choose one of minimal length, say r, and, given the length, of maximal exact degree.

There exists N such that

$$R' := \mathcal{S}((\tau_0^N(x_1)-1)(x_2-1)\dots(x_r-1)y) = \mathcal{S}((x_2-1)\dots(x_r-1)y)\cdot\mathcal{S}(x_1-1).$$

The length of both factors on the right-hand side is smaller that k, so, by the induction assumption, $R' \in E_k\mathcal{L}(n)$. If $\tau_0^N(x_1) = tx_1b$ we have

$$\begin{aligned} R'-R &= \mathcal{S}((tx_1b - x_1)(x_2-1)\dots(x_{m+1}-1)y) \\ &= \mathcal{S}(x_1(b-1)(x_2-1)\dots(x_{m+1}-1)y) \end{aligned}$$

Notice now that $(b-1)$ can be exchanged with (x_i-1) and y modulo closures of products having shorter length or higher degree. Indeed,

$$(b-1)y = y(b-1) + ([b,y]-1)yb$$

and

$$(b-1)(x_i-1) = (x_i-1)(b-1) + ([b,x_i]-1)(x_ib-1) + ([b,x_i]-1).$$

Thus, modulo elements of $E_k\mathcal{L}(n)$

$$\mathcal{S}(x_1(b-1)(x_2-1)\dots(x_{m+1}-1)y) = \mathcal{S}(x_1(x_2-1)\dots(x_{m+1}-1)y(b-1)) = 0.$$

□

Let us now recall some results from the theory of nilpotent groups.

12.4.9. Theorem. *Let*

$$G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_N = \{1\}$$

be a finite series of subgroups of a group G with the property that $[G_p, G_q] \subseteq G_{p+q}$, and such that G_i/G_{i+1} is torsion-free for all $1 \leqslant i < N$. Then for all $i \geqslant 1$

$$G_i = G \cap (1 + E_i G),$$

where $\{E_i G\}$ is the canonical filtration of $\mathbb{Q}G$ induced by $\{G_i\}$.

The most important case of this theorem, namely, the case when G_n is the dimension series of G, has been proved by Jennings [**Jen**], see also [**H**]. As stated above, this theorem can be found in [**PIB, Pas**].

We cannot apply Theorem 12.4.9 directly to the filtration of $\mathcal{L}(n)$ by the $\mathcal{L}(n)_i$, since we do not know whether the successive quotients $\mathcal{L}(n)_i/\mathcal{L}(n)_{i+1}$ are torsion-free. This can be dealt with in the following manner.

For a subset H of a group G let $\sqrt{H}$ be the set of all $x \in G$ such that $x^p \in H$ for some $p > 0$. If H is a normal subgroup, and G/H is nilpotent, then $\sqrt{H}$ is again a normal subgroup of G. The set $\sqrt{\{1\}}$ is precisely the set of all periodic (torsion) elements of G, it is a subgroup if G is nilpotent.

12.4.10. Lemma. *Let*

$$G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_N = \{1\}$$

be a finite series of subgroups of a group G with the property that $[G_p, G_q] \subseteq G_{p+q}$. Then $[\sqrt{G_p}, \sqrt{G_q}] \subseteq \sqrt{G_{p+q}}$ and the canonical filtration of $\mathbb{Q}G$ induced by $\{\sqrt{G_i}\}$ coincides with the filtration induced by $\{G_i\}$.

For the proof see the proofs of Lemmas 1.3 and 1.4 in Chapter IV of [**PIB**].

Now we can prove Theorem 12.4.3.

Write $\overline{\mathcal{L}(n)}$ for $\mathcal{L}(n)/\sqrt{\{1\}}$, $\overline{\mathcal{L}(n)}_i$ for $\mathcal{L}(n)_i/\sqrt{\{1\}}$, and let $\{E_k\overline{\mathcal{L}(n)}\}$ be the canonical filtration on $\mathbb{Q}\overline{\mathcal{L}(n)}$ induced by $\{\overline{\mathcal{L}(n)}_i\}$. By Theorem 12.4.9

$$\overline{\mathcal{L}(n)} \cap (1 + E_{n+1}\overline{\mathcal{L}(n)}) = \{1\},$$

so it follows that

$$\mathcal{L}(n) \cap (1 + E_{n+1}\mathcal{L}(n)) \subseteq \sqrt{\{1\}}$$

inside $\mathcal{L}(n)$. In fact, this inclusion is an equality since any element of finite order lives in $1 + E_k\mathcal{L}(n)$ for all k. Indeed,

$$x^p - 1 = (x-1)^p + p(x-1)^{p-1} + \ldots + \binom{p}{2}(x-1)^2 + p(x-1),$$

so $x^p = 1$ implies that $x - 1$ lives in each $E_k\mathcal{L}(n)$.

By Proposition 12.4.8 the elements of $\mathcal{L}(n)$ that cannot be distinguished from the trivial link by the Vassiliev invariants of degree n form the subgroup $\mathcal{L}(n) \cap (1 + E_{n+1}\mathcal{L}(n))$ and we just saw that these are the elements of finite order in $\mathcal{L}(n)$. Finally, if the classes of two links L_1 and L_2 cannot be distinguished by invariants of order n, then $L_1 - L_2 \in E_{n+1}\mathcal{L}(n)$, and, hence, $L_1 L_2^{-1} - 1 \in E_{n+1}\mathcal{L}(n)$ and $L_1 L_2^{-1}$ is of finite order in $\mathcal{L}(n)$.

12.4.11. The case of knots. The situation simplifies considerably in the case of knots since the connected sum of knots is abelian and there is no need to appeal to Theorem 12.4.9. All we need to show is that the map

$$\mathcal{K} \to \mathcal{L}_1(n)$$

that sends a knot into its n-equivalence class is an invariant of degree n. But $\mathcal{L}_1(n)$ is the quotient of $\mathbb{Z}\mathcal{L}_1(n)$ by the additive subgroup spanned by (1) elements of the form $x - 1$ where x is n-trivial; (2) elements of the form $x_1 \# x_2 - x_1 - x_2$. This subgroup, however, is contained in the ideal $J_{n+1}\mathcal{L}_1(n)$ since any non-trivial product of elements of $J_{n+1}\mathcal{L}_1(n)$ is a linear combination of expressions of the form (2). Proposition 12.4.8 now gives the desired result.

12.4.12. Some comments.

Remark. Rational-valued Vassiliev invariants separate pure braids, and the Goussarov group of n-equivalence classes of pure braids on k strands is nothing but $P_k/\gamma_n P_k$, which is nilpotent of class n for $k > 2$. Since this group is a subgroup of $\mathcal{L}(n)$, we see that $\mathcal{L}(n)$ is nilpotent of class n for links on at least 3 strands. String links on 1 strand are knots, in this case $\mathcal{L}(n)$ is abelian. The nilpotency class of $\mathcal{L}(n)$ for links on 2 strands is unknown. Note that it follows from the results of [**DK**] that $\mathcal{L}(n)$ for links on 2 strands is, in general, non-abelian.

Remark. The relation of the Goussarov groups of string links on more than one strand to integer-valued invariants seems to be a much more difficult problem. While Proposition 12.4.8 gives information about the integer-valued invariants, Theorem 12.4.9 fails over $\mathbb{Z}$.

Remark. Proposition 12.4.8 shows that the map

$$\mathcal{L}_m \to L(n) \to \mathbb{Z}\mathcal{L}(n)/E_{n+1}\mathcal{L}(n)$$

is the *universal degree n Vassiliev invariant* in the following sense: each Vassiliev invariant of links in $\mathcal{L}_m$ of degree n can be extended uniquely to a linear function on $\mathbb{Z}\mathcal{L}(n)/E_{n+1}\mathcal{L}(n)$.

12.5. Braid invariants as string link invariants

A pure braid is a string link so every finite-type string link invariant is also a braid invariant of the same order (at least). It turns out that the converse is true:

12.5.1. Theorem. *A finite-type integer-valued pure braid invariant extends to a string link invariant of the same order.*

Corollary. *The natural map $\mathcal{A}^h(m) \to \mathcal{A}(m)$, where $\mathcal{A}^h(m)$ is the algebra of the horizontal chord diagrams and $\mathcal{A}(m)$ is the algebra of all string link chord diagrams, is injective.*

This was first proved in [**BN8**] by Bar-Natan. He considered quantum invariants of pure braids, which all extend to string link invariants, and showed that they span the space of all Vassiliev braid invariants.

Our approach will be somewhat different. We shall define a map

$$\mathcal{L}_m(n) \to P_m/\gamma_{n+1}P_m$$

from the Goussarov group of n-equivalence classes of string links to the group of n-equivalence classes of pure braids on m strands, together with a section $P_m/\gamma_{n+1}P_m \to \mathcal{L}_m(n)$. A Vassiliev invariant v of order n for pure braids is just a function on $P_m/\gamma_{n+1}P_m$, its pullback to $\mathcal{L}_m(n)$ gives the extension of v to string links.

Remark. Erasing one strand of a string link gives a homomorphism $\mathcal{L}_m \to \mathcal{L}_{m-1}$, which has a section. If $\mathcal{L}_m$ were a group, this would imply that string links can be combed, that is, that $\mathcal{L}_m$ splits as a semi-direct product of $\mathcal{L}_{m-1}$ with the kernel of the strand-erasing map. Of course, $\mathcal{L}_m$ is only a monoid, but it has many quotients that are groups, and these all split as iterated semi-direct products. For instance, string links form groups modulo concordance or link homotopy [**HL**]; here we are interested in the Goussarov groups.

Denote by $\mathcal{FL}_{m-1}(n)$ the kernel of the homomorphism $\mathcal{L}_m(n) \to \mathcal{L}_{m-1}(n)$ induced by erasing the last strand. We have semi-direct product decompositions

$$\mathcal{L}_m(n) \cong \mathcal{FL}_{m-1}(n) \ltimes \ldots \mathcal{FL}_2(n) \ltimes \mathcal{FL}_1(n).$$

We shall see that any element of $\mathcal{FL}_k(n)$ can be represented by a string link on $k+1$ strands whose first k strands are vertical. Moreover, taking the homotopy class of the last strand in the complement to the first k strands gives a well-defined map

$$\pi_k : \mathcal{FL}_k(n) \to F_k/\gamma_{n+1}F_k.$$

Modulo the $n+1$st term of the lower central series, the pure braid group has a semi-direct product decomposition

$$P_\infty/\gamma_{n+1}P_\infty \cong F_{m-1}/\gamma_{n+1}F_{m-1} \ltimes \ldots \ltimes F_1/\gamma_{n+1}F_1.$$

The homomorphisms π_i with $i < m$ can now be assembled into one surjective map

$$\mathcal{L}_m(n) \to P_m/\gamma_{n+1}P_m.$$

Considering a braid as a string link gives a section of this map; this will establish the theorem stated above as soon as we justify the our claims about the groups $\mathcal{FL}_k(n)$.

12.5.2. String links with one non-trivial component. The fundamental group of the complement of a string link certainly depends on the link. However, it turns out that all this dependence is hidden in the intersection of all the lower central series subgroups.

Let X be a string link on m strands and $\widetilde{X}$ be its complement. The inclusions of the top and bottom planes of X, punctured at the endpoints, into $\widetilde{X}$ give two homomorphisms i_t and i_b of F_m into $\pi_1\widetilde{X}$.

Lemma. [**HL**] *For any n the homomorphisms*

$$F_m/\gamma_n F_m \to \pi_1\widetilde{X}/\gamma_n\pi_1\widetilde{X}$$

induced by i_t and i_b, are isomorphisms.

A corollary of this lemma is that for any n there is a well-defined map

$$\mathcal{L}_m(n) \to F_{m-1}/\gamma_{n+1}F_{m-1}$$

given by taking the homotopy class of the last strand of a string link in the complement to the first $m-1$ strands. We must prove that if two string links represent the same element of $\mathcal{FL}_{m-1}(n)$, their images under this map coincide.

In terms of braid closures, erasing the last strand of a string link corresponds to erasing all strands of P_∞ with ends at the points (m, i) for all $i \geqslant 0$. Erasing these strands, we obtain the group which we denote by P_∞^{m-1}; write Φ for the kernel of the erasing map. We have a semi-direct product decomposition

$$P_\infty = \Phi \ltimes P_\infty^{m-1},$$

and the product is almost direct. In particular, this means that $\gamma_k P_\infty = \gamma_k\Phi \ltimes \gamma_k P_\infty^{m-1}$ for all k.

Lemma. *Let $x \in \Phi$, and $h \in \gamma_{n+1}\Phi$. The string links $\mathcal{S}(x)$ and $\mathcal{S}(xh)$ define the same element of $F_{m-1}/\gamma_{n+1}F_{m-1}$.*

Proof. Each braid in Φ can be combed: Φ is an almost direct product of the free groups G_i which consist of braids whose all strands, apart from the one with the endpoints at (m, i), are straight, and whose strands with endpoints at (m, j) with $j < i$ do not interact. Each element a of G_i gives a path in the complement to the first $m - 1$ strands of the string link, and, hence, an element $[a]$ of F_{m-1}. Notice that this correspondence is a homomorphism of G_i to F_{m-1}. (Strictly speaking, these copies of F_{m-1} for different i are only isomorphic, since these are fundamental groups of the same space with different basepoints. To identify these groups we need a choice of paths connecting the base points. Here we shall choose intervals of straight lines.)

Given $x \in \Phi$ we can write it as $x_1 x_2 \ldots x_r$ with $x_i \in G_i$. Then the homotopy class of the last strand of $\mathcal{S}_n(x)$ produces the element

$$[x_1][x_2]^{-1} \ldots [x_r]^{(-1)^{r-1}} \in F_{m-1}.$$

Let $x' = xh$ with $h \in \gamma_{n+1}\Phi$. Then the fact that Φ is an almost direct product of the G_i implies that if $x' = x_1' x_2' \ldots x_r'$ with $x_i \in G_i$, then $x_i \equiv x_i'$ mod $\gamma_{n+1}G_i$. It follows that the elements of F_{m-1} defined by $\mathcal{S}(x)$ and $\mathcal{S}_n(x')$ differ by multiplication by an element of $\gamma_{n+1}F_{m-1}$. □

Lemma. *Let $x \in \Phi$, and $y \in \gamma_{n+1}P_\infty^{m-1}$. The string links $\mathcal{S}(x)$ and $\mathcal{S}(xy)$ define the same element of $F_{m-1}/\gamma_{n+1}F_{m-1}$.*

Proof. Denote by $\widetilde{X}$ the complement of $\mathcal{S}(y)$. We shall write a presentation for the fundamental group of $\widetilde{X}$. It will be clear from this presentation that the element of

$$F_{m-1}/\gamma_{n+1}F_{m-1} = \pi_1\widetilde{X}/\gamma_{n+1}\pi_1\widetilde{X}$$

given by the homotopy class of the last strand of $\mathcal{S}(xy)$ does not depend on y.

Let us assume that both x and y lie in the braid group $P_{m(2N+1)}$. Let H be the horizontal plane coinciding with the top plane of the braid y. The plane H cuts the space $\widetilde{X}$ into the upper part H_+ and the lower part H_-. The fundamental groups of H_+, H_- and $H_+ \cap H_-$ are free. Let us denote by $\{\alpha_{i,j}\}$, $\{\beta_{i,j}\}$ y $\{\gamma_{i,k}\}$ the corresponding free sets of generators (here $1 \leqslant i < m$, $1 \leqslant j \leqslant N+1$ and $1 \leqslant k \leqslant 2N+1$) as in Figure 12.5.2.1. By the Van Kampen Theorem, $\pi_1\widetilde{X}$ has a presentation

$$\begin{array}{lll} \langle \alpha_{i,j}, \beta_{i,j}, \gamma_{i,k} \mid & \theta_y^{-1}(\gamma_{i,2q-1}) = \beta_{i,q}, & \theta_y^{-1}(\gamma_{i,2q}) = \beta_{i,q}^{-1} \\ & \gamma_{i,2q-1} = \alpha_{i,q}, & \gamma_{i,2q} = \alpha_{i,q+1}^{-1} \quad \rangle, \end{array}$$

where $1 \leqslant q \leqslant N+1$ and θ_y is the automorphism of $F_{(m-1)(2N+1)}$ given by the braid y. Since $y \in \gamma_{n+1}P_{(m-1)(2N+1)}$, it is easy to see that

$$\theta_y^{-1}(\gamma_{i,j}) \equiv \gamma_{i,j} \mod \gamma_{m+1}\pi_1\widetilde{X}.$$

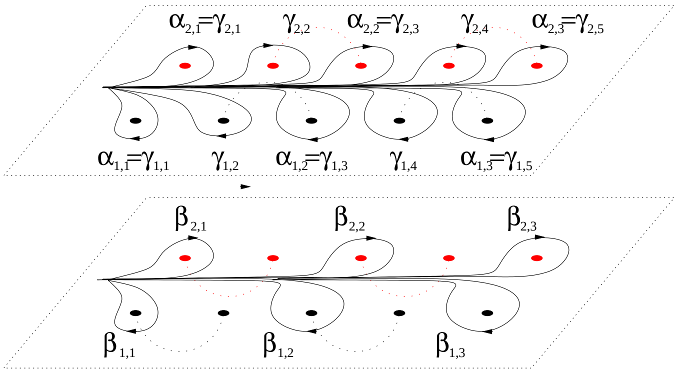


Figure 12.5.2.1

Replacing $\theta_y^{-1}(\gamma_{i,j})$ by $\gamma_{i,j}$ in the presentation of $\pi_1\widetilde{X}$ we obtain a presentation of the free group F_{m-1}. $\square$

Now, a string link that gives rise to an element of $\mathcal{FL}_{m-1}(n)$ can be written as $\mathcal{S}(xy)$ where $x \in \Phi$ and $y \in \gamma_{n+1}P_\infty^{m-1}$. Any link n-equivalent to it is of the form $\mathcal{S}(txyb \cdot h)$ where $t \in H^T$, $b \in H^B$ and $h \in \gamma_{n+1}P_\infty$. We have

$$\mathcal{S}(txyb \cdot h) = \mathcal{S}(xy \cdot bhb^{-1}) = \mathcal{S}(xh'yh''),$$

where $h' \in \gamma_{n+1}\Phi$ and $h'' \in \gamma_{n+1}P_\infty^{m-1}$. It follows from the two foregoing lemmas that $\mathcal{S}(xh'yh'')$ and $\mathcal{S}(xy)$ define the same element of $F_{m-1}/\gamma_{n+1}F_{m-1}$.

Exercises

(1) Show that reducing the coefficients of the Magnus expansion of an element of F_n modulo m, we obtain the universal $\mathbb{Z}_m$-valued Vassiliev invariant for F_n. Therefore, all mod m Vassiliev invariants for F_n are mod m reductions of integer-valued invariants.

(2) Let $\mathcal{M}' : F_n \to \mathbb{Z}[[X_1, \ldots, X_n]]$ be any multiplicative map such that for all x_i we have $\mathcal{M}'(x_i) = 1 + \alpha_i X_i + \ldots$ with $\alpha_i \neq 0$. Show that $\mathcal{M}'$ is a universal Vassiliev invariant for F_n.

(3) (a) Show that the semi-direct product in the decomposition $P_3 = F_2 \ltimes \mathbb{Z}$ given by combing is not direct.
(b) Find an isomorphism between P_3 and $F_2 \times \mathbb{Z}$.

(4) Show that if a semi-direct product $A \ltimes B$ is almost direct, then $\gamma_k A \ltimes B$ coincides with $\gamma_k A \ltimes \gamma_k B$ inside $A \ltimes B$ for all k.

Chapter 13

Gauss diagrams

In this chapter we shall show how the finite-type invariants of a knot can be read off its Gauss diagram. It is not surprising that this is possible in principle, since the Gauss diagram encodes the knot completely. However, the particular method we describe, invented by Polyak and Viro and whose efficiency was proved by Goussarov, turns out to be conceptually very simple. For a given Gauss diagram, it involves only counting its subdiagrams of some particular types.

We shall prove that each finite-type invariant arises in this way and describe several examples of such formulas.

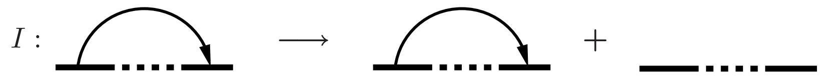
13.1. The Goussarov theorem

Recall that in Chapter 12 we have constructed a universal Vassiliev invariant for the free group by sending a word to the sum of all of its subwords. A similar construction can be performed for knots if we think of a knot as being "generated by its crossings".

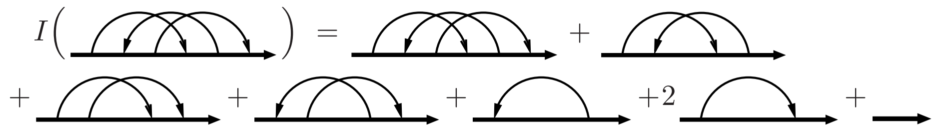
Let **GD** be the set of all Gauss diagrams (we shall take them to be based, or long, even though for the moment it is of little importance). There is a subset $\mathbf{GD}^{re} \subset \mathbf{GD}$ that consists of all *realizable Gauss diagrams*, that is the diagrams of (long) classical knots. Denote by $\mathbb{Z}\mathbf{GD}$ the set of all finite linear combinations of the elements of **GD**. We define the map $I : \mathbb{Z}\mathbf{GD} \to \mathbb{Z}\mathbf{GD}$ by simply sending a diagram to the sum of all its subdiagrams:

$$I(D) =: \sum_{D' \subseteq D} D'$$

and continuing this definition to the whole of $\mathbb{Z}\mathbf{GD}$ by linearity. In other terms, the effect of this map can be described as



For example, we have



Here all signs on the arrows are assumed to be, say, positive.

The map I is an isomorphism, the inverse being

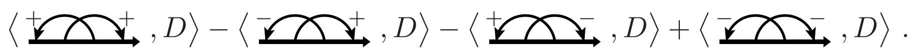
$$I^{-1}(D) = \sum_{D' \subseteq D} (-1)^{|D-D'|} D',$$

where $|D - D'|$ is the number of arrows of D not contained in D'.

Let us write the map $I : \mathbb{Z}\mathbf{GD} \to \mathbb{Z}\mathbf{GD}$ as

$$I(D) = \sum_{A \in \mathbf{GD}} \langle A, D \rangle A.$$

This equality provides a definition of the pairing $\langle A, D\rangle$. In principle, the integers $\langle A, D\rangle$ change if a Reidemeister move is performed on D. However, one can find invariant linear combinations of these integers. For example, in Section 3.6.7 we have proved that the Casson invariant c_2 of a knot can be expressed as



We shall see more examples of such invariants in Section 13.3. Here, we shall prove that for each Vassiliev invariant of classical knots there exists a formula of this type.

Each linear combination with integer coefficients of the form

$$\sum_{A \in \mathbf{GD}} c_A \langle A, D \rangle$$

as a function of D is just the composition $c \circ I$, where $c : \mathbb{Z}\mathbf{GD} \to \mathbb{Z}$ is the linear map with $c(A) = c_A$.

Recall that by $\mathcal{K}$ we denote the set of (isotopy classes of) classical knots. A Gauss diagram uniquely determines the corresponding knot, therefore, a function $v : \mathcal{K} \to \mathbb{Z}$ (the range may be arbitrary) defines a map $\mathbb{Z}\mathbf{GD} \to \mathbb{Z}$, which we denote by the same letter v.

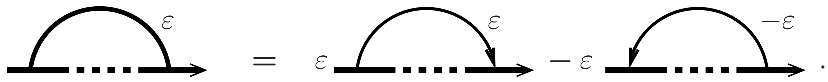
Theorem (Goussarov). *For each integer-valued Vassiliev invariant v of classical knots of order n there exists a linear map $c : \mathbb{Z}\mathbf{GD} \to \mathbb{Z}$ such that $v = c \circ I\,|_{\mathbb{Z}\mathbf{GD}^{re}}$ and such that c is zero on each Gauss diagram with more than n arrows.*

The proof of the Goussarov Theorem is the main goal of this section.

13.1.1. Gauss diagrams with chords. One can, of course, also consider Gauss diagrams for singular knots with double points. These, apart from arrows, have solid undirected chords on them, each chord labeled with a sign. The *sign of a chord* is positive if in the positive resolution of the double point the overcrossing is passed first. (Recall that we are dealing with long Gauss diagrams, and that the points on a long knot are ordered.)

Gauss diagrams with at most n chords span the space $\mathbb{Z}\mathbf{GD}_n$, which is mapped to $\mathbb{Z}\mathbf{GD}$ by a version of the Vassiliev skein relation:

(13.1.1.1)



Using this relation, any knot invariant, or, indeed, any function on Gauss diagrams can be extended to diagrams with chords. Note that the map $\mathbb{Z}\mathbf{GD}_n \to \mathbb{Z}\mathbf{GD}$ is not injective; in particular, changing the sign of a chord in a diagram from $\mathbf{GD}_n$ multiplies its image in $\mathbb{Z}\mathbf{GD}$ by -1. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}\mathbf{GD}_n & \to & \mathbb{Z}\mathbf{GD} \\ \downarrow I & & \downarrow I \\ \mathbb{Z}\mathbf{GD}_n & \to & \mathbb{Z}\mathbf{GD} \end{array}$$

where $I : \mathbb{Z}\mathbf{GD}_n \to \mathbb{Z}\mathbf{GD}_n$ is the isomorphism that sends a diagram to the sum of all its subdiagrams that contain the same chords.

13.1.2. Descending diagrams and "canonical actuality tables". In Section 3.7 we have described a procedure of calculating a Vassiliev invariant using the actuality table. This procedure involves some choices. Firstly, in order to build the table, we have to choose for each chord diagram a singular knot representing it. Secondly, when calculating the knot invariant we have to choose repeatedly sequences of crossing changes that will express our knot as a linear combination of singular knots from the table.

It turns out that for long knots these choices can be eliminated. We shall now define something that can be described as a canonical actuality table and describe a calculation procedure for Vassiliev invariants that only depends on the initial Gauss diagram representing a knot. Strictly speaking, our "canonical actuality tables" are not actuality tables, since they contain one singular knot for each long chord diagram *with signed chords*.

We shall draw the diagrams of the long knots in the plane (x, y), assuming that the knot coincides with the x-axis outside some ball.

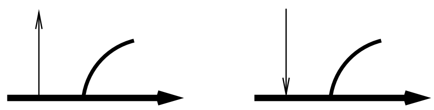
A diagram of a (classical) long knot is *descending* if for each crossing the overcrossing comes first. A knot whose diagram is descending is necessarily trivial. The Gauss diagram corresponding to a descending knot diagram has all its arrows pointed in the direction of the increase of the coordinate x (i.e. to the right).

The notion of a descending diagram can be generalized to diagrams of knots with double points. A Gauss diagram of a long knot with double points is called *descending* if

(1) *all the arrows are directed to the right*, and

(2) *no endpoint of an arrow can be followed by the left endpoint of a chord.*

In other words, the following situations are forbidden:

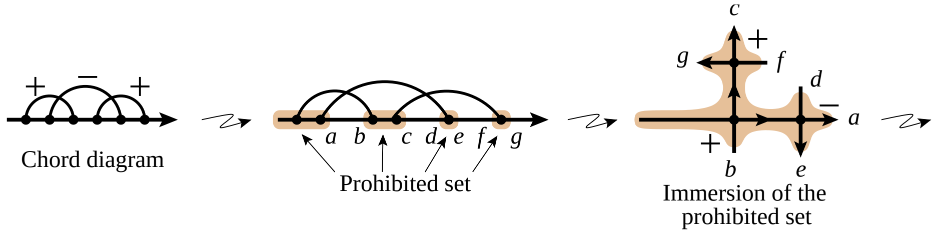


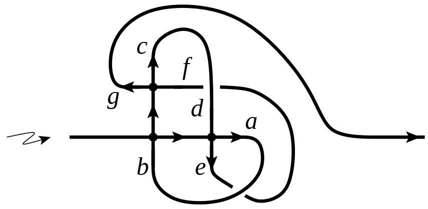
For these two conditions to make sense the Gauss diagram with double points need not be realizable; we shall speak of descending diagrams irrespective of whether they can be realized by classical knots with double points.

Descending diagrams are useful because of the following fact.

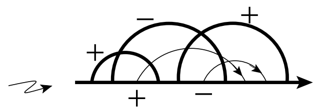
13.1.3. Lemma. *Each long chord diagram with signed chords underlies a unique (up to isotopy) singular long knot that has a descending Gauss diagram.*

Proof. The endpoints of the chords divide the line of the parameter into intervals, two of which are semi-infinite. Let us say that such an interval is *prohibited* if it is bounded from the right by a left end of a chord. Clearly, of the two semi-infinite intervals the left one is prohibited while the right one is not. If a chord diagram D underlies a descending Gauss diagram G_D, then G_D has no arrow endpoints on the prohibited intervals. We shall refer to the union of all prohibited intervals with some small neighbourhoods of the chord endpoints (which do not contain endpoints of other chords or arrows) as the *prohibited set*.





Descending singular knot

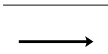
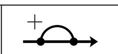
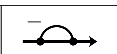
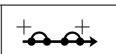
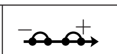
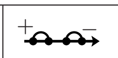
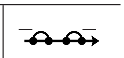
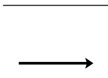
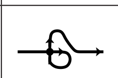
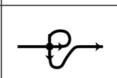
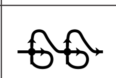
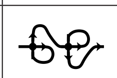
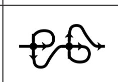
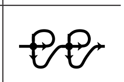


Its Gauss diagram

The prohibited set can be immersed into the plane with double points corresponding to the chords respecting the signs of the chords, the chords themselves being contracted to points. The image of the prohibited set will be an embedded tree T. Such an immersion is uniquely defined up to isotopy.

The leaves of T are numbered in the order given by the parameter along the knot. Note that given T, the rest of the plane diagram can be reconstructed as follows: the leaves of T are joined, in order, by arcs lying outside of T; these arcs only touch T at their endpoints and each arc lies below all the preceding arcs; the last arc extends to infinity. Such reconstruction is unique since the complement of T is homeomorphic to a 2-disk, so all possible choices of arcs are homotopic. □

13.1.4. Now, a *canonical actuality table* for an invariant of order n is the set of its values on all singular long knots with descending diagrams and at most n double points. For example, here is the canonical actuality table for the second coefficient c_2 of the Conway polynomial.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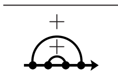	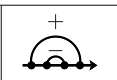	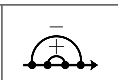	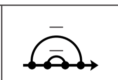	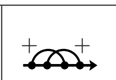	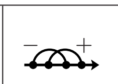	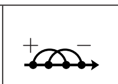	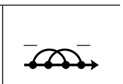
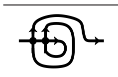	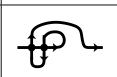	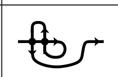	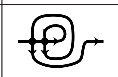	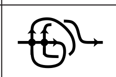	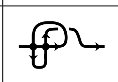	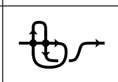	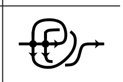
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13.1.5. Algorithm of calculating. Now we remove the second ambiguity in the procedure of calculating a Vassiliev invariant mentioned in 13.1.2. For given n we give an algorithm how to express a (not necessarily descending) Gauss diagram with chords as a linear combination of descending diagrams, modulo diagrams with more than n chords. The algorithm consists in the repetition of a certain map P of Gauss diagrams which makes a diagram in a sense "more descending". The map P works as follows.

Take a diagram D. Replace all the arrows of D that point to the left by the arrows that point to the right (possibly creating new chords in the process), using relation (13.1.1.1).

Denote by $\sum a_i D_i'$ the resulting linear combination. Now, each of the D_i' may contain "prohibited pairs": these are arrow endpoints which are followed by a left endpoint of a chord. Using the Reidemeister moves a prohibited pair can be transformed as follows:

On a Gauss diagram this transformation can take one of the forms shown in Figure 13.1.5.1 where the arrows corresponding to the new crossings are thinner.

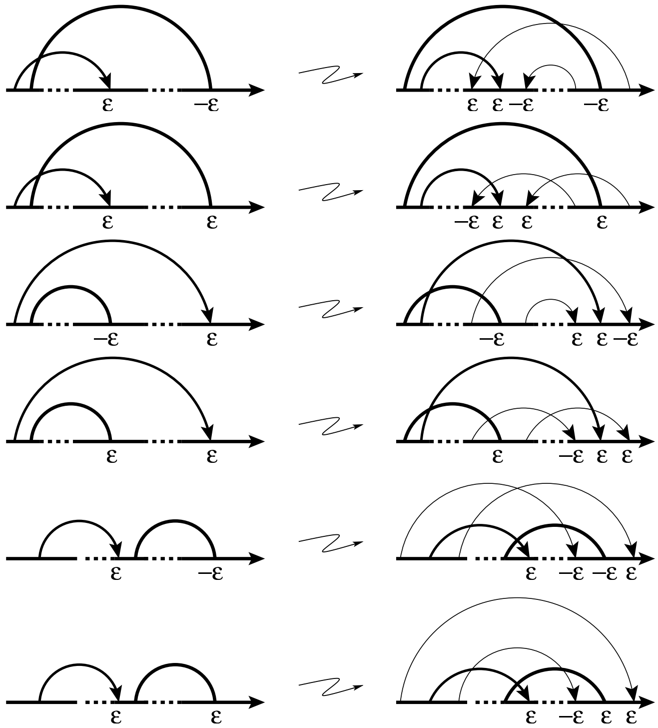


Figure 13.1.5.1

For each D_i' consider the leftmost prohibited pair, and replace it with the corresponding configuration of arrows and chords as in Figure 13.1.5.1; denote the resulting diagram by D_i''. Set $P(D) := \sum a_i D_i''$ and extend P linearly to the whole $\mathbb{Z}\mathbf{GD}_\infty = \bigcup_n \mathbb{Z}\mathbf{GD}_n$.

If D is descending, then $P(D) = D$. We claim that applying P repeatedly to any diagram we shall eventually arrive to a linear combination of descending diagrams, modulo the diagrams with more than n chords.

Let us order the chords in a diagram by their left endpoints. We say that a diagram is *descending up to the k-th chord* if the closed interval from $-\infty$ up to the left end of the kth chord contains neither endpoints of leftwards-pointing arrows, nor prohibited pairs.

If D is descending up to the kth chord, each diagram in $P(D)$ also is. Moreover, applying P either decreases the number of arrow heads to the left of the left end of the $k+1$-st chord, or preserves it. In the latter case, it decreases the number of arrow tails in the same interval. It follows that for some finite m each diagram in $P^m(D)$ will be decreasing up to the $k+1$st chord. Therefore, repeating the process, we obtain after a finite number of steps a combination of diagrams descending up to the $n+1$st chord. Those of them that have at most n chords are descending, and the rest can be disregarded.

Remark. By construction, P respects the realizability of the diagrams. In particular, the above algorithm expresses a long classical knot as a linear combination of singular classical knots with descending diagrams.

13.1.6. Constructing the map c. Let v be a Vassiliev knot invariant of order $\leqslant n$, that is, a linear function $v : \mathbb{Z}\mathbf{GD}^{re} \to \mathbb{Z}$. We are going to define a map $c : \mathbb{Z}\mathbf{GD} \to \mathbb{Z}$ such that on the subgroup $\mathbb{Z}\mathbf{GD}^{re}$ the equality $c \circ I = v$ holds. The definition is obvious:

$$\boxed{c = v \circ I^{-1}}\,. \tag{13.1.6.1}$$

However, for this equation to make sense we need to extend v from $\mathbb{Z}\mathbf{GD}^{re}$ to the whole of $\mathbb{Z}\mathbf{GD}$.

If D is a descending Gauss diagram with signed chords, there exists precisely one singular classical knot K which has a descending diagram with the same signed chords. We set $v(D) := v(K)$. Now, if D is an arbitrary diagram, then we apply the algorithm of the previous subsection to obtain a linear combination $\sum a_i D_i$ of descending diagrams. Set $v(D) := \sum a_i v(D_i)$. It is indeed an extension of v because if D is a realizable Gauss diagram then the value $v(D)$ can be calculated from the canonical actuality table for v. This calculation is exactly expressed in the formula $v(D) = \sum a_i v(D_i)$ given above.

To prove the Goussarov Theorem we now need to show that c vanishes on Gauss diagrams with more than n arrows.

13.1.7. Proof of the Goussarov Theorem. Let us evaluate c on a descending Gauss diagram A whose total number of chords and arrows is greater than n. We have

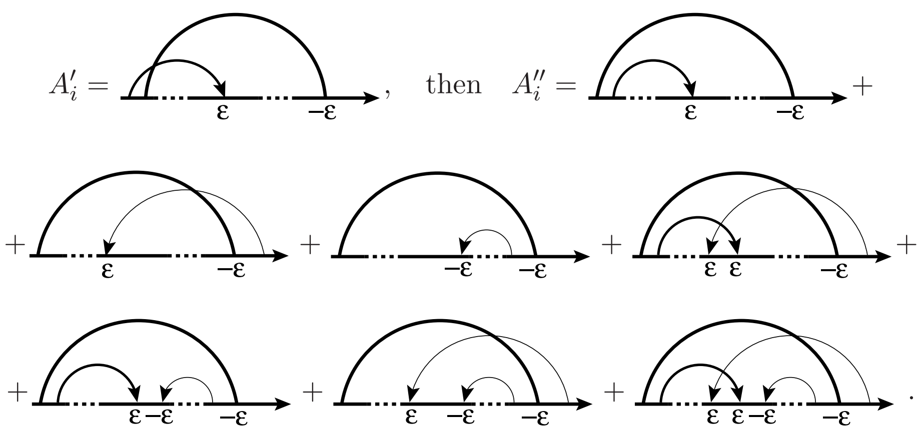
$$c(A) = v(I^{-1}(A)) = \sum_{A' \subseteq A} (-1)^{|A-A'|} v(A').$$

All the subdiagrams A' of A have the same chords as A and therefore are descending. Hence, by the construction of the extension of v to $\mathbb{Z}\mathbf{GD}$, the values of v on all the A' are equal to $v(A)$. If A has more than n chords, then $v(A) = 0$. If A has at most n chords, it has at least one arrow. It is easy to see that in this case $\sum_{A' \subseteq A} (-1)^{|A-A'|} = 0$, and it follows that $c(A) = 0$. In particular, c vanishes on all descending Gauss diagrams with more than n arrows.

In order to treat non-descending Gauss diagrams, we shall introduce an algorithm, very similar to that of Section 13.1.5 that converts any long Gauss diagram with chords into a combination of descending diagrams with at least the same total number of chords and arrows. The algorithm consists in the repetition of a certain map Q, similar to P, which also makes a diagram "more descending". We shall prove that the map Q preserves c in the sense that $c \circ Q = c$ and does not decrease the total number of chords and arrows. Then applying Q to a Gauss diagram A enough number of times we get a linear combination of descending diagrams without altering the value of c. Then the arguments of the previous paragraph show that $c(A) = 0$ which will conclude the proof of the Goussarov Theorem.

Take a Gauss diagram A. Like in Section 13.1.5, we replace all the arrows of A that point leftwards by the arrows that point to the right, using relation (13.1.1.1).

Denote by $\sum a_i A'_i$ the resulting linear combination and check if the summands A'_i contain prohibited pairs. Here is where our new construction differs from the previous one. For each A'_i consider the leftmost prohibited pair, and replace it with the sum of the seven non-empty subdiagrams of the corresponding diagram from the right column of Figure 13.1.5.1 containing at least one of the three arrows. Denote the sum of these seven diagrams by A''_i. For example, if A'_i is the first diagram from the left column of Figure 13.1.5.1,



Now, set $Q(A) = \sum a_i A_i''$ and extend Q linearly to the whole $\mathbb{Z}\mathbf{GD}_\infty$.

As before, applying Q repeatedly to any diagram we shall eventually arrive to a linear combination of descending diagrams, modulo the diagrams with more than n chords. Note that Q does not decrease the total number of chords and arrows.

It remains to prove that Q preserves c. Since $I : \mathbb{Z}\mathbf{GD} \to \mathbb{Z}\mathbf{GD}$ is epimorphic, it is sufficient to check that on diagrams of the form $I(D)$. Assume that we have established that $c(Q(I(D)))) = c(I(D))$ for all Gauss diagrams D with some chords and at most k arrows. If there are no arrows at all then D is descending and $Q(I(D)) = I(D)$. Let now D have $k+1$ arrows. If D is descending, than again $Q(I(D)) = I(D)$ and there is nothing to prove. If D is not descending, then let us first assume for simplicity that all the arrows of D point to the right. Denote by l the arrow involved in the leftmost prohibited pair, and let D_l be the diagram D with l removed. We have

$$I(P(D)) = Q\big(I(D) - I(D_l)\big) + I(D_l).$$

Indeed, $P(D)$ is a diagram from the right column of Figure 13.1.5.1. Its subdiagrams fall into two categories depending on whether they contain at least one of the three arrows indicated on Figure 13.1.5.1 or none of them. The latter are subdiagrams of D_l and they are included in $I(D_l)$. The former can be represented as $Q\big(I(D) - I(D_l)\big)$.

By the induction assumption, $c(Q(I(D_l))) = c(I(D_l))$. Therefore,

$$c(Q(I(D))) = c(I(P(D))) = v(P(D)) \ .$$

But applying P does not change the value of v because of our definition of the extension of v from Section 13.1.6. So

$$c(Q(I(D))) = v(P(D)) = v(D) = c(I(D)) \ .$$

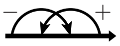
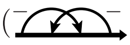
Hence $c(Q(A)) = c(A)$ for any Gauss diagram A.

If some arrows of D point to the left, the argument remains essentially the same and we leave it to the reader. □

13.1.8. Example. Casson invariant. We exemplify the proof of the Goussarov theorem deriving the Gauss diagram formula for the Casson invariant, aka the second coefficient of the Conway polynomial c_2. At the beginning of this chapter we already mentioned a formula for it first given in Section 3.6.7. However the formula we are going to derive following the proof of the Goussarov theorem will be different.

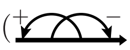
Let $v = c_2$. We will use the definition $c = v \circ I^{-1}$ to find the function $c : \mathbb{Z}\mathbf{GD} \to \mathbb{Z}$.

If a Gauss diagram A consists of 0 or 1 chord then obviously $c(A) = 0$. Also if A consists of two non-intersecting arrows then $c(A) = 0$. So we need to consider the only situation when A consists of two intersecting arrows. The are 16 such diagrams differing by the direction of arrows and signs on them. The next table shows the answers.

$c($$) = 0$	$c($$) = 0$	$c($$) = 0$	$c($$) = 0$
$c($$) = 0$	$c($$) = 0$	$c($$) = 0$	$c($$) = 0$
$c($$) = 0$	$c($$) = 0$	$c($$) = 0$	$c($$) = 0$
$c($$) = 1$	$c($$) = -1$	$c($$) = -1$	$c($$) = 1$

Let us do in detail the calculation of some of these values.

Take $A =$ . According to the definition of I^{-1} from page 364 we have

$c(A) = v($$) - v($$) - v($$) + v($$)$

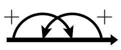
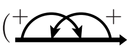
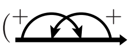
The first three values vanish. This follows from the actuality table for $v = c_2$ in Section 13.1.4: the first and third Gauss diagrams are descending, so they represent a trivial long knot; for the second value one should use the Vassiliev skein relation (13.1.1.1)

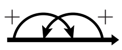
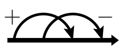
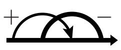
$v($$) = v($$) + v($$)$

and then the actuality table values. Thus we have

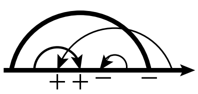
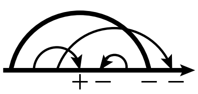
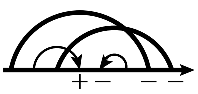
$c(A) = v($$) = v($$) + v($$)$.

The last two Gauss diagrams are descending. From the canonical actuality table the values of c_2 on them are zeros. So $c(A) = 0$.

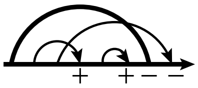
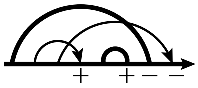
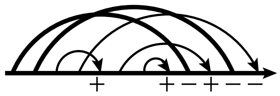
Now let us take $A =$ . Applying I^{-1} to A and using the canonical actuality table for c_2 as before we get $c($$) = v($$)$. To express the last Gauss diagram as a combination of descending diagrams first we should reverse its right arrow with the relation (13.1.1.1):

 $=$  $+$  .

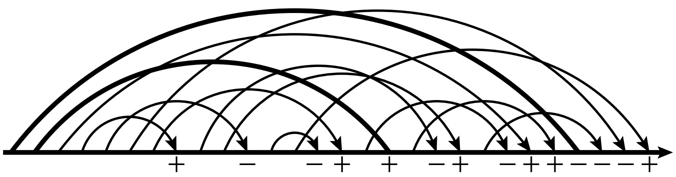
The first Gauss diagram here is descending. But the second one is not, it has a prohibited pair. So we have to apply the map P from Section 13.1.5 to it. According to the first case of Figure 13.1.5.1 we have

 $=$  $=$  $+$  .

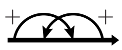
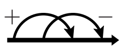
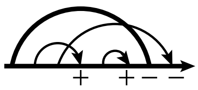
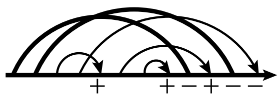
In the first diagram we have to reverse one more arrow. And to the second we need to apply the map P again. After that the reversion of arrows in it would not create any problem because the additional terms would have 3 chords, and we can ignore them if we are interested in the second order invariant $v = c_2$ only.

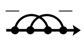
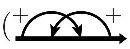
 $=$  $-$  $+$ 

modulo diagrams with three chords. The first and third diagrams here are descending. But with the second one we have a little problem because it has a prohibited interval with many (three) arrow ends on it. We need to apply P five times in order to make it descending modulo diagrams with three chords. The result will be a descending diagram B with two non intersecting chords, one inside another. So the value of v on it would be zero and we may ignore this part of the calculation (see problem (2) on page 386). But we give the answer to make the interested readers to be able to check their understanding of the procedure:

$B =$  .

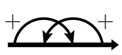
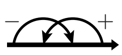
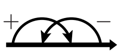
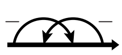
Combining all these results we have

 $=$  $- B +$  $+$ 

modulo diagrams with at least three chords. The value of v on the last Gauss diagram is equal to its value on the canonical descending knot with the same chord diagram, , from the actuality table. This value is 1. The values of v on the other three descending Gauss diagrams are zero. Thus we have $c($$) = 1$.

We leave to the reader to check all other values of c from the table as an exercise.

This table means that the value of c_2 on a knot K with the Gauss diagram D is

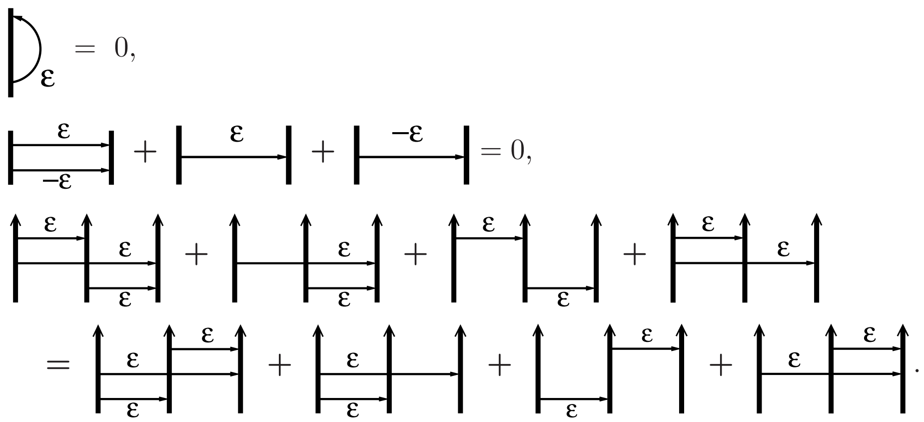
$c_2(K) = \langle$  $-$  $-$  $+$  $, D\rangle$.

This formula differs from the one at the beginning of the chapter by the orientation of all its arrows.

13.2. The Polyak algebra for virtual knots

There are two different notions of Vassiliev invariants for virtual knots: that of [**GPV**] and that of [**Ka5**]. We are only interested in virtual knots so far as they allow to speak of "subknots" of a knot and the definition of [**GPV**] is tailored for this purpose.

13.2.1. The universal invariant of virtual knots. The map $I : \mathbb{Z}\mathbf{GD} \to \mathbb{Z}\mathbf{GD}$ from Section 13.1, sending a diagram to the sum of all its subdiagrams $I(D) = \sum_{D' \subseteq D} D'$, is clearly not invariant under the Reidemeister moves. However, we can make it invariant by simply taking the quotient of the image of I by the images of the Reidemeister moves, or their *linearizations*. These linearizations have the following form:

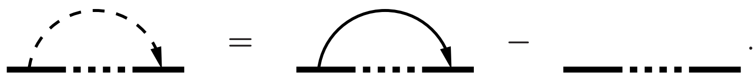


The space $\mathbb{Z}\mathbf{GD}$ modulo the linearized Reidemeister moves is called the *Polyak algebra*. The structure of an algebra comes from the connected sum of long Gauss diagrams; we shall not use it here. The Polyak algebra, which we denote by $\mathcal{P}$, looks rather different from the quotient of $\mathbb{Z}\mathbf{GD}$ by the usual Reidemeister moves, the latter being isomorphic to the free Abelian group spanned by the set of all virtual knots $V\mathcal{K}$. Note, however, that by construction, the resulting invariant $I^* : \mathbb{Z}V\mathcal{K} \to \mathcal{P}$ is an isomorphism, and, therefore, contains the complete information about the virtual knot.

It is not clear how to do any calculations in $\mathcal{P}$. It may be more feasible to consider the (finite-dimensional) quotient $\mathcal{P}_n$ of $\mathcal{P}$ which is obtained by setting all the diagrams with more than n arrows equal to zero. Quite remarkably, the map $I_n : \mathbb{Z}V\mathcal{K} \to \mathcal{P}_n$ obtained by composing I^* with the quotient map, turns out to be an order n Vassiliev invariant for virtual knots, universal in the sense that any other order n Vassiliev invariant is obtained by composing I_n with some linear function on $\mathcal{P}_n$.

Let us now define the Vassiliev invariants so that the previous sentence makes sense.

While the simplest operation on plane knot diagrams is the crossing change, for Gauss diagrams there is a similar, but even simpler manipulation: deleting/inserting of an arrow. An analogue of a knot with a double point for this operation is a diagram with a dashed arrow. A dashed arrow can be resolved by means of the following "virtual Vassiliev skein relation":



An invariant of virtual knots is said to be of finite type (or Vassiliev) of order n if it vanishes on all Gauss diagrams with more than n dashed arrows.

Observe that the effect of I on a diagram all of whose arrows are dashed, is just making all the arrows solid. More generally, the image under I of a Gauss diagram with some dashed arrows is a sum of Gauss diagrams all of which contain these arrows. It follows that I_n is of order n: indeed, if a Gauss diagram has more than n dashed arrows it is sent by I to a Gauss diagram with at least n arrows, which is zero in $\mathcal{P}_n$.

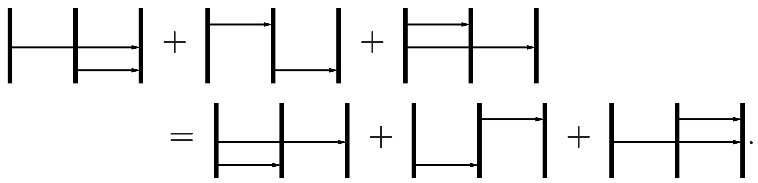
13.2.2. Open problems. A finite type invariant of order n for virtual knots gives rise to a finite type invariant of classical knots of at least the same order. Indeed, a crossing change can be thought of as deleting an arrow followed by inserting the same arrow with the direction reversed.

Exercise. Show that if Vassiliev invariants are defined as above for closed (unbased) virtual knots, the space $\mathcal{P}_2$ is 0-dimensional. Hence, the Casson knot invariant cannot be extended to a Vassiliev invariant of order 2 for closed virtual knots.

It is not clear, however, whether a finite type invariant of classical knots can be extended to an invariant of virtual *long* knots of the same order. The calculation of [**GPV**] show that this is true in orders 2 and 3.

Given that I^* is a complete invariant for virtual knots, one may hope that each virtual knot is detected by I_n for some n. It is not known whether this is the case. A positive solution to this problem would also mean that Vassiliev invariants distinguish classical knots.

It would be interesting to describe the kernel of the natural projection $\mathcal{P}_n \to \mathcal{P}_{n-1}$ which kills the diagrams with n arrows. First of all, notice that using the linearization of the second Reidemeister move, we can get rid of all signs in the diagrams in $\mathcal{P}_n$ that have exactly n arrows: changing the sign of an arrow just multiplies the diagram by -1. Now, the diagrams that have exactly n arrows satisfy the following 6T-relation in $\mathcal{P}_n$:



Consider the space $\vec{\mathcal{A}}_n$ of chord diagrams with n *oriented* chords, modulo the 6T-relation. There is a map $i_n : \vec{\mathcal{A}}_n \to \mathcal{P}_n$, whose image is the kernel of the projection to $\mathcal{P}_{n-1}$. It is not clear, however if i_n is an inclusion. The spaces $\vec{\mathcal{A}}_n$ were introduced in [**Po**] where their relation with usual chord diagrams is discussed.

One more open problem is as follows. Among the linear combinations of Gauss diagrams of the order no greater than n there are some that produce a well defined invariant of degree n. Obviously, such combinations form a vector space, call it L_n. The combinations that lead to the identically zero invariant form a subspace L'_n. The quotient space L_n/L'_n is isomorphic to the space of Vassiliev invariants $\mathcal{V}_n$. The problem is to obtain a description of (or some information about) the spaces L_n and L'_n and in these terms learn something new about $\mathcal{V}_n$. For example, we have seen that the Casson invariant c_2 can be given by two different linear combinations k_1, k_2 of Gauss diagrams of order 2. It is not difficult to verify that these two combinations, together with the empty Gauss diagram k_0 that corresponds to the constant 1, span the space L_2. The subspace L'_2 is spanned by the difference $k_1 - k_2$. We see that $\dim L_2/L'_2 = 2 = \dim \mathcal{V}_2$.

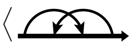
13.3. Examples of Gauss diagram formulas

13.3.1. Highest part of the invariant. Let us start with one observation that will significantly simplify our formulas.

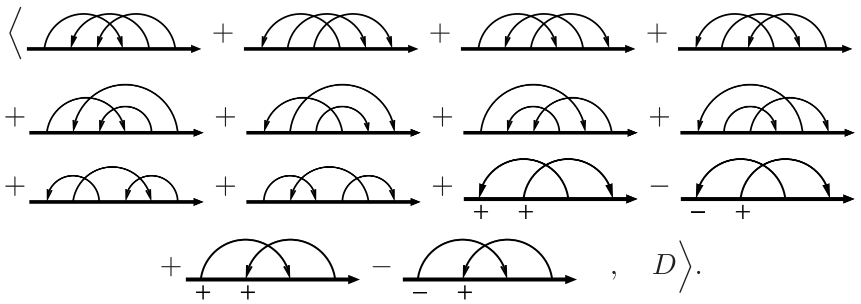
Lemma. *Let $c : \mathbb{Z}\mathbf{GD} \to \mathbb{Z}$ be a linear map representing an invariant of order n. If $A_1, A_2 \in \mathbf{GD}$ are diagrams with n arrows obtained from each other by changing the sign of one arrow, then $c(A_1) = -c(A_2)$.*

Proof. As we noted before, a knot invariant c vanishes on all linearized Reidemeister moves of the form $I(R)$, where $R = 0$ is a usual Reidemeister move on realizable diagrams. Consider a linearized second Reidemeister move involving one diagram A_0 with $n+1$ arrows and two diagrams A_1 and A_2 with n arrows. Clearly, c vanishes on A_0, and therefore $c(A_1) = -c(A_2)$. $\square$

This observation gives rise to the following notation. Let A be a Gauss diagram with n arrows *without signs*, *an unsigned Gauss diagram*. Given a Gauss diagram D, we denote by $\langle A, D\rangle$ the alternating sum $\sum_i (-1)^{\mathrm{sign} A_i}\langle A_i, D\rangle$, where the A_i are all possible Gauss diagrams obtained from A by putting signs on its arrows, and $\mathrm{sign} A_i$ is the number of chords of A_i whose sign is negative. Since the value of c on all the A_i coincides, up to sign, we can speak of the value of c on A.

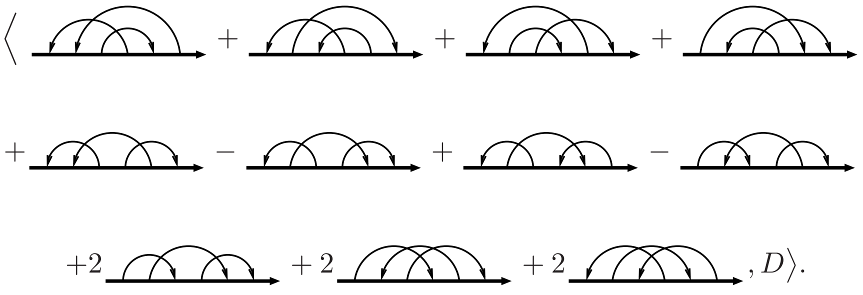
For example, the formula for the Casson invariant of a knot K with the Gauss diagram D can be written as $c_2(K) = \Big\langle$  $, D\Big\rangle$.

13.3.2. Invariants of degree 3. Apart from the Casson invariant, the simplest Vassiliev knot invariant is the coefficient $j_3(K)$ in the power series expansion of the Jones polynomial (see Section 3.6). There are many formulas for $j_3(K)$; the first such formula was found by M. Polyak and O. Viro in terms of unbased diagrams, see [**PV1**]. In [**GPV**] the following expression is given for an invariant of degree 3 (equal to $2v_3 + v_2$ in terms of the basic invariants of Table 3.7.5.1 (see page 92), or to $-j_3/3 + c_2$ in terms of the coefficients of the Jones and Conway polynomials:



(In this formula a typo of [**GPV**] is corrected.) Here the bracket $\langle\cdot,\cdot\rangle$ is assumed to be linear in its first argument.

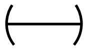
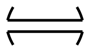
S. Willerton in his thesis [**Wil3**] found the following formula for $-j_3/3$:



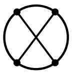
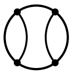
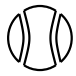
A third Gauss diagram formula for j_3 will be given in Section 13.3.4.

Other combinatorial formulas for $c_2(K)$ and $j_3(K)$ were found earlier by J. Lannes [**Lan**]: they are not Gauss diagram formulas.

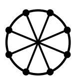
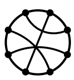
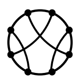
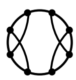
13.3.3. Coefficients of the Conway polynomial. Besides the Gauss diagram formulas for the low degree invariants, two infinite series of such formulas are currently known: those for the coefficients of the Conway and the HOMFLY polynomials. The former can be, of course, derived from the latter, but we start from the discussion of the Conway polynomial, because it is easier. We will follow the original exposition of [**CKR**].

Definition. A chord diagram D is said to be k-*component* if after parallel doubling of each chord according to the picture  $\rightsquigarrow$  , the resulting curve will have k components. We use the notation $|D| = k$.

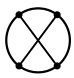
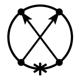
Example. For chord diagram with two chords we have:

$\Big|$  $\Big| = 1 \Longleftarrow$  , $\Big|$  $\Big| = 3 \Longleftarrow$  .

We will be interested in one-component diagrams only. With four chords, there are four one-component diagrams (the notation is borrowed from [**?**]):

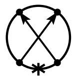
$d_1^4 =$  , $d_5^4 =$  , $d_6^4 =$  , and $d_7^4 =$  .

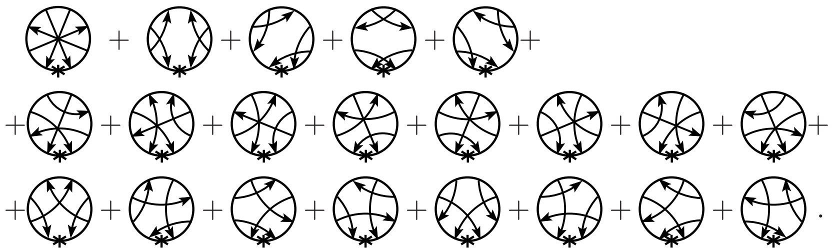
Definition. We can turn a one-component chord diagram with a base point into an arrow diagram according to the following rule. *Starting from the base point we travel along the diagram with doubled chords. During this journey we pass both copies of each chord in opposite directions. Choose an arrow on each chord which corresponds to the direction of the first passage of the chord.* Here is an example.

 $\rightsquigarrow$  $\rightsquigarrow$  .

We call the Gauss diagram obtained in this way the *ascending* arrow diagram.

Definition. The *Conway combination* $\mathfrak{C}_{2n}$ is the sum of all based one-component ascending Gauss diagrams with $2n$ arrows. For example,

$\mathfrak{C}_2 :=$  ,

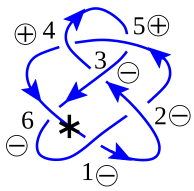
$\mathfrak{C}_4 :=$  .

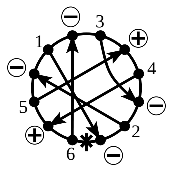
Note that for a given one-component chord diagram we have to consider all possible choices for the base point. However, some choices may lead to the same Gauss diagram. In $\mathfrak{C}_{2n}$ we list them without repetitions. For instance, all choices of a base point for the diagram d_1^4 give the same Gauss diagram. So d_1^4 contributes only one Gauss diagram to $\mathfrak{C}_4$. The diagram d_7^4 contributes four Gauss diagrams because of its symmetry, while d_5^4 and d_6^4 contribute eight Gauss diagrams each.

Theorem. *For $n \geqslant 1$, the coefficient c_{2n} of z^{2n} in the Conway polynomial of a knot K with the Gauss diagram G is equal to*

$$c_{2n} = \langle \mathfrak{C}_{2n}, G \rangle \ .$$

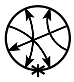
Example. Consider the knot $K := 6_2$ and its Gauss diagram $G := G(6_2)$:

knot 6_2 

$G =$ 

Gauss diagram of 6_2

To compute the pairing $\langle \mathfrak{C}_4, G \rangle$ we must match the arrows of each diagram of $\mathfrak{C}_4$ with the arrows of G. One common property of all terms in $\mathfrak{C}_{2n}$ is that in each term both endpoints of the arrows that are adjacent to the base point are the *arrowtails*. This follows from our construction of $\mathfrak{C}_{2n}$. Hence the arrow $\{1\}$ of G can not participate in the matching with any diagram of $\mathfrak{C}_4$. The only candidates to match with the first arrow of a diagram of $\mathfrak{C}_4$ are the arrows $\{2\}$ and $\{4\}$ of G. If it is $\{4\}$, then $\{1, 2, 3\}$ cannot participate in the matching, and there remain only 3 arrows to match with the four arrows of $\mathfrak{C}_4$. Therefore the arrow of G which matches with the first arrow of a diagram of $\mathfrak{C}_4$ must be $\{2\}$. In a similar way we can find that the arrow of G which matches with the last arrow of a diagram of $\mathfrak{C}_4$ must be $\{6\}$. This leaves three possibilities to match with the four arrows of $\mathfrak{C}_4$: $\{2,3,4,6\}$, $\{2,3,5,6\}$, and $\{2,4,5,6\}$. Checking them all we find only one quadruple, $\{2,3,5,6\}$, which constitute a diagram equal to the second diagram in the second row of $\mathfrak{C}_4$. The product of the local writhes of the arrows $\{2,3,5,6\}$ is equal to $(-1)(-1)(+1)(-1) = -1$. Thus,

$\langle \mathfrak{C}_4, G\rangle = \langle$  $, G\rangle = -1$,

which coincides with the coefficient c_4 of the Conway polynomial $\nabla(K) = 1 - z^2 - z^4$.

13.3.4. Coefficients of the HOMFLY polynomial. Let $P(K)$ be the HOMFLY polynomial of the knot K. Substitute $a = e^h$ and take the Taylor expansion in h. The result will be a Laurent polynomial in z and a power series in h. Let $p_{k,l}(K)$ be the coefficient of $h^k z^l$ in that expression. The numbers $p_{0,l}$ coincide with the coefficients of the Conway polynomial, because the latter is obtained from HOMFLY by fixing $a = 1$.

Remark. From Exercise (21) on page 95 follow that

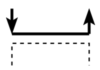
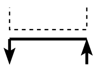
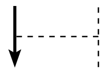
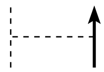
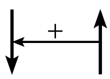
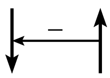
(1) for all nonzero terms the sum $k + l$ is non-negative;

(2) $p_{k,l}$ is a Vassiliev invariant of degree no greater than $k + l$;

(3) if l is odd, then $p_{k,l} = 0$.

We will describe a Gauss diagram formula for $p_{k,l}$ following [**CP**].

Let A be a (based, or long) Gauss diagram, S a subset of its arrows (referred to as a *state*) and α an arrow of A. Doubling all chords in A that belong to S, we obtain a diagram consisting of one or several circles with some signed arrows attached to them. Denote by $\langle\alpha|A|S\rangle$ the expression in two variables h and z that depends on the sign of the chord α and the type of the first passage of α (starting from the basepoint) according to the following table:

First passage:				
	$e^{-h}z$	0	$e^{-2h}-1$	0
	$-e^{h}z$	0	$e^{2h}-1$	0

To the Gauss diagram A we then assign a power series $W(A)$ in h and z defined by

$$W(A) = \sum_S \langle A|S\rangle \left(\frac{e^h - e^{-h}}{z}\right)^{c(S)-1},$$

where $\langle A|S\rangle = \prod_{\alpha\in A}\langle\alpha|A|S\rangle$ and $c(S)$ is the number of components obtained after doubling all the chords in S. Denote by $w_{k,l}(A)$ the coefficient of $h^k z^l$ in this power series and consider the following linear combination of Gauss diagrams: $A_{k,l} := \sum w_{k,l}(A)\cdot A$. Note that the number $w_{k,l}(A)$ is non-zero only for a finite number of diagrams A.

Theorem. *Let G be a Gauss diagram of a knot L. Then*

$$p_{k,l}(K) = \langle A_{k,l}, G\rangle\ .$$

For a proof of the theorem, we refer the reader to the original paper [**CP**]. Here, we will only give one example. To facilitate the practical application of the theorem, we start with some general remarks.

A state S of a Gauss diagram A is called *ascending*, if in traversing the diagram with doubled chords we approach the neighborhood of every arrow (not only the ones in S) first at the arrow head. As follows directly from the construction, only ascending states contribute to $W(A)$.

Note that since $e^{\pm 2h} - 1 = \pm 2h +$ (higher degree terms) and $\pm e^{\mp h}z = \pm z +$ (higher degree terms), the power series $W(A)$ starts with terms of degree at least $|A|$, the number of arrows of A. Moreover, the z-power of $\langle A|S\rangle \left(\frac{e^h-e^{-h}}{z}\right)^{c(S)-1}$ is equal to $|S| - c(S) + 1$. Therefore, for fixed k and l, the weight $w_{k,l}(A)$ of a Gauss diagram may be non-zero only if A satisfies the following conditions:

(i) $|A|$ is at most $k + l$;

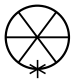
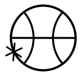
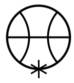
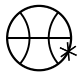
(ii) there is an ascending state S such that $c(S) = |S| + 1 - l$.

For diagrams of the highest degree $|A| = k + l$, the contribution of an ascending state S to $w_{k,l}(A)$ is equal to $(-1)^{|A|-|S|}2^k\varepsilon(A)$, where $\varepsilon(A)$ is the product of signs of all arrows in A. If two such diagrams A and A' with $|A| = k+l$ differ only by signs of the arrows, their contributions to $A_{k,l}$ differ by the sign $\varepsilon(A)\varepsilon(A')$. Thus all such diagrams may be combined to the unsigned diagram A, appearing in $A_{k,l}$ with the coefficient $\sum_S(-1)^{|A|-|S|}2^k$ (where the summation is over all ascending states of A with $c(S) = |S|+1-l$).

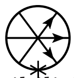
Exercise. Prove that Gauss diagrams with isolated arrows do not contribute to $A_{k,l}$. (Hint: all ascending states cancel out in pairs.)

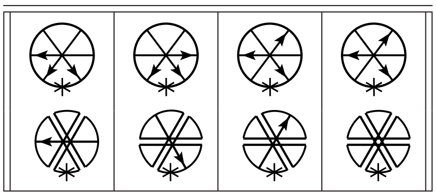
Now, by way of example, let us find an explicit formula for $A_{1,2}$. The maximal number of arrows is equal to 3. To get z^2 in $W(A)$ we need ascending states with either $|S| = 2$ and $c(S) = 1$, or $|S| = 3$ and $c(S) = 2$. In the first case the equation $c(S) = 1$ means that the two arrows of S must intersect. In the second case the equation $c(S) = 2$ does not add any restrictions on the relative position of the arrows. In the cases $|S| = |A| = 2$ or $|S| = |A| = 3$, since S is ascending, A itself must be ascending as well.

For diagrams of the highest degree $|A| = 1 + 2 = 3$, we must count ascending states of unsigned Gauss diagrams with the coefficient $(-1)^{3-|S|}2$, i.e. -2 for $|S| = 2$ and $+2$ for $|S| = 3$. There are only four types of (unsigned) 3-arrow Gauss diagrams with no isolated arrows:

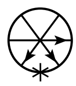
 ;  ,  ,  .

Diagrams of the same type differ by the directions of arrows.

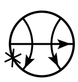
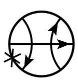
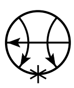
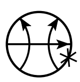
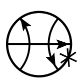
For the first type, recall that the first arrow should be oriented towards the base point; this leaves 4 possibilities for the directions of the remaining two arrows. One of them, namely  , does not have ascending states with $|S| = 2, 3$. The remaining possibilities, together with their ascending states, are shown in the table:



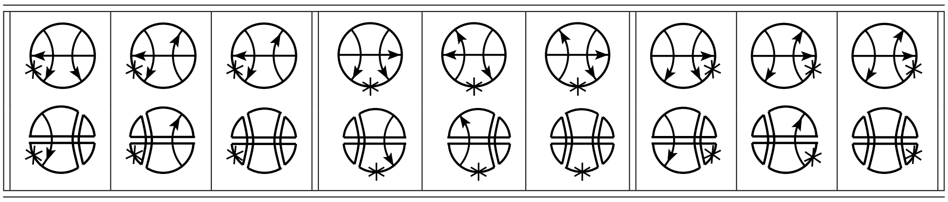
The final contribution of this type of diagrams to $A_{1,2}$ is equal to

-2  $- 2$  .

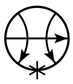
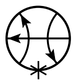
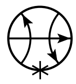
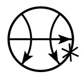
The other three types of degree 3 diagrams differ by the location of the base point. A similar consideration shows that 5 out of the total of 12 Gauss diagrams of these types, namely

 ,  ,  ,  , 

do not have ascending states with $|S| = 2, 3$. The remaining possibilities, together with their ascending states, are shown in the table:

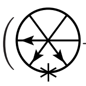
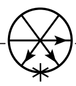
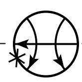
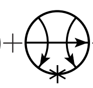
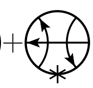
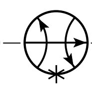
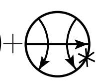
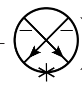


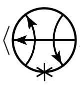
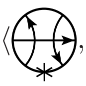
The contribution of this type of diagrams to $A_{1,2}$ is thus equal to

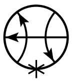
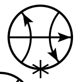
-2  $-\,2$  $-\,2$  $+\,2$  $-\,2$  .

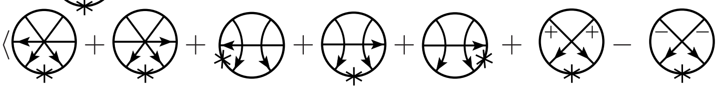
Besides diagrams of degree 3, some degree 2 diagrams contribute to $A_{1,2}$ as well. Since $|A| = 2 < k + l = 3$, contributions of 2-diagrams depend also on their signs. Such diagrams must be ascending (since $|S| = |A| = 2$) and should not have isolated arrows. There are four such diagrams: , with all choices of the signs ε_1, ε_2 for the arrows. For each choice we have $\langle A|S\rangle = \varepsilon_1\varepsilon_2 e^{-(\varepsilon_1+\varepsilon_2)h} z^2$. If $\varepsilon_1 = -\varepsilon_2$, then $\langle A|S\rangle = -z^2$, so the coefficient of hz^2 vanishes and such diagrams do not occur in $A_{1,2}$. For the two remaining diagrams with $\varepsilon_1 = \varepsilon_2 = \pm$, the coefficients of hz^2 in $\langle A|S\rangle$ are equal to ∓ 2 respectively.

Combining all the above contributions, we finally get

$A_{1,2} = -2\Big($  $+$  $+$  $+$  $+$  $-$  $+$  $+$  $-$  $\Big)$.

At this point we can see the difference between virtual and classical long knots. For classical knots the invariant $\mathcal{I}_{A_{1,2}} = \langle A_{1,2}, \cdot\rangle$ can be simplified further. Note that any classical Gauss diagram G satisfies $\langle$  $, G\rangle =$ $\langle$  $, G\rangle$. This follows from the symmetry of the linking number. Indeed, suppose we have matched two vertical arrows (which are the same in both diagrams) with two arrows of G. Let us consider the orientation preserving smoothings of the corresponding two crossings of the link diagram D associated with G. The smoothened diagram $\widetilde{D}$ will have three components. Matchings of the horizontal arrow of our Gauss diagrams with an arrow of G both measure the linking number between the first and the third components of $\widetilde{D}$, using crossings when the first component overpasses (respectively, underpasses) the third one. Thus, as functions on classical Gauss diagrams,

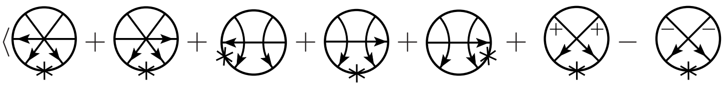
 is equal to  and we have

$p_{1,2}(G) = -2\langle$  $, G\rangle$.

For virtual Gauss diagrams this is no longer true.

In a similar way one may check that $A_{3,0} = -4A_{1,2}$.

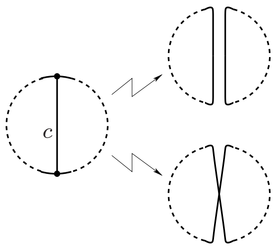
The obtained result implies one more formula for the invariant j_3. Indeed, $j_3 = -p_{3,0} - p_{1,2} = 3p_{1,2}$, therefore

$j_3(K) = -6\langle$  $, G\rangle$.

13.4. The Jones polynomial via Gauss diagrams

The description of the Jones polynomial given in this section is essentially a reformulation of the construction from a paper [**Zul**] by L. Zulli.

Let G be a Gauss diagram of a plane diagram of a knot K. Denote by $[G]$ the set of chords of G. The sign sign(c) of a chord $c \in [G]$ can be considered as a value of the function sign : $[G] \to \{-1, +1\}$. A *state* s for G is an arbitrary function $s : [G] \to \{-1, +1\}$. So for a Gauss diagram with n chords there are 2^n states. The function sign$(\cdot)$ is one of them. With each state s we associate an immersed plane curve in the following way. We double every chord c according to the rule:



, if $s(c) = 1$

, if $s(c) = -1$

Let $|s|$ denote the number of connected components of the curve obtained by doubling all the chords of G (compare with the definition of $\mathfrak{so}_N$-weight system in Sec.6.1.8). Also for a state s we define an integer

$$p(s) := \sum_{c\in[G]} s(c) \cdot \mathrm{sign}(c)\,.$$

The defining relations for the Kauffman bracket from Sec. 2.4 lead to the following expression for the Jones polynomial.

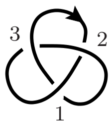
Theorem.

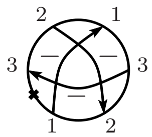
$$J(K) = (-1)^{w(K)} t^{3w(K)/4} \sum_s t^{-p(s)/4} \big(-t^{-1/2} - t^{1/2}\big)^{|s|-1} \ ,$$

where the sum is taken over all 2^n *states for* G *and* $w(K) = \sum\limits_{c\in[G]} \mathrm{sign}(c)$ *is the writhe of* K*.*

We extend the Jones polynomial to virtual knots by means of the same formula.

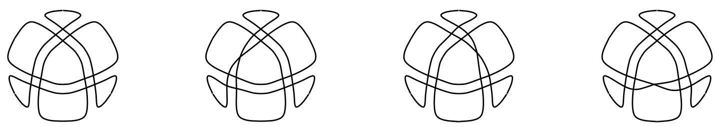
Example. For the left trefoil knot 3_1 we have the following Gauss diagram.



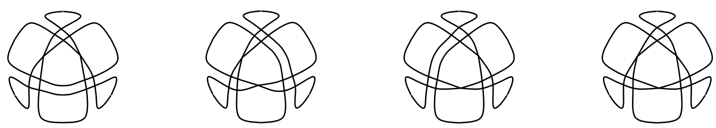
$G =$ 

$w(G) = -3$

There are eight states for such a diagram. Here are the corresponding curves and numbers $|s|$, $p(s)$.



$\|s\|=2$	$\|s\|=1$	$\|s\|=1$	$\|s\|=1$
$p(s)=-3$	$p(s)=-1$	$p(s)=-1$	$p(s)=-1$



$\|s\|=2$	$\|s\|=2$	$\|s\|=2$	$\|s\|=3$
$p(s)=1$	$p(s)=1$	$p(s)=1$	$p(s)=3$

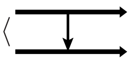
Therefore,

$$\begin{aligned} J(3_1) &= -t^{-9/4}\Big(t^{3/4}\big(-t^{-1/2}-t^{1/2}\big)+3t^{1/4}+3t^{-1/4}\big(-t^{-1/2}-t^{1/2}\big) \\ &\qquad\qquad +t^{-3/4}\big(-t^{-1/2}-t^{1/2}\big)^2\Big) \\ &= -t^{-9/4}\Big(-t^{1/4}-t^{5/4}-3t^{-3/4}+t^{-3/4}\big(t^{-1}+2+t\big)\Big) \\ &= t^{-1}+t^{-3}-t^{-4}\ , \end{aligned}$$

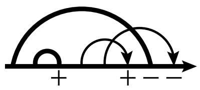
as we had before in Chapter 2.

Exercises

(1) Gauss diagrams and Gauss diagram formulas may be defined for links in a similar way. Prove that for a link L with two components K_1 and K_2

$lk(K_1, K_2) = \langle$  $, G(L)\rangle\,.$

(2) Find a sequence of Reidemeister moves that transforms the Gauss diagram B from page 373 to the diagram



.

Show that this diagram is not realizable. Calculate the value of the extension, according to 13.1.6, of the invariant c_2 on it.

(3) Let $\mathcal{A}$ be the algebra of chord diagrams, $\vec{\mathcal{A}}$ being the algebra of oriented (arrow) chord diagrams. Prove that the natural mapping $\mathcal{A} \to \vec{\mathcal{A}}$ is well-defined, i. e. that the 6T relation implies the 4T relation.

(4) Let $\vec{\mathcal{A}}$ be the algebra of Gauss diagrams, $\vec{\mathcal{C}}$ being the algebra of acyclic arrow graphs (oriented closed diagrams). Prove that the natural mapping $\vec{\mathcal{A}} \to \vec{\mathcal{C}}$ is well-defined, i.e. that the $S\vec{T}U$ relation implies the 6T relation.

(5) * Define an analog of the algebra of oriented closed diagrams $\vec{\mathcal{C}}$ spanned by all graphs, not only acyclic. Denote it by $\vec{\mathcal{C}}_{\mathrm{all}}$. Is it true that the natural mapping $\vec{\mathcal{C}} \to \vec{\mathcal{C}}_{\mathrm{all}}$ is an isomorphism?

(6) * Construct an analogue of the algebra of open diagrams $\mathcal{B}$ consisting of graphs with oriented edges.

Chapter 14

Miscellany

14.1. The Melvin–Morton conjecture

14.1.1. Formulation. Roughly speaking the Melvin–Morton conjecture says that the Alexander-Conway polynomial can be read from the highest order part of the colored Jones polynomial.

According to exercise (28) of Chapter 6 (see also [**MeMo, BNG**]) the coefficients J^k_n of the unframed colored Jones polynomial J^k (Section 11.2.4) are polynomials in k of degree at most $n+1$ without free terms. So we may write

$$\frac{J^k_n}{k} = \sum_{0\leqslant j\leqslant n} b_{n,j}k^j \qquad \text{and} \qquad \frac{J^k}{k} = \sum_{n=0}^{\infty} \sum_{0\leqslant j\leqslant n} b_{n,j}k^j h^n\,,$$

where $b_{n,j}$ are Vassiliev invariants of order $\leqslant n$. The highest order part of the colored Jones polynomial is the Vassiliev power series invariant

$$\mathrm{MM} := \sum_{n=0}^{\infty} b_{n,n}h^n\,.$$

The Melvin–Morton conjecture. [**MeMo**] *The highest order part of the colored Jones polynomial MM is inverse to the version of the Alexander–Conway power series A defined by equations (11.2.5.1-11.2.5.2). In other words,*

$$MM(K)\cdot A(K) = 1$$

for any knot K.

14.1.2. Historical remarks. In [**Mo**] H. Morton proved the conjecture for torus knots. After this L. Rozansky [**Roz1**] proved the Melvin–Morton conjectures on a level of rigor of Witten's path integral interpretation of Jones invariant. The first complete proof was done by D. Bar-Natan and S. Garoufalidis [**BNG**]. They invented a remarkable reduction of the conjecture to a certain equation for weight systems via canonical invariants. We review this reduction in Section 14.1.3. They checked the equation using calculations of the weight systems on chord diagrams. Also they proved a more general theorem [**BNG**] relating the highest order part of an arbitrary quantum invariant to the Alexander-Conway polynomial. Following [**Ch2**] we shall represent another proof of this generalized Melvin–Morton conjecture in Section 14.1.7. A. Kricker, B. Spence and I. Aitchison [**KSA**] proved the Melvin–Morton conjecture using the cabling operations described in Section 9.7. Later, developing this method, A. Kricker [**Kri1**] also proved the generalization. In the paper [**Vai1**] A. Vaintrob gives another proof of the Melvin–Morton conjectures. He used calculations on chord diagrams and the Lie superalgebra $\mathfrak{gl}(1|1)$ which is responsible for the Alexander–Conway polynomial. An idea to use the restriction of the equation for weight systems to the primitive space was explored in [**Ch1, Vai2**]. We shall follow [**Ch1**] in the direct calculation of the Alexander–Conway weight system in Section 14.1.5.

B. I. Kurpita and K. Murasugi found a different proof of the Melvin–Morton conjecture which does not use Vassiliev invariants and weight systems [**KuM**].

The works on the Melvin–Morton conjecture stimulated L. Rozansky [**Roz2**] to formulate an exiting conjecture about the fine structure of the Kontsevich integral. The conjecture was proved by A. Kricker [**Kri2**]. We shall discuss this in Section 11.5.

14.1.3. Reduction to weight systems. Both power series Vassiliev invariants MM and A are canonical, so does their product (exercise (6)). The invariant of the right-hand side that is identically equal to 1 on all knots is also a canonical invariant. The Melvin–Morton conjecture is thus an equality for canonical invariants, hence it is enough to prove the equality of their symbols.

Introduce the notation

$$\begin{aligned} S_{\mathrm{MM}} &:= \operatorname{symb}(\mathrm{MM}) = \sum_{n=0}^{\infty} \operatorname{symb}(b_{n,n})\,; \\ S_A &:= \operatorname{symb}(A) = \operatorname{symb}(C) = \sum_{n=0}^{\infty} \operatorname{symb}(c_n)\,. \end{aligned}$$

The Melvin–Morton conjecture in equivalent to the relation

$$S_{\mathrm{MM}} \cdot S_A = \mathbf{I}_0\,.$$

It is obvious in degrees 0 and 1. So basically we must prove that in degree $\geqslant 2$ the product $S_{\mathrm{MM}} \cdot S_A$ equals zero. To prove this we have to prove that $S_{\mathrm{MM}} \cdot S_A(p_1 \cdot \dots \cdot p_n) = 0$ on any product $p_1 \cdot \dots \cdot p_n$ of primitive elements of degree > 1.

The weight system S_{MM} is the highest part of the weight system $\varphi'^{V_k}_{\mathfrak{sl}_2}/k$ from exercise (28) of the Chapter 6. The last one is multiplicative as it was explained in Section 6.1.4. Hence S_{MM} is multiplicative too. Exercise (16) of Chapter 3 implies that the weight system S_A is also multiplicative. In other words, both weight systems S_{MM} and S_A are group-like elements of the Hopf algebra of weight systems $\overline{\mathcal{W}}$. A product of two group-like elements is group-like which shows that the weight system $S_{\mathrm{MM}} \cdot S_A$ is multiplicative. Therefore, it is sufficient to prove that

$$S_{\mathrm{MM}} \cdot S_A\big|_{\mathcal{P}_{>1}} = 0\,.$$

By the definitions of the multiplication of weight systems and primitive elements,

$$S_{\mathrm{MM}} \cdot S_A(p) = (S_{\mathrm{MM}} \otimes S_A)(\delta(p)) = S_{\mathrm{MM}}(p) + S_A(p)\,.$$

Therefore we have reduced the Melvin–Morton conjecture to the equality

$$S_{\mathrm{MM}}\big|_{\mathcal{P}_{>1}} + S_A\big|_{\mathcal{P}_{>1}} = 0\,.$$

Now we shall exploit the filtration

$$0 = \mathcal{P}^1_n \subseteq \mathcal{P}^2_n \subseteq \mathcal{P}^3_n \subseteq \dots \subseteq \mathcal{P}^n_n = \mathcal{P}_n\ .$$

from Section 5.5.2 and the wheel $\overline{w}_n$ that spans $\mathcal{P}^n_n/\mathcal{P}^{n-1}_n$ for even n and belongs to $\mathcal{P}^{n-1}_n$ for odd n.

The Melvin–Morton conjecture follows from the next theorem.

14.1.4. Theorem. *The weight systems S_{MM} and S_A have the properties*

1) $S_{MM}\big|_{\mathcal{P}^{n-1}_n} = S_A\big|_{\mathcal{P}^{n-1}_n} = 0$*;*
2) $S_{MM}(w_{2m}) = 2, \quad S_A(w_{2m}) = -2.$

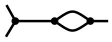
The equation $S_A(w_{2m}) = -2$ is a particular case of exercise (34) of Chapter 6. Exercise (26) of the same chapter implies that for any $p \in \mathcal{P}^{n-1}_n$ the degree in c of the polynomial $\rho_{\mathfrak{sl}_2}(p)$ is less than or equal to $[(n-1)/2]$. After the substitution $c = \frac{k^2-1}{2}$ corresponding to the weight systems of the colored Jones polynomial, the degree of the polynomial $\rho^{V_k}_{\mathfrak{sl}_2}(p)/k$ in k will be at most $n-1$. Therefore, its n-th term vanishes and $S_{\mathrm{MM}}\big|_{\mathcal{P}^{n-1}_n} = 0$. Exercise (24) of Chapter 6 asserts that the highest term of the polynomial

$\rho_{\mathfrak{sl}_2}(\overline{w_{2m}})$ is $2^{m+1}c^m$. Again the substitution $c = \frac{k^2-1}{2}$ (taking the trace of the corresponding operator and dividing the result by k) gives that the highest term of $\rho_{\mathfrak{sl}_2}^{V_k}(\overline{w_{2m}})/k$ is $\frac{2^{m+1}k^{2m}}{2^m} = 2k^{2m}$. Hence $S_{\mathrm{MM}}(w_{2m}) = 2$. The only equation which remains to prove is that $S_A\big|_{\mathcal{P}_n^{n-1}} = 0$. We shall prove it in the next section.

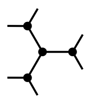
14.1.5. Alexander–Conway weight system. Using the state sum formula for S_A from the exercise (34) of Chapter 6 we are going to prove that $S_A(p) = 0$ for any closed diagram $p \in \mathcal{P}_n^{n-1}$.

First of all note that any such $p \in \mathcal{P}_n^{n-1}$ has an internal vertex connected with three other internal vertices. Indeed, each external vertex is connected with only one internal vertex. The number of external vertices is not greater than $n-1$. The total number of vertices of p is $2n$, so there must be at least $n+1$ internal vertices, and only $n-1$ of them can be connected with external vertices.

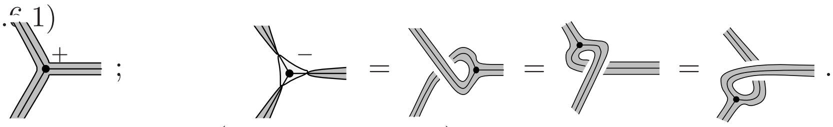
Pick such a vertex connected with three other internal vertices. There are two possible cases: either all the three vertices are different or two of them coincide.

The second case is easier, so let us start with it. Here we have a "bubble" . After resolving the vertices of this fragment and erasing the curves with more than one component we are left with the linear combination of curves -2  $+$ 2 
which cancel each other. So $S_A(p) = 0$.

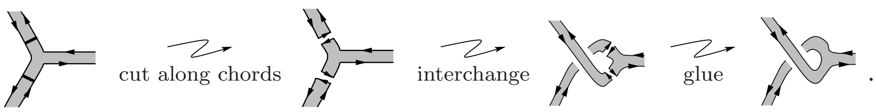
For the first case we formulate our claim as a lemma.

14.1.6. Lemma. $S_A\Bigg($  $\Bigg) = 0.$

We shall utilize the state surfaces $\Sigma_s(p)$ from problem 30 of Chapter 6. Neighbourhoods of "+"- and "−"-vertices of our state look on the surface like three meeting bands:

(14.1.6.1)


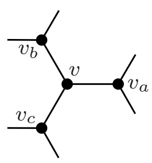
Switching a mark (value of the state) at a vertex means regleuing of the three bands along two chords on the surface:

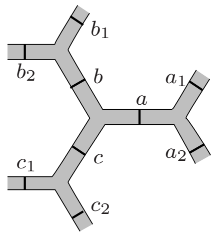


Proof. We are going to divide the set of all those states s for which the state surface $\Sigma_s(p)$ has one boundary component into pairs in such a way that the states s and s' of the same pair differ by an odd number of marks. The terms of the pairs will cancel each other and will contribute zero to $S_A(p)$.

In fact, to do this we shall adjust only marks of the four vertices of the fragment pictured in the lemma. The marks $\varepsilon_1, \dots, \varepsilon_l$ and $\varepsilon'_1, \dots, \varepsilon'_l$ in the states s and s' will be the same except for some marks of the four vertices of the fragment. Denote the vertices by v, v_a, v_b, v_c and their marks in the state s by ε, ε_a, ε_b, ε_c, respectively.

Assume that $\Sigma_s(p)$ has one boundary component. Modifying the surface as in (14.1.6.1) we can suppose that the neighbourhood of the fragment has the form

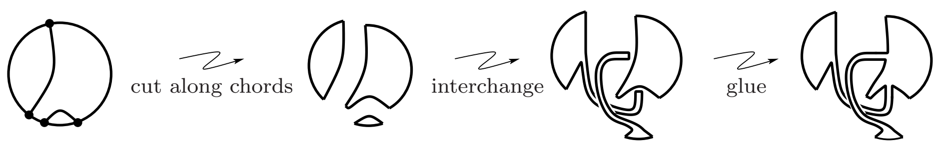




Draw nine chords a, a_1, a_2, b, b_1, b_2, c, c_1, c_2 on our surface as shown on the picture. The chords a, b, c are located near the vertex v; a, a_1, a_2 near the vertex v_a; b, b_1, b_2 near v_b and c, c_1, c_2 near v_c.

Since our surface has only one boundary component, we can draw it as a plane circle and a, a_1, a_2, b, b_1, b_2, c, c_1, c_2 as chords inside it. Look at the possible chord diagrams thus obtained.

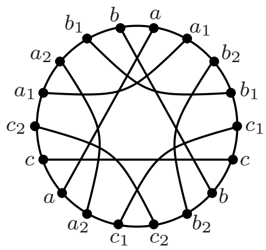
If two, say b and c, of three chords located near a vertex, say v, do not intersect, then the surface $\Sigma_{\dots,-\varepsilon,\varepsilon_a,\varepsilon_b,\varepsilon_c,\dots}(P)$ obtained by switching the mark ε to $-\varepsilon$ has only one boundary component too. Indeed, the regluing effect along two non-intersecting chords can be seen on chord diagrams as follows:



So, in this case, the state $s = \{\dots, -\varepsilon, \varepsilon_a, \varepsilon_b, \varepsilon_c, \dots\}$ should be paired with $s' = \{\dots, \varepsilon, \varepsilon_a, \varepsilon_b, \varepsilon_c, \dots\}$.

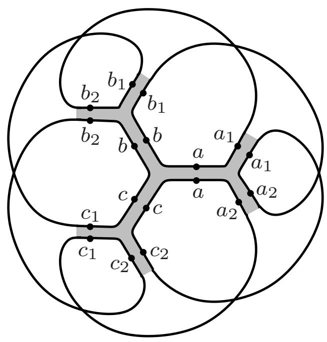
Therefore, switching of a mark at a vertex increases the number of boundary components (and then such a marked diagram may give a non-zero contribution to $S_A(D)$) if and only if the three chords located near the vertex intersect pairwise.

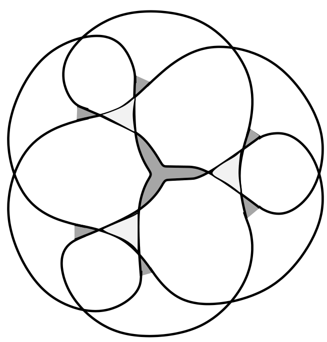
Now we can suppose that any two of the three chords in each triple (a,b,c), (a,a_1,a_2), (b,b_1,b_2), (c,c_1,c_2) intersect. This leaves us with only one possible chord diagram:



So the boundary curve of the surface connects the ends of our fragment as in the left picture below.

$\Sigma_{\ldots,\varepsilon,\varepsilon_a,\varepsilon_b,\varepsilon_c,\ldots}(p)$ $\Sigma_{\ldots,\varepsilon,-\varepsilon_a,-\varepsilon_b,-\varepsilon_c,\ldots}(p)$





Switching marks at v_a, v_b, v_c gives a surface also with one boundary component as in the right picture above. Pairing the state $s = \{\ldots,\varepsilon,\varepsilon_a,\varepsilon_b,\varepsilon_c,\ldots\}$ up with $s' = \{\ldots,\varepsilon,-\varepsilon_a,-\varepsilon_b,-\varepsilon_c,\ldots\}$ we get the desired result.

The Lemma and thus the Melvin–Morton conjecture are proved. □

14.1.7. Generalization of the Melvin–Morton conjecture to other quantum invariants. Let $\mathfrak{g}$ be a semi-simple Lie algebra and let V_λ be an irreducible representation of $\mathfrak{g}$ of the highest weight λ. Denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, by R the set of all roots and by R^+ the set of positive roots. Let $\langle\cdot,\cdot\rangle$ be the scalar product on $\mathfrak{h}^*$ induced by the Killing form. These data define the unframed quantum invariant $\theta_{\mathfrak{g}}^{V_\lambda}$ which after the substitution $q = e^h$ and the expansion into a power series in h can be written as (see Section 11.2.3)

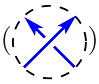
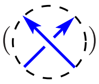
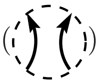
$$\theta_{\mathfrak{g}}^{V_\lambda} = \sum_{n=0}^{\infty} \theta_{\mathfrak{g},n}^{\lambda} h^n.$$

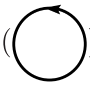
Theorem. ([**BNG**]).
1) *The invariant* $\theta_{\mathfrak{g},n}^{\lambda}/\dim(V_\lambda)$ *is a polynomial in* λ *of degree at most* n.

2) *Define the Bar-Natan–Garoufalidis function* BNG *as a power series in* h *whose coefficient at* h^n *is the degree* n *part of the polynomial* $\theta^{\lambda}_{\mathfrak{g},n}/\dim(V_\lambda)$. *Then for any knot* K,

$$BNG(K) \cdot \prod_{\alpha \in R^+} A_\alpha(K) = 1\,,$$

where A_α *is the following normalization of the Alexander–Conway polynomial:*

$A_\alpha($  $)$ $-$ $A_\alpha($  $)$ $=$ $\left(e^{\frac{\langle\lambda,\alpha\rangle h}{2}} - e^{-\frac{\langle\lambda,\alpha\rangle h}{2}}\right) A_\alpha($  $)\,;$

$A_\alpha($  $)$ $=$ $\dfrac{\langle\lambda,\alpha\rangle h}{e^{\frac{\langle\lambda,\alpha\rangle h}{2}} - e^{-\frac{\langle\lambda,\alpha\rangle h}{2}}}\,.$

Proof. The symbol S_{BNG} is the highest part (as a function of λ) of the Lie algebra weight system $\varphi'^{V_\lambda}_{\mathfrak{g}}$ associated with the representation V_λ. According to exercise (4), the symbol of A_α in degree n equals $\langle\lambda,\alpha\rangle^n \mathrm{symb}(c_n)$.

The relation between invariants can be reduced to the following relation between their symbols:

$$S_{BNG}\big|_{\mathcal{P}_n} + \sum_{\alpha\in R^+} \langle\lambda,\alpha\rangle^n \mathrm{symb}(c_n)\big|_{\mathcal{P}_n} = 0\,,$$

for $n > 1$.

As above, $S_{BNG}\big|_{\mathcal{P}_n^{n-1}} = \mathrm{symb}(c_n)\big|_{\mathcal{P}_n^{n-1}} = 0$, and $\mathrm{symb}(c_n)(w_{2m}) = -2$. Thus it remains to prove that

$$S_{BNG}(w_{2m}) = 2 \sum_{\alpha\in R^+} \langle\lambda,\alpha\rangle^{2m}\,.$$

To prove this equality we shall use the method of Section 6.2. First, we take the Weyl basis of $\mathfrak{g}$ and write the Lie bracket tensor J in this basis.

Fix the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \big(\bigoplus_{\alpha\in R} \mathfrak{g}_\alpha\big)$. The Cartan subalgebra $\mathfrak{h}$ is orthogonal to all the $\mathfrak{g}_\alpha$'s and $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ for $\beta \neq -\alpha$. Choose the elements $e_\alpha \in \mathfrak{g}_\alpha$ and $h_\alpha = [e_\alpha, e_{-\alpha}] \in \mathfrak{h}$ for each $\alpha \in R$ in such a way that $\langle e_\alpha, e_{-\alpha}\rangle = 2/\langle\alpha,\alpha\rangle$, and for any $\lambda \in \mathfrak{h}^*$, $\lambda(h_\alpha) = 2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle$.

The elements $\{h_\beta, e_\alpha\}$, where β belongs to a basis $B(R)$ of R and $\alpha \in R$, form the Weyl basis of $\mathfrak{g}$. The Lie bracket $[\cdot,\cdot]$ as an element of $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$

can be written as follows:

$$
\begin{aligned}
[\cdot,\cdot] \quad = \quad & \sum_{\substack{\beta\in B(R)\\ \alpha\in R}} \Big(h^*_\beta \otimes e^*_\alpha \otimes \alpha(h_\beta) e_\alpha - e^*_\alpha \otimes h^*_\beta \otimes \alpha(h_\beta) e_\alpha \Big) \\
& + \sum_{\alpha\in R} e^*_\alpha \otimes e^*_{-\alpha} \otimes h_\alpha + \sum_{\substack{\alpha,\gamma\in R\\ \alpha+\gamma\in R}} e^*_\alpha \otimes e^*_\gamma \otimes N_{\alpha,\gamma} e_{\alpha+\gamma}\,,
\end{aligned}
$$

where the stars indicate elements of the dual basis. The second sum is most important because the first and third ones give no contribution to the Bar-Natan–Garoufalidis weight system S_{BNG}.

After identification of $\mathfrak{g}^*$ and $\mathfrak{g}$ via $\langle\cdot,\cdot\rangle$ we get $e^*_\alpha = (\langle\alpha,\alpha\rangle/2)e_{-\alpha}$. In particular, the second sum of the tensor J is

$$
\sum_{\alpha\in R} \Big(\langle\alpha,\alpha\rangle/2\Big)^2 e_{-\alpha} \otimes e_\alpha \otimes h_\alpha\,.
$$

According to Secction 6.2, to calculate $S_{BNG}(w_{2m})$ we must assign a copy of the tensor $-J$ with each internal vertex and then make all contractions corresponding to internal edges. After that take the product $\rho_{\mathfrak{g}}^{V_\lambda}(w_{2m})$ of all operators in V_λ corresponding to external vertices. $\rho_{\mathfrak{g}}^{V_\lambda}(w_{2m})$ is a scalar operator of multiplication by a certain constant. This constant is a polynomial in λ of degree at most $2m$. Its part of degree $2m$ is $S_{BNG}(w_{2m})$.

We associate the tensor $-J$ with an internal vertex according to the cyclic ordering of the three edges in such a way that the third tensor factor of $-J$ corresponds to the edge connecting the vertex with an external vertex. After that we take the product of operators corresponding to these external vertices. This means that we take the product of operators corresponding to the third tensor factor of $-J$. Of course, we are interested only in those operators which are linear in λ. One can show (see, for example, [**BNG**, Lemma 5.1]) that it is possible to choose a basis in the space of the representation V_λ in such a way that the Cartan operators h_α and raising operators e_α ($\alpha\in R^+$) will be linear in λ while the lowering operators $e_{-\alpha}$ ($\alpha\in R^+$) will not depend on λ. So we have to take into account only those summands of $-J$ that have h_α or e_α ($\alpha\in R^+$) as the third tensor factor. Further, to calculate the multiplication constant of our product it is sufficient to act by the operator on any vector. Let us choose the highest weight vector v_0 for this. The Cartan operators h_α multiply v_0 by $\lambda(h_\alpha) = 2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle$. So indeed they are linear in λ. But the raising operators e_α ($\alpha\in R^+$) send v_0 to zero. This means that we have to take into account only those summands of $-J$ whose third tensor factor is one of the h_α's. This is exactly the second sum of J with the opposite sign:

$$\sum_{\alpha\in R}\Big(\langle\alpha,\alpha\rangle/2\Big)^2 e_\alpha\otimes e_{-\alpha}\otimes h_\alpha.$$

Now making all contractions corresponding to the edges connecting internal vertices of w_{2m} we get the tensor:

$$\sum_{\alpha\in R}\Big(\langle\alpha,\alpha\rangle/2\Big)^{2m}\underbrace{h_\alpha\otimes\ldots\otimes h_\alpha}_{2m\ \text{times}}.$$

The corresponding element of $U(\Gamma)$ acts on the highest weight vector v_0 as multiplication by

$$S_{BNG}(w_{2m})=\sum_{\alpha\in R}\langle\lambda,\alpha\rangle^{2m}=2\sum_{\alpha\in R^+}\langle\lambda,\alpha\rangle^{2m}.$$

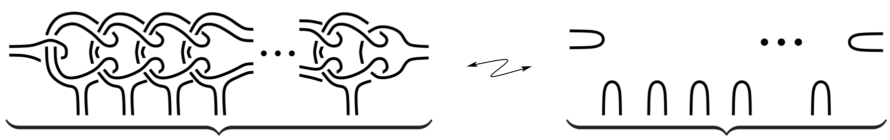
The theorem is proved. □

14.2. The Goussarov–Habiro theory

14.2.1. Formulation of the Goussarov–Habiro theorem.

14.2.1.1. In September 1995, at a conference in Oberwolfach, Michael Goussarov reported about a theorem describing the pairs of knots indistinguishable by Vassiliev invariants of order $\leqslant n$. As usual with Goussarov's results the corresponding publication [**G4**] appeared several years later. K. Habiro independently found the same theorem [**Ha1, Ha2**]. In this chapter we will discuss a version of this theorem and related results. Other approaches to the Goussarov–Habiro type theorems can be found in [**CT, Sta3, TY**].

Theorem (Goussarov–Habiro)**.** *Let K_1 and K_2 be two knots. Then $v(K_1) = v(K_2)$ for any $\mathbb{Z}$-valued Vassiliev invariant v of order $\leqslant n$ if and only if K_1 and K_2 are related by a finite sequence of moves $\mathcal{M}_n$:*



$n+2$ components *$n+2$ components*

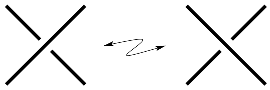
14.2.1.2. Denote by $\mathcal{B}_n$ (resp. $\mathcal{T}_n$) the tangle on the left (resp. right) side of the move $\mathcal{M}_n$. The tangle $\mathcal{B}_n$ is an example of *Brunnian tangles* characterized by the property that removing any of its components makes the remaining tangle to be isotopic to the trivial tangle $\mathcal{T}_{n-1}$ with $n+1$ components.

We start the series of moves $\mathcal{M}_n$ with $n = 0$:

$\mathcal{M}_0 : \quad \mathcal{B}_0 =$  $= \mathcal{T}_0$

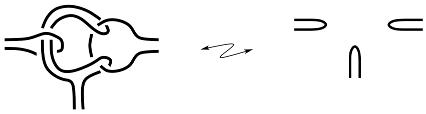
which is equivalent to the usual change of crossing.

14.2.2. Exercise. Represent the change of crossing move

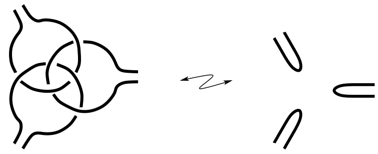


as a composition of M_0 and an appropriate Reidemeister move of second type.

For $n = 1$, the move $\mathcal{M}_1$ looks like

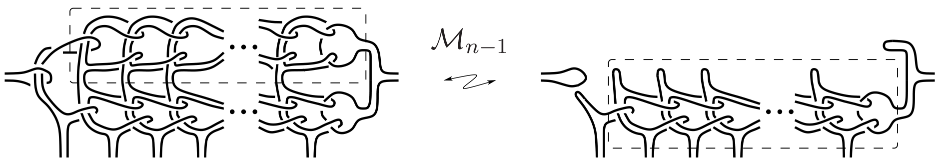
$\mathcal{M}_1 : \quad \mathcal{B}_1 =$  $= \mathcal{T}_1$

It is also known as the *Borromean move*



Since there are no invariants of order $\leqslant 1$ except constants (Proposition 3.3.2), the Goussarov–Habiro theorem implies that any knot can be transformed to the unknot by a finite sequence of Borromean moves $\mathcal{M}_1$, i.e. $\mathcal{M}_1$ is an *unknotting operation*.

14.2.3. Remark. Coincidence of all Vassiliev invariants of order $\leqslant n$ implies coincidence of all Vassiliev invariants of order $\leqslant n-1$. This means that one can accomplish a move $\mathcal{M}_n$ by a sequence of moves $\mathcal{M}_{n-1}$. Indeed, let us deform the tangle $\mathcal{B}_n$ to the tangle on the left in the following picture:



(to see that this is indeed $\mathcal{B}_n$, look at the picture closely and try to untangle what can be untangled, working from right to left). Then the tangle in the dashed rectangle is $\mathcal{B}_{n-1}$. To perform the move $\mathcal{M}_{n-1}$ we must cut it and paste $\mathcal{T}_{n-1}$ instead. This gives us the tangle on the right also containing $\mathcal{B}_{n-1}$. Now performing once more the move $\mathcal{M}_{n-1}$ we obtain the trivial tangle $\mathcal{T}_n$.

14.2.4. Reformulation of the Goussarov–Habiro theorem.

14.2.4.1. Recall that we denoted by $\mathcal{K}$ the set of all (isotopy classes of) knots, by $\mathbb{Z}\mathcal{K}$ the free $\mathbb{Z}$-module (even an algebra) consisting of all finite formal $\mathbb{Z}$-linear combinations of knots (see Section 1.6) and by $\mathcal{K}_n$ the singular knot filtration in $\mathbb{Z}\mathcal{K}$ (see 3.2.1). Using the moves $\mathcal{M}_n$, we can introduce another filtration in the module $\mathbb{Z}\mathcal{K}$.

Definition. Let $\mathcal{H}_n$ denote the $\mathbb{Z}$-submodule of $\mathbb{Z}[\mathcal{K}]$ spanned by the differences of two knots obtained one from another by a single move $\mathcal{M}_n$. For example, the difference $3_1 - 6_3$ belongs to $\mathcal{H}_2$.

14.2.4.2. Goussarov–Habiro's theorem can be reformulated as follows.

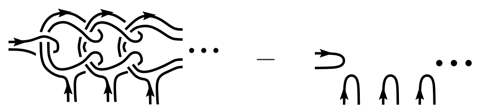
Theorem. *For all n the submodules $\mathcal{K}_n$ and $\mathcal{H}_n$ coincide.*

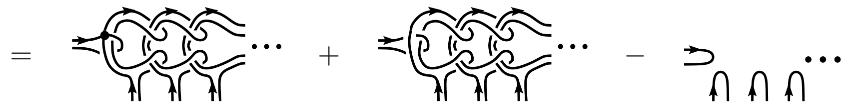
Proof. (Of the equivalence of the two statements.) Indeed, if one knot K_1 can be obtained from another one K_2 by a finite sequence of moves $\mathcal{M}_n$, then the difference $K_1 - K_2$ belongs to $\mathcal{H}_n$. Therefore, by theorem 2.2.2, it belongs to $\mathcal{K}_n$, i.e., it is equal to a linear combination of knots with $> n$ double points. Hence $v(K_1) = v(K_2)$ for any Vassiliev invariant v of order $\leqslant n$. Conversely, if $v(K_1) = v(K_2)$ for any Vassiliev invariant v of order $\leqslant n$, then $K_1 - K_2$ belongs to $\mathcal{K}_n$ and, by theorem 14.2.4.2, to $\mathcal{H}_n$. Therefore $K_1 - K_2$ is equal to a sum of differences of pairs of knots obtained one from another by a single $\mathcal{M}_n$ move. This means that all summands in this sum can be canceled except for the two knots K_1 and K_2. This allows us to rearrange the pairs in this sum in such a way that K_1 occurs in the first pair and $-K_2$ occurs in the last pair and two consecutive pairs contain the isotopic knots, one with plus one and another with minus one coefficients. Obviously such an order of pairs gives us a sequence of $\mathcal{M}_n$ moves that transform K_1 to K_2. Hence Goussarov–Habiro's theorem 14.2.1.1 follows from theorem 14.2.4.2. In the same way theorem 14.2.4.2 follows from Goussarov–Habiro's theorem. So they are equivalent. $\square$

To prove theorem 14.2.4.2 we have to prove two inclusions: $\mathcal{H}_n \subseteq \mathcal{K}_n$ and $\mathcal{K}_n \subseteq \mathcal{H}_n$. The first one is an easy task. We do it in the next section. The hard part — proof of the second inclusion — can be found in [**G4, Ha1, Ha2, Sta3, TY**].

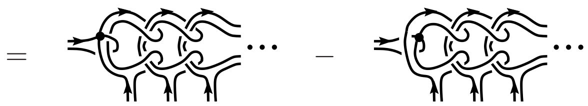
14.2.5. Proof of sufficiency.

14.2.5.1. In this section we prove that $\mathcal{H}_n \subseteq \mathcal{K}_n$. For this it is enough to represent the difference $\mathcal{B}_n - \mathcal{T}_n$ as a linear combination of tangles with $n+1$ double points in each. Let us choose the orientations of the components of our tangles as shown. We are going to use the Vassiliev skein relation and gradually transform the difference $\mathcal{B}_n - \mathcal{T}_n$ into the required form.

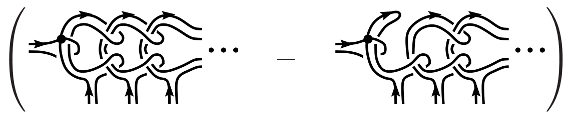
$\mathcal{B}_n - \mathcal{T}_n =$ 

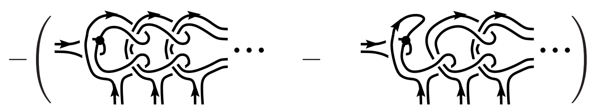


But the difference of the last two tangles can be expressed as a singular tangle:

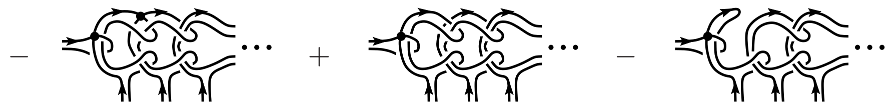


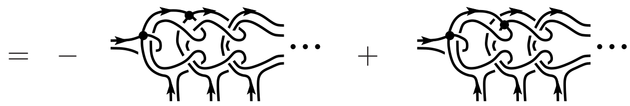
We got a presentation of $\mathcal{B}_n - \mathcal{T}_n$ as a linear combination of two tangles with one double point in each between the first and the second components of the tangles. Now we add and subtract isotopic singular tangles with one double point:

$\mathcal{B}_n - \mathcal{T}_n =$ 

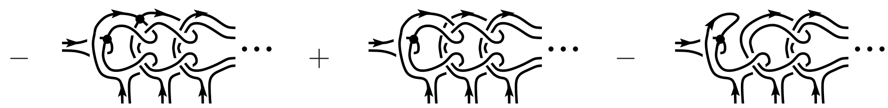


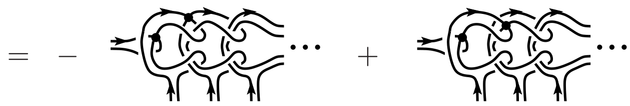
Then using the Vassiliev skein relation we can see that the difference in the first pair of parentheses is equal to



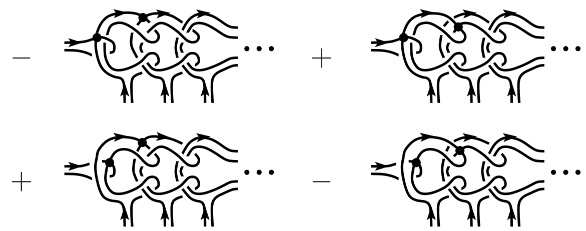


Similarly the difference in the second pair of parentheses would be equal to



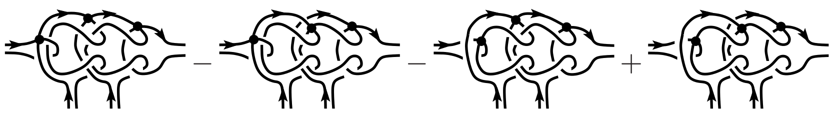


So we have represented $\mathcal{B}_n - \mathcal{T}_n$ as a linear combination of four tangles with two double points in each, one is between the first and the second components of the tangles, and another one is between the second and third components:

$\mathcal{B}_n - \mathcal{T}_n \quad =$ 

Continuing in the same way we come to a linear combination of 2^n tangles with $n+1$ double points in each occurring between consecutive components. It is easy to see that if we change the orientations of arbitrary k components of our tangles $\mathcal{B}_n$ and $\mathcal{T}_n$, then the whole linear combination will be multiplied by $(-1)^k$.

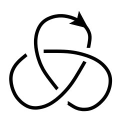
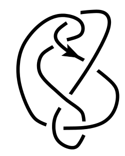
14.2.6. Example.

$\mathcal{B}_2 - \mathcal{T}_2 =$ 

14.2.7. Invariants of order 2.

14.2.8. Example. There is only one (up to proportionality and adding a constant) nontrivial Vassiliev invariant of order $\leqslant 2$. It is the coefficient c_2 of the Conway polynomial defined in Sec. 3.7.2.

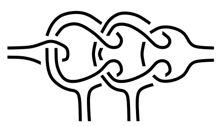
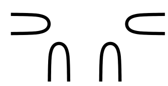
Consider two knots

$3_1 \quad =$  , $\qquad 6_3 \quad =$  .

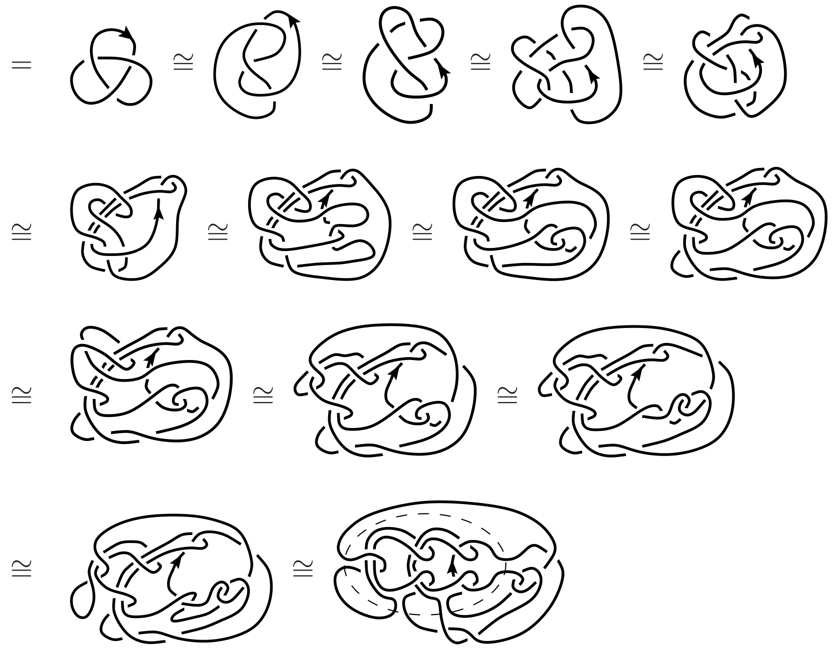
We choose the orientations as indicated. Their Conway polynomials

$$C(3_1) = 1 + t^2\ , \qquad\qquad C(6_3) = 1 + t^2 + t^4.$$

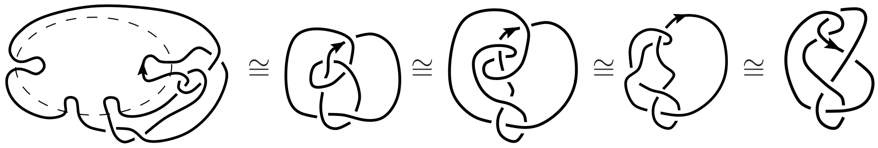
have equal coefficients of t^2. Therefore for any Vassiliev invariant v of order $\leqslant 2$ we have $v(3_1) = v(6_3)$. In this case the Goussarov–Habiro theorem states that it is possible to obtain the knot 6_3 from the knot 3_1 by moves $\mathcal{M}_2 : \mathcal{B}_2 \leftrightsquigarrow \mathcal{T}_2$

$\mathcal{M}_2 :$  $\leftrightsquigarrow$ 

Let us show this. We start with the standard diagram of 3_1, and then transform it in order to specify the tangle $\mathcal{B}_2$.

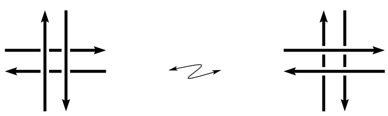
$3_1 =$ 

Now we have the tangle $\mathcal{B}_2$ in the dashed oval. To perform the move $\mathcal{M}_2$ we must replace it by the trivial tangle $\mathcal{T}_2$:

 $= 6_3$

14.2.8.1. The *mod* 2 reduction of c_2 is called the *Arf invariant* of a knot. A description of the Arf invariant similar to the Goussarov–Habiro description of c_2 was obtained by L. Kauffman.

Theorem. *(L.Kauffman* [**Ka1, Ka2**]*). K_1 and K_2 have the same Arf invariant if and only if K_1 can be obtained from K_2 by a finite number of so called pass-moves:*



The orientations are important. Allowing pass-moves with arbitrary orientations we obtain an unknotting operation (see [**Kaw2**]).

14.2.8.2. *Open problem.* (L. Kauffman) Find a set of moves relating the knots with the same c_2 modulo n, for $n = 3, 4, \dots$.

14.2.8.3. *Open problem.* Find a set of moves relating any two knots with the same Vassiliev invariants modulo 2 (3, 4, ...) up to the order n.

14.2.8.4. *Open problem.* Find a set of moves relating any two knots with the same Conway polynomial.

The paper by S. Naik and T. Stanford [**NaS**] might be useful in studying these problems. It describes the moves relating two knots with the same Seifert matrix.

14.2.9. The Goussarov groups.

14.2.10. Definition. Two knots K_1 and K_2 are called *n-equivalent* if they cannot be distinguished by $\mathbb{Z}$-valued Vassiliev invariants of order $\leqslant n$, i.e., $v(K_1) = v(K_2)$ for any $v \in \mathcal{V}_n^{\mathbb{Z}}$.

Let $\mathcal{G}_n$ be the set of n-equivalence classes of knots.

14.2.11. Theorem (Goussarov [**G1**])**.** *The set $\mathcal{G}_n$ is an Abelian group with respect to the connected sum of knots.*

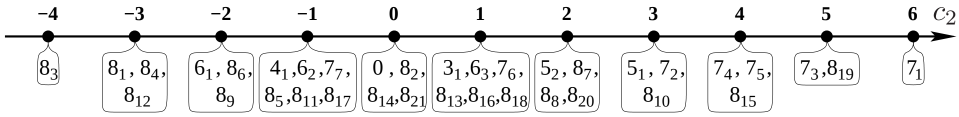
We call the group $\mathcal{G}_n$ the n-th Goussarov group.

Since there are no Vassiliev invariants of order $\leqslant 1$ except constants, the zeroth and the first Goussarov groups are trivial. The proof of the theorem in [**G1**] is an explicit construction of an n-inverse to a given knot K, i.e. such a knot K_n^{-1} that the connected sum $K\#K_n^{-1}$ has all trivial Vassiliev invariants up to the order n. The proof proceeds by induction on n and gives the presentation of the n-inverse K_n^{-1} as a connected sum $K_n^{-1} = K_2\#K_3\#\dots\#K_n$ such that for any $j = 2,\dots,n$ the truncated connected sum $K_2\#K_3\#\dots\#K_j$ is a j-inverse to K.

Exercise. Following the proofs given in [**G1**] and [**Sta3**], construct the 2-inverse of the trefoil 3_1.

In this section we confine ourselves with a description of the groups $\mathcal{G}_2$, $\mathcal{G}_3$.

14.2.11.1. *The second Goussarov group $\mathcal{G}_2$.* Consider the coefficient c_2 of t^2 in the Conway polynomial $C(K)$. According to exercise 6 at the end of Chapter 2, $C(K_1\#K_2) = C(K_1)\cdot C(K_2)$ and $C(K)$ has the form $C(K) = 1 + c_2(K)t^2 + \dots$. These facts imply that $c_2(K_1\#K_2) = c_2(K_1) + c_2(K_2)$. Therefore c_2 is a homomorphism of $\mathcal{G}_2$ into $\mathbb{Z}$. Since c_2 is the only nontrivial invariant of order $\leqslant 2$ and takes value 1 on certain knots, the homomorphism $c_2 : \mathcal{G}_2 \to \mathbb{Z}$ is in fact an isomorphism. So $\mathcal{G}_2 \cong \mathbb{Z}$. From the table in Sec.2.3.3 we can see that $c_2(3_1) = 1$ and $c_2(4_1) = -1$. This means that the knot 3_1 represents a generator of $\mathcal{G}_2$, and 4_1 is 2-inverse of 3_1. The prime knots with up to 8 crossings are distributed in the second Goussarov group $\mathcal{G}_2$ as follows.



14.2.11.2. *The third Goussarov group $\mathcal{G}_3$.* In order 3 we have one more Vassiliev invariant j_3, the coefficient at h^3 in the power series expansion of the Jones polynomial with substitution $t = e^h$. The Jones polynomial is multiplicative, $J(K_1\#K_2) = J(K_1)\cdot J(K_2)$ (see exercise 7 at the end of Chapter 2) and its expansion has the form $J(K) = 1 + j_2(K)h^2 + j_3(K)h^3 + \dots$ (see Sec. 3.6). Thus we can write

$$J(K_1\#K_2) = 1 + (j_2(K_1) + j_2(K_2))\,h^2 + (j_3(K_1) + j_3(K_2))\,h^3 + \dots$$

In particular, $j_3(K_1\#K_2) = j_3(K_1) + j_3(K_2)$. According to exercise 6 at the end of Chapter 3, j_3 is divisible by 6. Then $j_3/6$ is a homomorphism from $\mathcal{G}_3$ to $\mathbb{Z}$. Together with c_2 we get the isomorphism

$$\mathcal{G}_3 \cong \mathbb{Z}\oplus\mathbb{Z} = \mathbb{Z}^2; \qquad\qquad K \mapsto (c_2(K), j_3(K)/6)$$

Let us identify $\mathcal{G}_3$ with the integral lattice on a plane. The distribution of prime knots on this lattice is shown in Figure 14.2.11.1 (numbers with a bar refer to knots that differ from the corresponding table knots (see page 26) by a mirror reflection).

In particular, the 3-inverse of the trefoil 3_1 can be represented by 6_2, or $\overline{7_7}$. Also we can see that $3_1\#4_1$ is 3-equivalent to $\overline{8_2}$. Therefore $3_1\#4_1\#8_2$ is 3-equivalent to the unknot, and $4_1\#8_2$ also represents the 3-inverse to 3_1. The knots 6_3 and $\overline{8_2}$ represent the standard generators of $\mathcal{G}_3$.

14.2.11.3. *Open problem.* Is there any torsion in the group $\mathcal{G}_n$?

14.3. Bialgebra of graphs

It turns out that the natural mapping that assigns to every chord diagram its intersection graph, can be converted into a homomorphism of bialgebras $\gamma : \mathcal{A} \to \mathcal{L}$, where $\mathcal{A}$ is the algebra of chord diagrams and $\mathcal{L}$ is an algebra generated by graphs modulo certain relations which was introduced by S. Lando [**Lnd1**]. Here is his construction.

Let $\mathfrak{G}$ be the graded vector space (over a field $\mathbb{F}$) spanned by all simple graphs (without loops nor multiple edges) as free generators:

$$\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1 \oplus \mathfrak{G}_2 \oplus \dots,$$

It is graded by the order (the number of vertices) of a graph. This space is easily turned into a bialgebra:

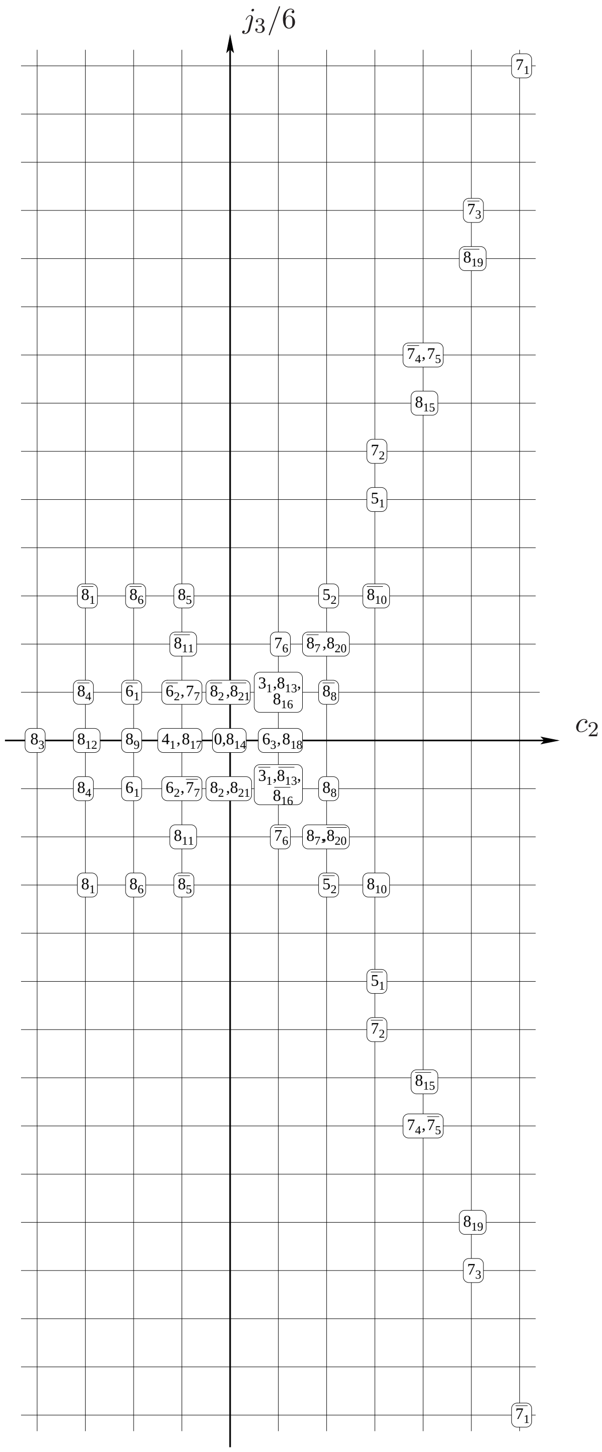


Figure 14.2.11.1. Values of Vassiliev invariants c_2 and j_3 on small prime knots

1. The multiplication is defined as the disjoint union of graphs, then extended by linearity. The empty graph plays the role of the unit in this algebra.

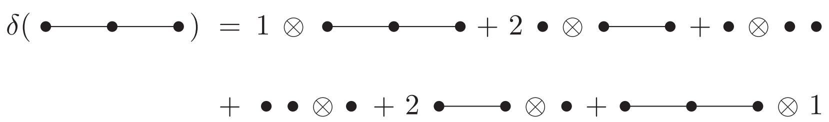
2. The comultiplication is defined similarly to the comultiplication in the bialgebra of chord diagrams. If G is a graph, let $V = V(G)$ be the set of its vertices. For any subset $U \subset V$ denote by $G(U)$ the graph with the set of vertices U and those vertices of the graph G whose both endpoints belong to G. We put by definition:

$$\delta(G) = \sum_{U \subseteq V(G)} G(U) \otimes G(V \setminus U), \tag{14.3.0.1}$$

and extend by linearity to the whole of $\mathfrak{G}$.

The sum in (14.3.0.1) is taken over all subsets $U \subset V$ and contains as many as $2^{\#(V)}$ summands.

Example of a coproduct.



Exercise. Check the axioms of a Hopf algebra for $\mathfrak{G}$.

The mapping from chord diagrams to intersection graphs does not extend to a linear operator $\mathcal{A} \to \mathfrak{G}$, because the combinations of graphs that correspond to 4-term relations for chord diagrams, do not vanish in $\mathfrak{G}$. To obtain a linear mapping, it is necessary to mod out the space $\mathfrak{G}$ by relations, consistent with the 4 term relations. Here is an appropriate definition.

Let G be an arbitrary graph and u, v an ordered pair of its vertices. The pair u, v defines two transformations of the graph G: $G \mapsto G'_{uv}$ and $G \mapsto \widetilde{G}_{uv}$. Both graphs G'_{uv} and $\widetilde{G}_{uv}$ have the same set of vertices as G. They are obtained as follows.

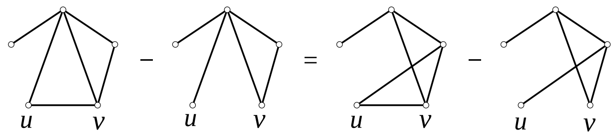
If uv is an edge in G, then the graph G'_{uv} is obtained from G by deleting the edge uv; otherwise this edge should be added (thus, $G \mapsto G'_{uv}$ is toggling the adjacency of u and v).

The graph $\widetilde{G}_{uv}$ is obtained from G in a more tricky way. Consider all vertices $w \in V(G) \setminus \{u, v\}$ which are adjacent in G with v. Then in the graph $\widetilde{G}_{uv}$ vertices u and w are joined by an edge if and only if they are not joined in G. For all other pairs of vertices their adjacency in G and in $\widetilde{G}_{uv}$ is the same. Note that the two operations applied at the same pair of vertices, commute and hence the graph G'_{uv} is well-defined.

14.3.1. Definition. A four-term relation for graphs is

$$G - G'_{uv} = \widetilde{G}_{uv} - \widetilde{G}'_{uv} \tag{14.3.1.1}$$

Example.



Exercises. 1. Check that, passing to intersection graphs, the four-term relation for chord diagram carries over exactly into this four-term relation for graphs. 2. Find the four-term relation of chord diagrams which is the preimage of the relation shown in the previous example.

14.3.2. Definition. The graph bialgebra of Lando $\mathcal{L}$ is the quotient of the graph algebra $\mathfrak{G}$ over the ideal generated by all 4-term relations (14.3.1.1).

14.3.3. Theorem. *The multiplication and the comultiplication defined above induce a bialgebra structure in the quotient space $\mathcal{L}$.*

Proof. The only thing that needs checking is that the multiplication and the comultiplication both respect the 4-term relation 14.3.1.1. For the multiplication (disjoint union of graphs) this statement is obvious. In order to verify it for the comultiplication, it is sufficient to consider two different cases. Namely, let $u, v \in V(G)$ be two distinct vertices of a graph G. The right-hand side summands in the comultiplication formula (14.3.0.1) split into two groups: those where both vertices u and v belong either to the subset $U \subset V(G)$ or to its complement $V(G) \setminus U$; and those where u and v belong to different subsets. By cleverly grouping the terms of the first kind for the coproduct $\delta(G - G'_{uv} - \widetilde{G}_{uv} + \widetilde{G}'_{uv})$ they all cancel out in pairs. The terms of the second kind can be paired to mutually cancel already for each of the two summands $\delta(G - G'_{uv})$ and $\delta(\widetilde{G}_{uv} - \widetilde{G}'_{uv})$. The theorem is proved. $\square$

Relations 14.3.1.1 are homogeneous with respect to the number of vertices, therefore $\mathcal{L}$ is a graded algebra. By Theorem A.2.25 (p. 430), the algebra $\mathcal{L}$ is polynomial over its space of primitive elements.

Now we have a well-defined bialgebra homomorphism

$$\gamma : \mathcal{A} \to \mathcal{L}$$

which extends the assignment of the intersection graph to a chord diagram. It is defined by the linear mapping between the corresponding primitive spaces $P(\mathcal{A}) \to P(\mathcal{L})$.

According to S. Lando [**Lnd1**], the dimensions of the homogeneous components of $P(\mathcal{L})$ are known up to degree 7. It turns out that the homomorphism γ is an isomorphism in degrees up to 6, while the mapping $\gamma : P_7(\mathcal{A}) \to P_7(\mathcal{L})$ has a 1-dimensional kernel. See [**Lnd1**] for further details and open problems related to the algebra $\mathcal{L}$.

14.4. Estimates for the number of Vassiliev knot invariants

The knowledge of $\dim \mathcal{P}_i$ for $i \leqslant d$ is equivalent to the knowledge of $\dim \mathcal{A}_i$, $i \leqslant d$, or $\dim \mathcal{V}_i$, $i \leqslant d$. At present, the exact asymptotic behavior of these numbers as d tends to infinity is not known. Below, we summarize the known results on the exact enumeration as well as on the asymptotic lower and upper bounds.

14.4.1. Historical remarks.

14.4.1.1. *Exact results.* The exact dimensions of the spaces related to Vassiliev invariants are known up to $n = 12$. The results are shown in the table below. They were obtained by V. Vassiliev for $n \leqslant 5$ in 1990 [**Va2**], then by D. Bar-Natan for $n \leqslant 9$ in 1993 [**BN1**] and by J. Kneissler (for $n = 10, 11, 12$) in 1997 [**Kn0**]. More detail information about the dimensions of the primitive spaces see in the table on page 139.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{P}_n$	0	0	1	1	2	3	5	8	12	18	27	39	55
$\dim \mathcal{A}_n$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{V}_n$	1	1	2	3	6	10	19	33	60	104	184	316	548

14.4.1.2. *Upper bounds.* A priori it was obvious that $\dim \mathcal{A}_d < (2d-1)!! = 1 \cdot 3 \cdots (2d-1)$, because this is the total number of chord diagrams before factorization over the rotations.

Then, there appeared five papers where this estimate was successively improved:

(1) (1993) Chmutov and Duzhin [**CD1**] proved that $\dim \mathcal{A}_d < (d-1)!$

(2) (1995) K. Ng in [**Ng**] replaced $(d-1)!$ by $(d-2)!/2$.

(3) (1996) A. Stoimenow [**Sto1**] proved that $\dim \mathcal{A}_d$ grows slower than $d!/a^d$, where $a = 1.1$.

(4) (2000) B. Bollobás and O. Riordan [**BR1**] obtained the asymptotical bound $d!/(2\ln(2) + o(1))^d$ (approximately $d!/1.38^d$).

(5) (2001) D. Zagier [**Zag1**] improved the last result to $\frac{6^d\sqrt{d}\cdot d!}{\pi^{2d}}$, which is asymptotically smaller than $d!/a^d$ for any constant $a < \pi^2/6 = 1.644...$

14.4.1.3. *Lower bounds.* In the story of lower bounds for the number of Vassiliev knot invariants there is a funny episode. The first paper by Kontsevich about Vassiliev invariants ([**Kon1**], section 3) contains the following passage:

"Using this construction [1], one can obtain the estimate

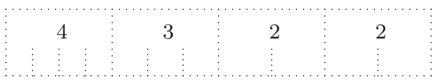
$$\dim(\mathcal{V}_n) > e^{c\sqrt{n}}, \quad n \to +\infty$$

for any positive constant $c < \pi\sqrt{2/3}$ (see [**BN1a**], Exercise 6.14)."

Here $\mathcal{V}_n$ is a slip of the pen, instead of $\mathcal{P}_n$, because of the reference to Exercise 6.14 where primitive elements are considered. Exercise 6.14 was present, however, only in the first edition of Bar-Natan's preprint and eliminated in the following editions as well as in the final published version of his text [**BN1**]. In [**BN1a**] it reads as follows (page 43):

"*Exercise* 6.14. (Kontsevich, [24]) Let $P_{\geqslant 2}(m)$ denote the number of partitions of an integer m into a sum of integers bigger than or equal to 2. Show that $\dim \mathcal{P}_m \geqslant P_{\geqslant 2}(m+1)$.

Hint 6.15. Use a correspondence like



$$\longleftrightarrow \qquad 10 + 1 = 4 + 3 + 2 + 2,$$

and ..."

The reference [24] was to "M. Kontsevich. Private communication."! Thus, both authors referred to each other, and none of them gave any proof. Later, however, Kontsevich explained what he had in mind (see item 5 below).

Arranged by time, the history of world records in asymptotic lower bounds for the dimension of the primitive space $\mathcal{P}_d$ looks as follows.

(1) (1994) $\dim \mathcal{P}_d \geqslant 1$ ("forest elements" found by Chmutov, Duzhin and Lando [**CDL3**]).
(2) (1995) $\dim \mathcal{P}_d \geqslant [d/2]$ (given by colored Jones function — see Melvin–Morton [**MeMo**] and Chmutov–Varchenko [**ChV**]).
(3) (1996) $\dim \mathcal{P}_d \gtrsim d^2/96$ (see Duzhin [**Du1**]).
(4) (1997) $\dim \mathcal{P}_d \gtrsim d^{\log d}$, i. e. the growth is faster than any polynomial (Chmutov–Duzhin [**CD2**]).
(5) (1997) $\dim \mathcal{P}_d > e^{\pi\sqrt{d/3}}$ (Kontsevich [**Kon2**]).
(6) (1997) $\dim \mathcal{P}_d > e^{C\sqrt{d}}$ for any constant $C < \pi\sqrt{2/3}$ (Dasbach [**Da3**]).

[1] Of Lie algebra weight systems.

14.4.2. Proof of the lower bound. We will sketch the proof of the lower bound for the number of Vassiliev knot invariants, following our paper [**CD2**] and then explain how O. Dasbach [**Da3**], using the same method, managed to improve the estimate and establish the bound which is still (2006) the best.

The idea of the proof is simple: we construct a large family of open diagrams whose linear independence in the algebra $\mathcal{B}$ follows from the linear independence of the values on these diagrams of a certain polynomial invariant P, which is obtained from the universal $\mathfrak{gl}_N$ invariant by means of a simplification.

As we know from Chapter 6, the $\mathfrak{gl}_N$ invariant $\psi_{\mathfrak{gl}_N}$, evaluated on an open diagram C, is a polynomial in the generalized Casimir elements x_0, x_1, ..., x_N. This polynomial is homogeneous in the sense of the grading defined by setting $\deg x_m = m$. However, it is in general not homogeneous if x_m's are considered as simple variables of degree 1.

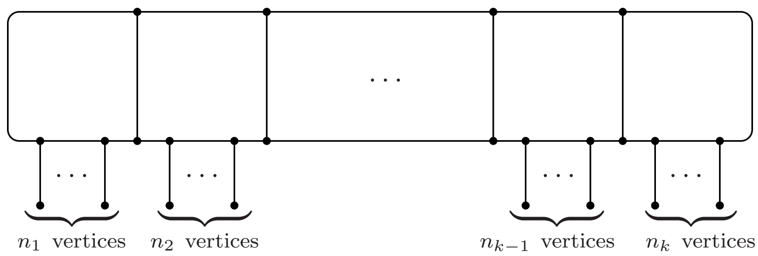
14.4.3. Definition. The polynomial invariant $P : \mathcal{B} \to \mathbb{Z}[x_0, ..., x_N]$ is the highest homogeneous part of $\psi_{\mathfrak{gl}_N}$ if all the variables are taken with degree 1.

For example, if we had $\psi_{\mathfrak{gl}_N}(C) = x_0^2 x_2 - x_1^2$, then we would have $P(C) = x_0^2 x_2$.

Now we introduce the family of primitive open diagrams whose linear independence we shall prove.

14.4.4. Definition. The baguette diagram $B_{n_1,\dots,n_k}$ is

$B_{n_1,\dots,n_k} \quad =$

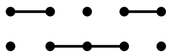


It has a total of $2(n_1 + \cdots + n_k + k - 1)$ vertices, out of which $n_1 + \cdots + n_k$ are univalent.

To write down the formula for the value $P(B_{n_1,\dots,n_k})$, we will need the following definitions.

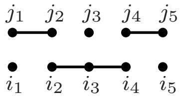
14.4.5. Definition. A *two-line scheme* of order k is a combinatorial object defined as follows. Consider k pairs of points arranged in two rows like $\vdots\ \vdots\ \cdots\ \vdots$. Choose one of the 2^{k-1} subsets of the set $\{1, \dots, k-1\}$. If s belongs to the chosen subset, then we connect the lower points of s-th and $(s+1)$-th pairs, otherwise we connect the upper points.

Example. Here is the scheme corresponding to $k = 5$ and the subset $\{2, 3\}$:

 .

The number of connected components in a scheme of order k is $k + 1$.

14.4.6. Definition. Let σ be a scheme; $i_1, \dots, i_k$ be non negative integers: $0 \leqslant i_1 \leqslant n_1, \dots, 0 \leqslant i_k \leqslant n_k$. We assign i_s to the lower vertex of the s-th pair of σ and $j_s = n_s - i_s$ — to the upper vertex. For example



.

Then the corresponding monomial is $x_{\sigma_0} x_{\sigma_1} \dots x_{\sigma_k}$ where σ_t is the sum of integers assigned to the vertices of t-th connected component of σ.

Example. For the above weighted scheme we get the monomial

$$x_{i_1} x_{j_1+j_2} x_{i_2+i_3+i_4} x_{j_3} x_{j_4+j_5} x_{i_5} \ .$$

Now the formula for P can be stated as follows.

14.4.7. Proposition. *If $N > n_1 + \dots + n_k$ then*

$$P_{gl_N}(B_{n_1,\dots,n_k}) = \sum_{i_1,\dots,i_k} (-1)^{j_1+\dots+j_k} \binom{n_1}{i_1} \dots \binom{n_k}{i_k} \sum_{\sigma} x_{\sigma_0} x_{\sigma_1} \dots x_{\sigma_k},$$

where the external sum ranges over all integers $i_1, \dots, i_k$ such that $0 \leqslant i_1 \leqslant n_1, \dots, 0 \leqslant i_k \leqslant n_k$; the internal sum ranges over all the 2^{k-1} schemes, $j_s = n_s - i_s$, and $x_{\sigma_0} x_{\sigma_1} \dots x_{\sigma_k}$ is the monomial associated with the scheme σ and integers $i_1, \dots, i_k$.

Examples.

For the baguette diagram B_2 we have $k = 1$, $n_1 = 2$. There is only one scheme:  . The corresponding monomial is $x_{i_1} x_{j_1}$, and

$$\begin{aligned} P_{gl_N}(B_2) &= \sum_{i_1=0}^{2} (-1)^{j_1} \binom{2}{i_1} x_{i_1} x_{j_1} \\ &= x_0 x_2 - 2x_1 x_1 + x_2 x_0 \quad = \quad 2(x_0 x_2 - x_1^2) \end{aligned}$$

which agrees with the example given in Chapter 6 on page 190.

For $B_{1,1}$: $k = 2$, $n_1 = n_2 = 1$. There are two schemes: and . The corresponding monomial are $x_{i_1} x_{i_2} x_{j_1+j_2}$ and $x_{i_1+i_2} x_{j_1} x_{j_2}$. We have

$$P_{gl_N}(B_{1,1}) = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} (-1)^{j_1+j_2} x_{i_1} x_{i_2} x_{j_1+j_2} + \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} (-1)^{j_1+j_2} x_{i_1+i_2} x_{j_1} x_{j_2}$$
$$= x_0x_0x_2 - x_0x_1x_1 - x_1x_0x_1 + x_1x_1x_0 + x_0x_1x_1 - x_1x_0x_1 - x_1x_1x_0 + x_2x_0x_0$$
$$= 2(x_0^2 x_2 - x_0 x_1^2)$$

Sketch of the proof of Proposition 14.4.7. The diagram $B_{n_1,\dots,n_k}$ has k parts separated by $k-1$ walls. Each wall is an edge connecting trivalent vertices to which we will refer as *wall vertices*. The s-th part has n_s outgoing legs. We will refer to the corresponding trivalent vertices as *leg vertices*.

The proof consists of three steps.

At the first step we study the effect of resolutions of the wall vertices. We prove that the monomial obtained by certain resolutions of these vertices has the maximal possible degree if and only if for each wall both resolutions of its vertices have the same sign. These signs are related to the above defined schemes in the following way. If we take the positive resolutions at both endpoints of the wall number s, then we connect the lower vertices of the s-th and the $(s+1)$-th pairs in the scheme. If we take the negative resolutions, then we connect the upper vertices.

At the second step we study the effect of resolutions of leg vertices. We show that the result depends only on the numbers of positive resolutions of leg vertices in each part and does not depend on which vertices in a part were resolved positively and which negatively. We denote by i_s the number of positive resolutions in part s. This yields the binomial coefficients $\binom{n_s}{i_s}$ in the formula of Proposition 14.4.7. The total number $j_1+\dots+j_k$ of negative resolutions of leg vertices gives the sign $(-1)^{j_1+\dots+j_k}$.

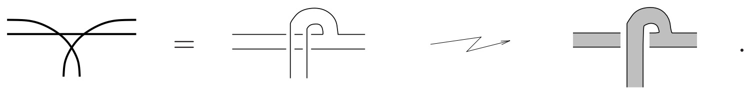
The first two steps allow us to consider only those cases where the resolutions of the left i_s leg vertices in the part s are positive, the rest j_s resolutions are negative and both resolutions at the ends of each wall have the same sign. At the third step we prove that such resolutions of wall vertices lead to monomials associated with corresponding schemes according to definition 14.4.6.

We will make some comments only about the first step, because it is exactly at this step where Dasbach found an improvement of the original argument of [**CD2**].

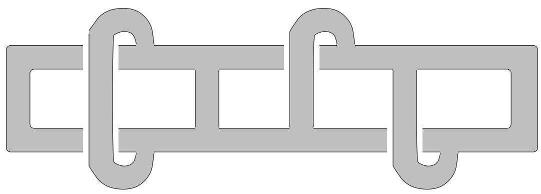
Let us fix certain resolutions of all trivalent vertices of $B_{n_1,\dots,n_k}$. We denote the obtained T-diagram (see p. 184) by T. It consists of $n = n_1 + \dots + n_k$ pairs of points and a number of lines connecting them. After a suitable permutation of the pairs T will look like a disjoint union of certain x_m's. Hence it defines a monomial in x_m's which we denote by $m(T)$.

Let us close all lines in the diagram by connecting the two points in every pair with an additional short line. We obtain a number of closed curves, and we can draw them in such a way that they have 3 intersection points in the vicinity of each negative resolution and do not have other intersections. Each variable x_m gives precisely one closed curve. Thus the degree of $m(T)$ is equal to the number of these closed curves.

Consider an oriented surface S which has our family of curves as its boundary (the Seifert surface):



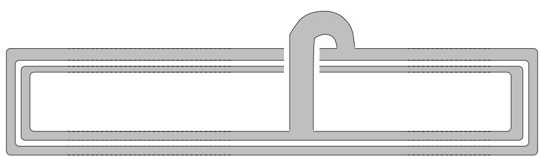
The degree of $m(T)$ is equal to the number of boundary components b of S. The whole surface S consists of an annulus corresponding to the big circle in $B_{n_1,\dots,n_k}$ and $k-1$ bands corresponding to the walls. Here is an example:



where each of the two walls on the left has the same resolutions at its endpoints, while the two walls on the right have different resolutions at their endpoints. The resolutions of the leg vertices do not influence the surface S.

The Euler characteristic χ of S can be easily computed. The surface S is contractible to a circle with $k-1$ chords, thus $\chi = -k+1$. On the other hand $\chi = 2 - 2g - b$, where g and b are the genus and the number of boundary components of S. Hence $b = k + 1 - 2g$. Therefore, the degree of $m(T)$, equal to b, attains its maximal value $k+1$ if and only if the surface S has genus 0.

We claim that if there exists a wall whose ends are resolved with the opposite signs then the genus of S is not zero. Indeed, in this case we can draw a closed curve in S which does not separate the surface (independently of the remaining resolutions):



Hence the contribution to $P(B_{n_1,\dots,n_k})$ is given by only those monomials which come from equal resolutions at the ends of each wall.

Now, with Proposition 14.4.7 in hand, we can prove the following result.

14.4.8. Theorem. *Let $n = n_1 + \cdots + n_k$ and $d = n + k - 1$. Baguette diagrams $B_{n_1,\ldots,n_k}$ are linearly independent in $\mathcal{B}$ if $n_1, \ldots, n_k$ are all even and satisfy the following conditions:*

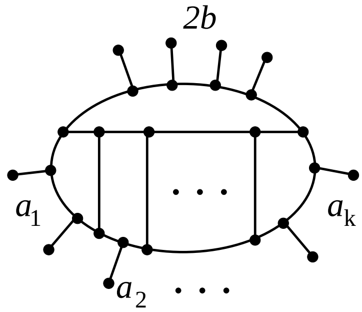
$$\begin{aligned}
&n_1 < n_2\\
&n_1 + n_2 < n_3\\
&n_1 + n_2 + n_3 < n_4\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&n_1 + n_2 + \cdots + n_{k-2} < n_{k-1}\\
&n_1 + n_2 + \cdots + n_{k-2} + n_{k-1} < n/3.
\end{aligned}$$

The proof is based on the study of the *supports* of polynomials $P(B_{n_1,\ldots,n_k})$ — the subsets of $\mathbb{Z}^k$ corresponding to non-zero terms of the polynomial.

Counting the number of elements described by the theorem, one arrives at the lower bound $n^{\log(n)}$ for the dimension of the primitive subspace $\mathcal{P}_n$ of $\mathcal{B}$.

The main difficulty in the above proof is the necessity to consider the 2^k resolutions for the wall vertices of a baguette diagram that correspond to the zero genus Seifert surface. O. Dasbach in [**Da3**] avoided this difficulty by considering a different family of open diagrams for which there are only two ways of resolution of the wall vertices leading to the surface of minimum genus. These are the *Pont-Neuf diagrams*:

$PN_{a_1,\ldots,a_k,b} =$



(the numbers a_1, ..., a_k, $2b$ refer to the number of legs attached to the corresponding edge of the inner diagram).

The reader may wish to check the above property of Pont-Neuf diagrams by way of exercise. It is remarkable that Pont Neuf diagrams not only lead to simpler considerations, but they are more numerous, too, and thus lead to a much better asymptotic estimate for $\dim \mathcal{P}_n$. The exact statement of Dasbach's theorem is as follows.

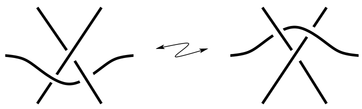
14.4.9. Theorem. *For fixed n and k, the diagrams $PN_{a_1,\ldots,a_k,b}$ with $0 \leqslant a_1 \leqslant \ldots \leqslant a_k \leqslant b$, $a_1 + \ldots + a_k + 2b = 2n$ are linearly independent.*

Counting the number of such partitions of $2n$, we obtain precisely the estimate announced by Kontsevich in [**Kon1**].

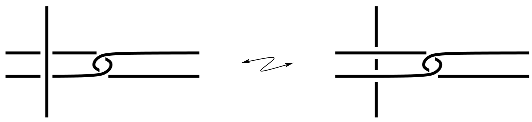
Corollary. $\dim \mathcal{P}_n$ is asymptotically greater than $e^{c\sqrt{n}}$ for any constant $c < \pi\sqrt{2/3}$.

Exercises

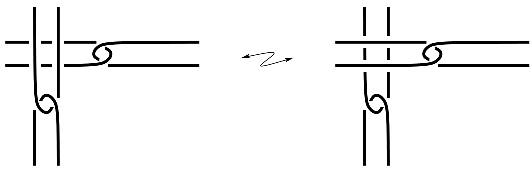
(1) Show that $\mathcal{M}_1$ is equivalent to the Δ *move* in the sense that, modulo Reidemeister moves, the $\mathcal{M}_1$ move can be accomplished by Δ moves and vise versa. The fact that Δ is an unknotting operation was proved in [**Ma, MN**].



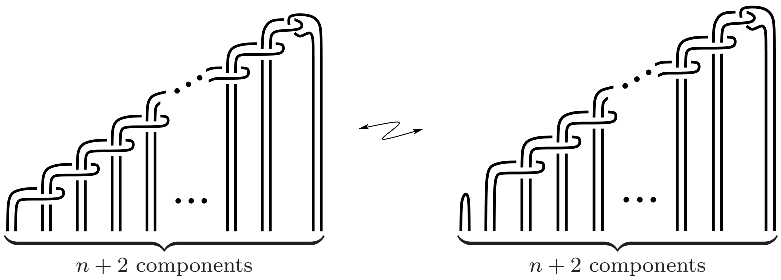
(2) Prove that $\mathcal{M}_1$ is equivalent to the move



(3) Prove that $\mathcal{M}_2$ is equivalent to the so called *clasp-pass* move



(4) Prove that $\mathcal{M}_n$ is equivalent to the move $\mathcal{C}_n$:



(5) Find the inverse element of the knot 3_1 in the group $\mathcal{G}_4$.

(6) (*S. Lando*). Let N be a formal variable. Prove that $N^{\operatorname{corank} A(G)}$ defines an algebra homomorphism $\mathcal{L} \to \mathbb{Z}[N]$, where $\mathcal{L}$ is the graph algebra of Lando and $A(G)$ stands for the adjacency matrix of the graph G considered over the field $\mathbb{F}_2$ of two elements.

(7) * Let $\lambda : A \to \mathcal{L}$ be the natural homomorphism from the algebra of chord diagrams into the graph algebra of Lando.

- Find $\ker \lambda$ (unknown in degrees greater than 7).
- Find $\operatorname{im} \lambda$ (unknown in degrees greater than 7).
- Describe the primitive space $P(\mathcal{L})$.
- $\mathcal{L}$ is the analog of the algebra of chord diagrams in the case of intersection graphs. Are there any counterparts of the algebras $\mathcal{C}$ and $\mathcal{B}$?

Chapter 15

The Vassiliev spectral sequence

Zdes budet gorod zalozhen.

Appendix

A.1. Lie algebras and their universal envelopes

A.1.1. Metrized Lie algebras. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$, that is, a $\mathbb{C}$-vector space equipped with a bilinear operation (*commutator*) $(x, y) \mapsto [x, y]$ subject to the rules

$$[x, y] = -[y, x]\,,$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

An *Abelian* Lie algebra is an arbitrary vector space with the commutator which is identically 0: $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

An *ideal* in a Lie algebra is a vector subspace stable under taking the commutator with an arbitrary element of the whole algebra. A Lie algebra is called *simple* if it is not Abelian and does not contain any proper ideal. Simple Lie algebras are classified (see, for example, [**FH, Hum**]). Over the field of complex numbers $\mathbb{C}$ there are four families of *classical algebras*:

Type	$\mathfrak{g}$	$\dim\mathfrak{g}$	description
A_n	$\mathfrak{sl}_{n+1}$	n^2+2n	$(n+1)\times(n+1)$ matrices with zero trace, $(n \geqslant 1)$
B_n	$\mathfrak{so}_{2n+1}$	$2n^2+n$	skew-symmetric $(2n+1)\times(2n+1)$ matrices, $(n \geqslant 2)$
C_n	$\mathfrak{sp}_{2n}$	$2n^2+n$	$2n\times 2n$ matrices X satisfying the relation $X^t\cdot M + M\cdot X = 0$, where M is the standard $2n\times 2n$ skew-symmetric matrix $M = \begin{pmatrix} O & Id_n \\ -Id_n & 0 \end{pmatrix}$, $(n \geqslant 3)$
D_n	$\mathfrak{so}_{2n}$	$2n^2-n$	skew-symmetric $2n\times 2n$ matrices, $(n \geqslant 4)$

and five *exceptional algebras*:

Type	E_6	E_7	E_8	F_4	G_2
$\dim\mathfrak{g}$	78	133	248	52	14

Apart from the low-dimensional isomorphisms

$$\mathfrak{sp}_2 \cong \mathfrak{so}_3 \cong \mathfrak{sl}_2; \qquad \mathfrak{sp}_4 \cong \mathfrak{so}_5; \qquad \mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2; \qquad \mathfrak{so}_6 \cong \mathfrak{sl}_4,$$

all the Lie algebras in the list above are different. The Lie algebra $\mathfrak{gl}_N$ of all $N \times N$ matrices is isomorphic to the direct sum of $\mathfrak{sl}_N$ and the trivial one-dimensional Lie algebra $\mathbb{C}$.

A *representation* of a Lie algebra $\mathfrak{g}$ in a vector space V is a Lie algebra homomorphism of $\mathfrak{g}$ into the Lie algebra $\mathrm{End}(V)$ of linear operators in V, i.e. a mapping $\rho : \mathfrak{g} \to \mathrm{End}(V)$ such that

$$\rho([x, y]) = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x).$$

The *standard representation* of a matrix Lie algebra, e.g. $\mathfrak{gl}_N$ or $\mathfrak{sl}_N$, is the one given by the identity mapping.

The *adjoint representation* is the action ad of $\mathfrak{g}$ on itself according to the rule

$$x \mapsto \mathrm{ad}_x \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})\,, \quad \mathrm{ad}_x(y) = [x, y]\,.$$

It is indeed a representation, because $\mathrm{ad}_{[x,y]} = \mathrm{ad}_x \cdot \mathrm{ad}_y - \mathrm{ad}_y \cdot \mathrm{ad}_x = [\mathrm{ad}_x, \mathrm{ad}_y]$.

— Add a little about representations: list of irreducible representations for sl_2, Casimir tensor and Casimir number! —

The *Killing form* on a Lie algebra $\mathfrak{g}$ is defined by the equality

$$\langle x, y\rangle^K = \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y).$$

Cartan's criterion says that this bilinear form is non-degenerate if and only if the algebra is *semi-simple*, i.e. is isomorphic to the direct sum of simple Lie algebras.

A.1.2. Exercise. Prove that the Killing form is ad-invariant in the sense of the following definition.

A bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ is said to be *ad-invariant* if it satisfies the identity

$$\langle \mathrm{ad}_z(x), y\rangle + \langle x, \mathrm{ad}_z(y)\rangle = 0,$$

or, equivalently,

$$\langle [x, z], y\rangle = \langle x, [z, y]\rangle. \tag{A.1.2.1}$$

for all $x, y, z \in \mathfrak{g}$.

This definition is justified by the fact described in the following exercise.

A.1.3. Exercise. Let $\mathfrak{G}$ be the connected Lie group corresponding to the Lie algebra $\mathfrak{g}$ and let $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ be its adjoint representation (see, e.g., [**AdJ**]). Then the ad-invariance of a bilinear form is equivalent to its Ad-invariance defined by the natural rule

$$\langle \mathrm{Ad}_g(x), \mathrm{Ad}_g(y) \rangle = \langle x, y \rangle$$

for all $x, y \in \mathfrak{g}$ and $g \in \mathfrak{G}$.

A Lie algebra is said to be *metrized*, if it is equipped with an ad-invariant symmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$. The class of metrizable algebras contains simple Lie algebras with (a multiple of) the Killing form, Abelian Lie algebras with an arbitrary non-degenerate bilinear form and their direct sums.

For the classical simple Lie algebras which consist of matrices, it is often more convenient to use, instead of the Killing form, a different bilinear form $\langle x, y \rangle = \mathrm{Tr}(xy)$, which is proportional to the Killing form with the coefficient $\frac{1}{2N}$ for $\mathfrak{sl}_N$, $\frac{1}{N-2}$ for $\mathfrak{so}_N$, and $\frac{1}{N+2}$ for $\mathfrak{sp}_N$.

A.1.4. Exercise. Prove that for the Lie algebra $\mathfrak{gl}_N$ the Killing form $\langle x, y \rangle =$ $(\mathrm{Tr}(\mathrm{ad}_x \cdot \mathrm{ad}_y)$ is degenerate with defect 1 and can be expressed as follows:

$$\langle x, y \rangle^K = 2N\mathrm{Tr}(xy) - 2\mathrm{Tr}(x)\mathrm{Tr}(y) \ .$$

A.1.5. Exercise. Prove that the form $\mathrm{Tr}(xy)$ on $\mathfrak{gl}_N$ is non-degenerate and ad-invariant.

Below, we will need the following lemma.

A.1.6. Lemma. *Let c_{ijk} be the structure constants of a metrized Lie algebra in a basis $\{e_i\}$, orthonormal with respect to the ad-invariant bilinear form $\langle \cdot, \cdot \rangle$:*

$$\langle e_i, e_j \rangle = \delta_{ij},$$
$$[e_i, e_j] = \sum_{k=1}^{d} c_{ijk} e_k.$$

Then the constants c_{ijk} are antisymmetric with respect to the permutations of the indices i, j and k.

Proof. The equality $c_{ijk} = -c_{jik}$ is the coordinate expression of the fact that the commutator is antisymmetric: $[x, y] = -[y, x]$. It remains to prove that $c_{ijk} = c_{jki}$. This follows immediately from equation (A.1.2.1), simply by setting $x = e_i$, $y = e_k$, $z = e_j$. □

Corollary. *Let $J \in \mathfrak{g}^{\otimes 3}$ be the structure tensor of a metrized Lie algebra transferred from $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ to $\mathfrak{g}^{\otimes 3}$ by means of the duality defined by the metric. Then J is totally antisymmetric: $J \in \wedge^3 \mathfrak{g}$.*

A.1.7. Universal enveloping algebras. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Denote by $T = \oplus_{n\geqslant 0} T_n$ the tensor algebra of the vector space $\mathfrak{g}$. Here $T_n = \mathfrak{g}\otimes\cdots\otimes\mathfrak{g}$ (n factors) is the vector space spanned by the elements $x_1\otimes\cdots\otimes x_n$ with all $x_i\in\mathfrak{g}$, subject to the relations that express the linear dependence of $x_1\otimes\cdots\otimes x_n$ on each x_i. Multiplication in T is given by the tensor product. We get the *universal enveloping algebra* of $\mathfrak{g}$, denoted by $U(\mathfrak{g})$, if, to these relations, we add the relations of the form $x\otimes y - y\otimes x = [x,y]$ where $[x,y]$ is the structure bracket in $\mathfrak{g}$. More formally, let I be the double-sided ideal of $T(\mathfrak{g})$ generated by all the elements $x\otimes y - y\otimes x - [x,y]$, $x,y\in\mathfrak{g}$. Then by definition

$$U(\mathfrak{g}) = T(\mathfrak{g})/I.$$

(Speaking informally, in $U(\mathfrak{g})$ we are allowed to treat the elements of $\mathfrak{g}$ as associative variables, always remembering the commutator relations that existed between themin $\mathfrak{g}$ itself.)

!! add: PBW and Harish-Chandra

A.2. Bialgebras and Hopf algebras

Here we provide background information about bialgebras and Hopf algebras necessary for the study of the algebras of knot invariants, chord diagrams, weight systems etc.

A.2.1. Coalgebras and bialgebras. To introduce the notions of a coalgebra and bialgebra, let us first recall what is an algebra.

A.2.2. Definition. An *algebra* over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space A equipped with a linear mapping $\mu : A\otimes A\to A$, called multiplication. We will only consider associative algebras with a unit. Associativity means that the diagram

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{\mu\otimes\mathrm{id}} & A\otimes A \\ {\scriptstyle \mathrm{id}\otimes\mu}\big\downarrow & & \big\downarrow{\scriptstyle\mu} \\ A\otimes A & \xrightarrow[\mu]{} & A \end{array}$$

is commutative. The unit is a linear mapping $\iota : \mathbb{F}\to A$ (uniquely defined by the element $\iota(1)\in A$) that makes commutative the diagram

$$\begin{array}{ccc} \mathbb{F}\otimes A & \xrightarrow{\iota\otimes\mathrm{id}} & A\otimes A \\ \big\uparrow & & \big\downarrow{\scriptstyle\mu} \\ A & = & A \end{array}$$

where the left vertical arrow is a natural isomorphism.

We have on purpose formulated the notion of algebra in this formal way: this allows us to derive the definition of a coalgebra by a mere reversion of arrows.

A.2.3. Definition. A *coalgebra* is a vector space A equipped with a linear mapping $\delta : A \to A\otimes A$, referred to as comultiplication, and a linear mapping $\varepsilon : A \to \mathbb{F}$, called the counit, such that the following two diagrams commute:

$$\begin{array}{ccc} A\otimes A\otimes A & \xleftarrow{\delta\otimes\mathrm{id}} & A\otimes A \\ {\scriptstyle \mathrm{id}\otimes\delta}\uparrow & & \uparrow{\scriptstyle\delta} \\ A\otimes A & \xleftarrow{\delta} & A \end{array} \qquad \begin{array}{ccc} \mathbb{F}\otimes A & \xleftarrow{\varepsilon\otimes\mathrm{id}} & A\otimes A \\ \downarrow & & \uparrow{\scriptstyle\delta} \\ A & = & A \end{array}$$

Algebras (resp. coalgebras) may or may not possess an additional property of commutativity (resp. cocommutativity), defined through the following commutative diagrams:

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\mu} & A \\ {\scriptstyle\tau}\uparrow & & \| \\ A\otimes A & \xrightarrow{\mu} & A \end{array} \qquad \begin{array}{ccc} A\otimes A & \xleftarrow{\delta} & A \\ {\scriptstyle\tau}\downarrow & & \| \\ A\otimes A & \xleftarrow{\delta} & A \end{array}$$

where $\tau : A\otimes A \to A\otimes A$ is the permutation of the tensor factors, $\tau : a\otimes b \mapsto b\otimes a$.

The following exercise is useful to get acquainted with the notion of a coalgebra.

A.2.4. Exercise. Let $\Delta_i^{(n)} : A^{\otimes n} \to A^{\otimes(n+1)}$ be defined as taking the coproduct of the i-th tensor factor, $\Delta_0^{(n)}(x) = 1\otimes x$, $\Delta_{n+1}^{(n)}(x) = x\otimes 1$. Set $\delta_n = \sum_{i=0}^{n+1}(-1)^i\Delta_i^{(n)}$. Prove that the sequence $\{\delta_n\}$ forms a complex, i.e. $\delta_{i+1}\circ\delta_i = 0$.

A.2.5. Definition. A *bialgebra* is a vector space A with the structure of an algebra given by μ, ι and the structure of a coalgebra given by δ, ε which agree in the sense that the following identities hold:

(1) $\varepsilon(1) = 1$

(2) $\delta(1) = 1\otimes 1$

(3) $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$

(4) $\delta(ab) = \delta(a)\delta(b)$

(in the last equation $\delta(a)\delta(b)$ denotes the component-wise product in $A\otimes A$ induced by the product μ in A).

Note that these conditions, taken in pairs, have the following meaning:

(1,3) $\Leftrightarrow$ ε is a (unitary) algebra homomorphism.

(2,4) $\Leftrightarrow$ δ is a (unitary) algebra homomorphism.

(1,2) $\Leftrightarrow$ ι is a (unitary) coalgebra homomorphism.

(3,4) $\Leftrightarrow$ μ is a (unitary) coalgebra homomorphism.

The coherence of the two structures making the definition of a bialgebra can thus be stated in either of the two equivalent ways:

- ε and δ are algebra homomorphisms,
- μ and i are coalgebra homomorphisms.

Example 1. The group algebra $\mathbb{F}[G]$ of a finite group G over a field $\mathbb{F}$ consists of formal linear combinations $\sum_{x\in G} \lambda_x x$ where $\lambda_x \in \mathbb{F}$ with the product defined on the basic elements by the group multiplication in G. The coproduct is defined as $\delta(x) = x \otimes x$ for $x \in G$ and then extended by linearity. Instead of a group G, in this example one can actually take a semigroup.

Example 2. The algebra $\mathbb{F}^G$ of $\mathbb{F}$-valued functions on a finite group G with pointwise multiplication

$$(fg)(x) = f(x)g(x)$$

and comultiplication defined by

$$\delta(f)(x,y) = f(xy)$$

where the element $\delta(f) \in \mathbb{F}^G \otimes \mathbb{F}^G$ is understood as a function on $G \times G$ due to the natural isomorphism $\mathbb{F}^G \otimes \mathbb{F}^G \cong \mathbb{F}^{G\times G}$.

Example 3. The polynomial algebra. Let A be the symmetric algebra of a finite-dimensional vector space X, i.e. $A = S(X) = \oplus_{n=0}^{\infty} S^k(X)$, where $S^k(X)$ is the symmetric part of the k-th tensor power of X. Then A is a bialgebra with the coproduct defined on the elements $x \in X = S^1(X) \subset A$ by setting $\delta(x) = 1\otimes x + x\otimes 1$ and then extended as an algebra homomorphism to the entire A.

Example 4. Universal enveloping algebras. Let $\mathfrak{g}$ be a Lie algebra, $A = U(\mathfrak{g})$ — its universal enveloping algebra with usual multiplication (see section A.1.7 for a definition and basic facts). Define $\delta(g) = 1\otimes g + g\otimes 1$ for $g \in \mathfrak{g}$ and extend it to all of A by the axioms of bialgebra. If $\mathfrak{g}$ is Abelian, then this example is reduced to the previous one.

Exercise. Define the appropriate unit and counit in each of the above examples.

A.2.6. Primitive and group-like elements. In a bialgebra, there are two remarkable classes of elements: primitive elements and group-like elements.

Definition. An element $a \in A$ of a bialgebra A is said to be *primitive* if

$$\delta(a) = 1 \otimes a + a \otimes 1.$$

The set of all primitive elements forms a vector subspace $\mathcal{P}(A)$ called the *primitive subspace* of the bialgebra A. The primitive subspace is closed under the commutator $[a, b] = ab - ba$, so it forms a Lie algebra (which is Abelian, if A is commutative). Indeed, if $a, b \in \mathcal{P}(A)$, then

$$\delta(a) = 1 \otimes a + a \otimes 1,$$
$$\delta(b) = 1 \otimes b + b \otimes 1,$$

whence

$$\delta(ab) = 1 \otimes ab + a \otimes b + b \otimes a + ab \otimes 1,$$
$$\delta(ba) = 1 \otimes ba + b \otimes a + a \otimes b + ba \otimes 1$$

and therefore

$$\delta([a, b]) = 1 \otimes [a, b] + [a, b] \otimes 1.$$

Definition. An element $a \in A$ is said to be *semigroup-like* if and

$$\delta(a) = a \otimes a.$$

If, in addition, a is invertible, then it is called *group-like*.

The set of all group-like elements $\mathcal{G}(A)$ of a bialgebra A is a multiplicative group. Indeed, $\mathcal{G}(A)$ consists of all invertible elements of the semigroup $\{a \in A \mid \delta(a) = a \otimes a\}$.

Among the examples of bialgebras given above, the notions of the primitive and group-like elements are especially transparent in the case $A = \mathbb{F}^G$ (Example 2). As follows from the definitions, primitive elements are *additive* functions ($f(xy) = f(x) + f(y)$) while group-like elements are *multiplicative* functions ($f(xy) = f(x)f(y)$).

In Example 3, using the isomorphism $S(X) \otimes S(X) \cong S(X \oplus X)$, we can rewrite the definition of the coproduct as $\delta(x) = (x, x) \in X \oplus X$ for $x \in X$. It is even more suggestive to view the elements of the symmetric algebra $A = S(X)$ as polynomial functions on the dual space X^* (where homogeneous subspaces $S^0(X)$, $S^1(X)$, $S^2(X)$, etc. correspond to constants, linear functions, quadratic functions etc. on X^*). In these terms, the product in A corresponds to the usual (pointwise) multiplication of functions, while the coproduct $\delta : S(X) \to S(X \oplus X)$ acts according to the rule

$$\delta(f)(\xi, \eta) = f(\xi + \eta), \quad \xi, \eta \in X^*.$$

Under the same identifications,

$$(f \otimes g)(\xi, \eta) = f(\xi)g(\eta),$$

in particular,

$$(f \otimes 1)(\xi, \eta) = f(\xi),$$
$$(1 \otimes f)(\xi, \eta) = f(\eta).$$

We see that an element of $S(X)$, cosidered as a function on X^*, is primitive (group-like) if and only if this function is additive (multiplicative):

$$f(\xi, \eta) = f(\xi) + f(\eta),$$
$$f(\xi, \eta) = f(\xi) f(\eta).$$

The first condition means that f is a linear function on X^*, i.e. it corresponds to an element of X itself; therefore,

$$\mathcal{P}(S(X)) = X.$$

Over a field of characteristic zero, the second condition cannot hold for polynomial functions except for the constant function equal to 1; thus

$$\mathcal{G}(S(X)) = \{1\}.$$

The *completed* symmetric algebra $\overline{S}(X)$, in contrast with $S(X)$, has a lot of group-like elements. As shows Quillen's theorem (see page 431),

$$\mathcal{G}(\overline{S}(X)) = \{\exp(x) \mid x \in X\},$$

where $\exp(x)$ is defined as a formal power series $1 + x + x^2/2! + \dots$

A.2.7. Exercise. Describe the primitive and group-like elements in examples 1 and 4.

(Answer to 1: $\mathcal{P} = 0$, $\mathcal{G} = G$.

Answer to 4: $\mathcal{P} = \mathfrak{g}$, $\mathcal{G} = 0$ (in the completed case $\mathcal{G} = \exp(\mathfrak{g})$.)

A.2.8. Dual bialgebras. The dual vector space of a bialgebra can be naturally equipped with a structure of a bialgebra, too.

Let A be a bialgebra over a field $\mathbb{F}$ and let $A^* = \mathrm{Hom}_{\mathbb{F}}(A, \mathbb{F})$. We define the *product* of two elements $f, g \in A^*$ as

(A.2.8.1) $$(f \cdot g)(a) = (f \otimes g)(\delta(a)) = \sum_i f(a'_i) g(a''_i),$$

if $\delta(a) = \sum_i a'_i \otimes a''_i$.

The *coproduct* of an element $f \in A^*$ is defined by the equation

(A.2.8.2) $$\delta(f)(a \otimes b) = f(ab).$$

where the evaluation of $\delta(f) \in A^* \otimes A^*$ on the element $a \otimes b \in A \otimes A$ comes from the natural pairing between $A^* \otimes A^*$ and $A \otimes A$.

Thus, the product (resp. coproduct) in A^* is defined through the coproduct (resp. product) in A. It is easy to define the unit and the counit in

A^* in terms of the counit and the unit of A and then check that the space A^* becomes indeed a bialgebra.

If read in the opposite direction, Equations (A.2.8.1) and (A.2.8.2) allow one to uniquely define the structure of a bialgebra in the initial space A, if such a structure is given in the dual space A^*. The uniqueness follows from the simple observation that an element a of a vector space A is completely defined by the values of all linear functionals $f(a)$, while an element of the tensor square $q \in A \otimes A$ is completely determined by values $(f \otimes g)(q)$ for all $f, g \in A^*$.

A.2.9. Exercise. Check that the axioms of a bialgebra in the dual space A^* imply that A becomes a bialgebra, too.

A.2.10. Exercise. Check that the bialgebra of Example 2 above is dual to the bialgebra of Example 1.

A.2.11. Exercise. Prove that for a finite-dimensional bialgebra A there is a natural isomorphism $(A^*)^* \cong A$.

If A is infinite-dimensional, then the assertion of the last Exercise does not hold: in this case the vector space $(A^*)^*$ is strictly bigger than A.

A.2.12. Proposition. *Primitive (resp. group-like) elements in the bialgebra A^* are linear functions on A which are additive (resp. multiplicative), i.e. satisfy the respective identities*

$$f(ab) = f(a) + f(b),$$
$$f(ab) = f(a)f(b)$$

for all $a, b \in A$.

Proof. An element $f \in A^*$ is primitive if $\delta(f) = 1 \otimes f + f \otimes 1$. Evaluating this on an arbitrary tensor product $a \otimes b$ with $a, b \in A$, we obtain

$$f(ab) = f(a) + f(b).$$

An element $f \in A^*$ is group-like if $\delta(f) = f \otimes f$. Evaluating this on an arbitrary tensor product $a \otimes b$, we obtain

$$f(ab) = f(a)f(b).$$

□

A.2.13. Filtrations and gradings in vector spaces. A decreasing filtration (*d-filtration*) in a vector space A over a field $\mathbb{F}$ is a sequence of subspaces A_i, $i = 0, 1, 2, \ldots$ such that

$$A = A_0 \supset A_1 \supset A_2 \supset \ldots$$

The *order* $\mathrm{ord}(a)$ of an element $a \in A$ is defined as the natural number n such that $a \in A_n$, but $a \notin A_{n-1}$. If no such number exists, then we set $\mathrm{ord}(a) = \infty$. The *factors* of a d-filtration are the quotient spaces $G_i A = A_i/A_{i+1}$.

An increasing filtration (*i-filtration*) in a vector space A is a sequence of subspaces A_i, $i = 0, 1, 2, ...$ such that

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A.$$

The *order* of an element $a \in A$ is defined as the natural number n such that $a \in A_n$, but $a \notin A_{n+1}$. If no such number exists, then we set $\mathrm{ord}(a) = \infty$. The *factors* of an i-filtration are the quotient spaces $G_i A = A_i/A_{i-1}$, where by definition $A_{-1} = 0$.

A filtration (either d- or i-) is said to be of *finite type* if all its factors are finite-dimensional. Note that in each case the whole space has a (possibly infinite-dimensional) 'part' not covered by the factors, viz. $\cap_{i=1}^{\infty} A_i$ for a d-filtration and $A/\cup_{i=1}^{\infty} A_i$ for an i-filtration. If these parts vanish, then we say that we deal with a reduced filtered space.

There is a simple way to obtain a reduced filtered space with the same factors:

- $A' = A/\cap_{i=1}^{\infty} A_i$ for a d-filtered space,
- $A' = \cup_{i=1}^{\infty} A_i$ for an i-filtered space.

In either case the reduced space inherits the filtration in a natural way.

A *filtered basis* of an i-filtered vector space A is constructed as follows. One takes a basis of A_0, then adds some vectors to form a basis of $A_1 \supset A_0$ etc. For a d-filtered space A, a filtered basis is defined as the disjoint union of a countable number of subsets $T = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}'$ where $\mathcal{T}'$ is a basis of $\cap_{i=1}^{\infty} A_i$ (possibly infinite), while all $\mathcal{T}_i$'s are finite sets with the property that for each n the union of $\cup_{i=n}^{\infty} \mathcal{T}_i$ and $\mathcal{T}'$ is a basis of A_n.

A vector space is said to be *graded* if it is represented as a direct sum of its subspaces

$$A = \bigoplus_{i=0}^{\infty} A_i.$$

With a filtered vector space A, one can associate a graded space $G(A)$ setting

$$G(A) = \sum_{i=0}^{\infty} G_i A = \sum_{i=0}^{\infty} A_i/A_{i+1}$$

in case of a d-filtration and

$$G(A) = \sum_{i=0}^{\infty} G_i A = \sum_{i=0}^{\infty} A_i/A_{i-1}$$

in case of an i-filtration.

If A is a filtered space of finite type, then the homogeneous components G_iA are also finite-dimensional; the 'size' of $G(A)$ has a compact description by means of the Poincaré series $\sum_{k=0}^{\infty} \dim(G_iA)t^k$, where t is an auxiliary formal variable.

A.2.14. Example. The Poincaré series of the algebra of polynomials in one variable is
$$1 + t + t^2 + ... = \frac{1}{1-t}.$$

A.2.15. Exercise. Find the Poincaré series of the polynomial algebra with n independent variables.

A.2.16. Graded completion.

Add a formal definiton of the graded completion!!

A.2.17. Dual filtration. Given a d-filtered vector space A (i.e. endowed with a decreasing filtration) one can introduce a natural structure of an i-filtered space in the dual space $A^* = \mathrm{Hom}(A, \mathbb{F})$ by the formula
$$(A^*)_i = \{f \in A^* \,:\, f|_{A_{i+1}} = 0\}.$$
Note that the subscript of A is $i+1$, not i. This convention is crucial to ensure the following important assertion.

A.2.18. Lemma. *Each factor of the dual filtered space is dual to the factor of the initial space with the same number: $G_i(A^*) = (G_iA)^*$. Hence, for an i-filtered space of finite type, we have an (unnatural) isomorphism $G_i(A^*) \cong G_iA$.*

Proof. Indeed, there is a natural map $\lambda : (A^*)_i \to (G_iA)^*$ defined by the formula $\lambda(f) = f|_{A_i}$. Since any linear function can be extended from a subspace to the whole space, this mapping is surjective. Its kernel is by definition $(A^*)_{i-1}$, whence the assertion. $\square$

A.2.19. Filtered bialgebras.

Definition. We will say that a bialgebra A is *d-filtered* (resp. *i-filtered*) if its underlying vector space has a decreasing (resp. increasing) filtration by subspaces A_i compatible with the algebraic operations in the following sense:
$$A_pA_q \subset A_{p+q} \quad \text{for} \quad p, q \geqslant 0, \qquad \delta(A_n) \subset \sum_{p+q=n} A_p \otimes A_q \quad \text{for} \quad n \geqslant 0.$$
The second condition can be equivalently rewritten as $\delta(A_n) \subset \sum_{p+q\geqslant n} A_p \otimes A_q$ for i-filtered bialgebras, and as $\delta(A_n) \subset \sum_{p+q\leqslant n} A_p \otimes A_q$ for d-filtered bialgebras.

Additional requirements imposed on the unit and the counit are: $1 \in A_0$ and $\varepsilon|_{A_1} = 0$.

A.2.20. Dual filtered bialgebra.

Theorem. *The bialgebra dual to a d-filtered bialgebra is an i-filtered bialgebra.*

Proof. Let A be a bialgebra with a decreasing filtration by subspaces A_i. The constructions of Sec. A.2.8 and A.2.17 define a structure of a bialgebra and that of a filtered vector space in the dual space A^*. We must check that these two structures are compatible, i.e. that

$$(A^*)_p(A^*)_q \subset (A^*)_{p+q}$$

and

$$\delta((A^*)_n) \subset \sum_{p+q=n} (A^*)_p \otimes (A^*)_q \,.$$

To prove the first assertion, let $f \in (A^*)_p$, $g \in (A^*)_q$ and $a \in A_{p+q+1}$. Write $\delta(a)$ as a sum of tensor products $a' \otimes a''$ where $\mathrm{ord}(a') + \mathrm{ord}(a'') \geqslant p + q + 1$. Then, by definition,

$$(fg)(a) = (f \otimes g)(\delta(a)) = \sum f(a')g(a'') = 0,$$

because in each summand either $\mathrm{ord}(a') \geqslant p + 1$ and then $f(a') = 0$ or $\mathrm{ord}(a'') \geqslant q + 1$ and then $g(a'') = 0$.

For the second assertion, pick an $f \in (A*)_n$. Choose a filtered basis (p. 426) in A and dual filtered basis in A^* and expand $\delta(f)$ over the corresponding basis of the tensor square $A^* \otimes A^*$. Let $\lambda f' \otimes f''$ be one of the terms of this expansion and let $a', a'' \in A$ be the vectors of the dual basis of A conjugate to f' and f''. Set $b = a'a''$. Then

$$f(b) = \delta(f)(a' \otimes a'') = \lambda \neq 0.$$

According to the definition of the dual filtration, this inequality implies that $\mathrm{ord}(b) \leqslant \mathrm{ord}(f) = n$. Note, finally, that $\mathrm{ord}(a') + \mathrm{ord}(a'') \leqslant \mathrm{ord}(a'a'')$ in a d-filtered algebra. □

A.2.21. Hopf algebras.

A.2.22. Definition. A *Hopf algebra* is a graded bialgebra. This means that the vector space A is graded by integer numbered subspaces

$$A = \bigoplus_{k\geqslant 0} A_k$$

and the grading is compatible with the operations μ, ι, δ, ε in the following sense:

$$\mu : A_m \otimes A_n \to A_{m+n},$$
$$\delta : A_n \to \bigoplus_{k+l=n} A_k \otimes A_l.$$

A Hopf algebra A is said to be of *finite type*, if all its homogeneous components A_n are finite-dimensional. A Hopf algebra is said to be *connected*, if (1) $\iota : \mathbb{F} \to A$ is an isomorphism of $\mathbb{F}$ onto $A_0 \subset A$ and (2) $\varepsilon\mid_{A_k} = 0$ for $k > 0$, while $\varepsilon\mid_{A_0}$ is an isomorphism between A_0 and $\mathbb{F}$, inverse to ι.

Remark 1. The above definition follows the classical paper [**MiMo**]. In more recent times, it became customary to include one more operation, called *antipode*, in the definition of a Hopf algebra. The antipode is a linear mapping $S : A \to A$ such that $\mu \circ (S \otimes 1) \circ \delta = \mu \circ (1 \otimes S) \circ \delta = \iota \circ \varepsilon$. Note that the bialgebras we will be mostly interested in (those that satisfy the premises of Theorem A.2.25 below) always have an antipode.

Remark 2. We will need to extend the definition of a Hopf algebra from graded algebras, i.e. direct sums of finite-dimensional homogeneous components, to *completed* graded algebras, i.e. direct products of such components (see p. 111).

A.2.23. Exercise. If A is a filtered bialgebra, then its associated graded vector space $G(A)$ carries the structure of a Hopf algebra with all algebraic operations naturally defined by the operations in A by passing to quotient spaces.

A.2.24. Dual Hopf algebra. Here we specialize the construction of Section A.2.8 to the case of Hopf algebras.

If $A = \oplus A_k$ is a Hopf algebra of finite type and $W_k = \mathrm{Hom}(A_k, \mathbb{F})$ are vector spaces, dual to the homogeneous components of A, then the dual space to the whole A is represented as the product $\prod_k W_k$ which is in general bigger than the direct sum $W = \sum_k W_k$. However, it is this smaller space W that we will call the dual Hopf algebra of A with the operations induced by the corresponding operations in A as follows:

$\mu^* : W_n \to \underset{k+l=n}{\oplus} \mathrm{Hom}(A_k \otimes A_l, \mathbb{F}) \cong \underset{k+l=n}{\oplus} W_k \otimes W_l$	is a comultiplication for W
$\delta^* : W_n \otimes W_m \to W_{m+n}$	is a multiplication for W
$\iota^* : W \to \mathbb{F}$	is a counit for W
$\varepsilon^* : \mathbb{F} \to W$	is a unit for W

A.2.25. Structure theorem for Hopf algebras. Is it easy to see that for a Hopf algebra the primitive subspace $P = P(A) \subset A$ is homogeneous in the sense that $P = \bigoplus_{n \geqslant 0} P \cap A_n$.

Theorem. (*Milnor–Moore* [**MiMo**]). *Any commutative cocommutative connected Hopf algebra of finite type over a field of characteristic 0 is equal to the symmetric algebra of its primitive subspace:*

$$A = S(P(A)).$$

The word 'equal' (not just isomorphic) means that if the inclusion $P(A) \to A$ is extended to a homomorphism $S(P(A)) \to A$ in the standard way, this gives a bialgebra isomorphism. In other words, if a linear basis is chosen in every homogeneous component $P_n = P \cap A_n$, then every element of A can be written uniquely as a polynomial in these variables.

Proof. There are two assertions to prove:

(I) that every element of A is polynomially expressible (i. e. as a linear combination of products) through primitive elements,

(II) that the value of a nonzero polynomial on a set of linearly independent homogeneous primitive elements cannot vanish in A.

Let us start to prove assertion (I) for homogeneous elements of the algebra by induction on their degree.

First note that under our assumptions the coproduct of a homogeneous element $a \in A_n$ has the form

$$\delta(x) = 1 \otimes x + \cdots + x \otimes 1, \tag{A.2.25.1}$$

where the dots stand for an element of $A_1 \otimes A_{n-1} + \cdots + A_{n-1} \otimes A_1$. Indeed, we can always write $\delta(x) = 1 \otimes y + \cdots + z \otimes 1$. By cocommutativity $y = z$. Then, $x = (\varepsilon \otimes \mathrm{id})(\delta(x)) = y + 0 + \cdots + 0 = y$.

In particular, for any element $x \in A_1$ equation (A.2.25.1) ensures that $\delta(x) = 1 \otimes x + x \otimes 1$, so that $A_1 = P_1$. (It may happen that $A_1 = 0$, but this does not interfere the subsequent argument!)

Take an element $x \in A_2$. We have

$$\delta(x) = 1 \otimes x + \sum \lambda_{ij} p_i^1 \otimes p_j^1 + x \otimes 1,$$

where p_i^1 constitute a basis of $A_1 = P_1$ and λ_{ij} is a symmetric matrix over the ground field. Let

$$x' = \frac{1}{2} \sum \lambda_{ij} p_i^1 p_j^1.$$

Then

$$\delta(x') = 1 \otimes x' + \sum \lambda_{ij} p_i^1 \otimes p_j^1 + x' \otimes 1.$$

It follows that

$$\delta(x - x') = 1 \otimes x' + x' \otimes 1,$$

i. e. $x - x'$ is primitive, and x is expressed through primitive elements as $(x - x') + x'$, which is a polynomial, linear in P_2 and quadratic in P_1.

Proceeding in this way, assertion I can be proved in grading 3, 4, ... We omit the formal inductive argument.

The uniqueness of the polynomial representation in terms of basic primitive elements (assertion II) is a consequence of the following observation:

Denote by $R_n \subset A_n$ the subspace spanned by all nontrivial products of homogeneous elements of positive degree. Then there is a direct sum decomposition

$$A_n = P_n \oplus R_n.$$

This completes the proof.

!!! correct the last ugly argument!!!

□

A.2.26. Corollary. *An algebra A satisfying the assumptions of the theorem*

1. *has no zero divisors,*
2. *has the antipode S defined on primitive elements (see p. 423) by*

$$S(p) = -p \ .$$

!!! Add the structure theorem for Hopf algebras that are not commutative, but cocommutative (they are universal envelops of their primitive spaces). Same corollaries hold. Also, add an example of a non-cocommutative algebra where the theorem does not hold at all!!!

A.2.27. Group-like elements in Hopf algebras.

A.2.28. Lemma. *(D. Quillen's theorem* [**Q**]*) For a completed connected Hopf algebra over a field of characteristic 0 the functions* exp *and* log *(defined by the usual Taylor expansions) establish one-to-one mappings between the set of primitive elements $P(A)$ and the set of group-like elements $G(A)$.*

Proof. Let $p \in P(A)$. Then

$$\delta(p^n) = (1 \otimes p + p \otimes 1)^n = \sum_{k+l=n} \frac{n!}{k!l!} p^k \otimes p^l$$

and therefore

$$\delta(e^p) = \delta(\sum_{n=0}^{\infty} \frac{p^n}{n!}) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k!l!} p^k \otimes p^l = \sum_{k=0}^{\infty} \frac{1}{k!} p^k \otimes \sum_{l=0}^{\infty} \frac{1}{l!} p^l = \delta(e^p) \otimes \delta(e^p)$$

which means that $e^p \in G(A)$.

Vice versa, assuming that $g \in G(A)$ we want to prove that $\log(g) \in P(A)$. By assumption, our Hopf algebra is connected which implies that the graded component $g_0 \in A_0 \cong \mathbb{F}$ is equal to 1. Therefore we can write $g = 1+h$ where $h \in \sum_{k>0} A_k$. The condition that g is group-like transcribes as

$$\delta(h) = 1 \otimes h + h \otimes 1 + h \otimes h. \tag{A.2.28.1}$$

Now,

$$p = \log(g) = \log(1+h) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} h^k$$

and an exercise in power series combinatorics shows that equation A.2.28.1 implies the required property

$$\delta(p) = 1 \otimes p + p \otimes 1.$$

□

A.3. Free associative and free Lie algebras

This section contains a compendium of results on free associative and free Lie algebras important for the study of the Drinfeld associator (Chapter 10). We give almost no proofs referring the interested reader to special literature, e.g. [**Reu**].

A.3.1. Definition. The *free (associative) algebra* with n generators $\mathcal{F}(n)$ over a field $\mathbb{F}$ is the algebra of non-commutative polynomials in n variables.

For example, the algebra $\mathcal{F}(2) = \mathbb{F}\langle x_1, x_2\rangle$ consists of finite linear combinations of the form $c + c_1x_1 + c_2x_2 + c_{11}x_1^2 + c_{12}x_1x_2 + c_{21}x_2x_1 + c_{22}x_2^2 + \ldots$, $c_\alpha \in \mathbb{F}$, with natural addition and multiplication.

The meaning of the word free is that, in $\mathcal{F}(n)$, there are no relations between the generators and thus the only identities that hold in $\mathcal{F}(n)$ are those that follow from the axioms of an algebra, for example, $(x_1 + x_2)^2 = x_1^2 + x_1x_2 + x_2x_1 + x_2^2$. An abstract way to define the free algebra uniquely up to isomorphism, is by means of the universal property:

> *if A is an arbitrary (associative, not necessarily commutative) algebra and $y_1, \ldots, y_n \in A$ a set of its elements, then there is a unique homomorphism $\mathcal{F}(n) \to A$ that takes each x_i into the corresponding y_i.*

We are interested in the relation between associative algebras and Lie algebras, and we begin with the simple observation that any associative algebra has the structure of a Lie algebra defined by the conventional rule $[a, b] = ab - ba$.

Definition. A commutator monomial of degree m and depth k in a free algebra $\mathcal{F}$ is a monomial in the generators $x_{i_1} \cdot \dots \cdot x_{i_m}$ with k pairs of brackets inserted in arbitrary position.

For example, $x_2[x_1, [x_1, x_3^2]]$ is a commutator monomial of degree 5 and depth 2. The maximal depth of a commutator monomial of degree m is $m-1$, and we will call monomials of maximal depth *full commutators*. It is easy to see that the linear span of full commutators is closed under the commutator, so it forms a Lie subalgebra $\mathcal{L}(n) \subset \mathcal{F}(n)$. Using the skew-symmetry and the Jacobi identity, one can rewrite any full commutator as a linear combination of *right-normalized* full commutators

$$[x_1 \cdot \dots \cdot x_m] := [x_1, [...[x_{m-1}, x_m]...]] \,.$$

The following table shows dimensions and indicates some bases of the homogeneous components of small degree for the algebra $\mathcal{L}(2)$:

m	$\dim \mathcal{L}(2)_m$	basis		
1	2	x, y		
2	1	$[x, y]$		
3	2	$[x, [x, y]]$	$[y, [x, y]]$	
4	3	$[x, [x, [x, [x, y]]]]$	$[y, [x, [x, [x, y]]]]$	$[y, [y, [x, [x, y]]]]$
5	6	$[x, [x, [x, [x, y]]]]$	$[y, [x, [x, [x, y]]]]$	$[y, [y, [x, [x, y]]]]$
		$[y, [y, [y, [x, y]]]]$	$[[x, y], [x, [x, y]]]$	$[[x, y], [y, [x, y]]]$

The free algebra is a graded algebra with the grading defined by the conventional degree of monomials: $\deg(x_{i_1} \cdot \dots \cdot x_{i_k}) = k$. The homogeneous component $\mathcal{F}(n)_k$ of degree k has dimension n^k, and the Poincaré series of $\mathcal{F}(n)$ is $\dim(\mathcal{F}(n)_k)t^k = 1/(1-nt)$.

Let A be an associative algebra or a Lie algebra. The (first) commutant of A, denoted by $A' = [A, A]$, is, by definition, the linear span of all commutators $[a, b]$ (in the associative case defined by the conventional rule $[a, b] = ab - ba$). There are two ways to iterate the operation $A \mapsto A'$:

- $A_{(1)} = A'$, $A_{(n+1)} = [A, A_{(n)}]$, leading to the decreasing filtration of A by subspaces $A_{(n)}$ called the *lower central series* of A,
- $A^{(1)} = A'$, $A^{(n+1)} = [A^{(n)}, A^{(n)}]$, leading to the filtration $A \supset A^{(1)} \supset A^{(2)} \supset \dots$ called the *derived series* of A.

Obviously, $A^{(n)} \subset A_{(n)}$ for any n, and this inclusion is in general strict if $n > 1$ (for example, this is so in the case of free algebras). The second subspace of the derived series $A^{(2)} = [[A, A], [A, A]]$ has a special name: it is called the *second commutant* of A.

In the case of the free associative algebra, the terms of both the derived series and the lower central series are homogeneous subspaces with respect to the grading.

A.3.2. Definition. The *completed* free algebra with n generators $\overline{\mathcal{F}}_n$ is the graded completion of $\mathcal{F}(n)$, i.e. the algebra of formal power series in n non-commuting variables, denoted also by $\mathbb{F}\langle\langle x_1, \dots, x_n\rangle\rangle$.

In the free algebra there is a coproduct $\delta : \mathcal{F}(n) \to \mathcal{F}(n) \otimes \mathcal{F}(n)$ defined on the generators by $\delta(x_i) = x_i \otimes x_i$ and then extended by linearity and multiplicativity to the entire $\mathcal{F}(n)$.

A.3.3. Theorem. *The primitive space $\mathcal{P}(\mathcal{F}(n))$ is equal to the linear span of all full commutators.*

There is a natural linear operator $\alpha : \mathcal{F}(n) \to \mathcal{F}(n)$ defined on the monomials by

$$\alpha(x_1 \cdot \dots \cdot x_k) = \frac{1}{k!}[x_1 \cdot \dots \cdot x_k],$$

where $[x_1 \cdot \dots \cdot x_k]$ denotes the full right-normalized commutator $[x_1, [...[x_{k-1}, x_k]...]]$.

A.3.4. Theorem. *The operator α is a projector onto the subspace $\mathcal{L}(n) \subset \mathcal{F}(n)$.*

—————————- (moved from chapter 9: brush up!!)

A.3.5. Free associative algebra and free Lie algebra. Let $\mathcal{C} = \mathbb{C}\langle\langle A, B\rangle\rangle$ be the algebra of formal power series in two non-commuting variables, i.e. the completed free associative algebra with two generators.

Coproduct in $\mathcal{C}$ — useful intuition.

If $\mathcal{C}$ is understood as the space of functions in two non-commuting variables A and B, then $\mathcal{C} \otimes \mathcal{C}$ can be be interpreted as the space of functions in four variables A_1, B_1, A_2, B_2 where each of A_1, B_1 commutes with each of A_2, B_2, but variables with the same subscript do not commute between themselves. In this setting, the coproduct has the following transparent meaning:

$$\delta(\varphi(A, B)) = \varphi(A_1 + A_2, B_1 + B_2).$$

In particular, the primitive elements of $\mathcal{C}$ are precisely the functions having the additivity property

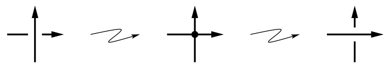
$$\varphi(A_1 + A_2, B_1 + B_2) = \varphi(A_1, B_1) + \varphi(A_2, B_2).$$

A.4. Discriminants and Vassiliev's spectral sequence

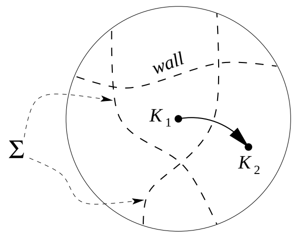
To be edited by JM; here follows the original fragment

A.4.1. Discriminant approach. Recall that a knot is a smooth embedding of the circle S^1 into 3-space $\mathbb{R}^3$. A smooth isotopy in the class of embeddings does not change the equivalence class of a knot. To change the equivalence class, one must make one or several crossing changes.

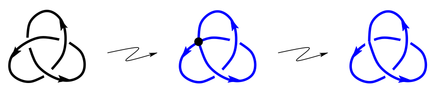
Doing a crossing change continuously, one must pass through a map from S^1 to $\mathbb{R}^3$ which is not an embedding, but an immersion $S^1 \to \mathbb{R}^3$ with a double point:



At this moment, it is quite instructive to imagine simultaneously the transformations of a closed curve in 3-space (the *physical space* in the terminology of Arnold [**Ar3**]) and corresponding paths in the space of maps from S^1 to $\mathbb{R}^3$ (the *functional space*). Here is a typical picture:



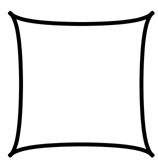
Path in the functional space



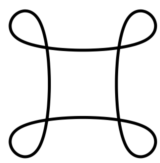
Isotopy of a knot

The disk on the left is the space *Imm* of immersions $S^1 \to \mathbb{R}^3$, the dashed lines are the *discriminant* Σ consisting of immersions which are not embeddings, and the connected components of the complement $Emb = Imm \setminus \Sigma$ correspond to equivalence classes of knots.

Remark. In the original Vassiliev's approach, he considered the space of *all* smooth maps $S^1 \to \mathbb{R}^3$ with both local and non-local singularities allowed:



Local singularities



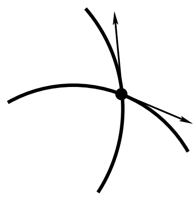
Non-local singularities

This is important to develop the general theory: the space of all maps is linear and hence contractible, therefore one can apply Alexander duality etc. (see details in [**Va3**]). However, it is clear that, to transform any knot into any other, it is enough to pass only through immersions with one transversal self-intersection, so in this elementary introduction we can confine ourselves with the class of immersions.

Embeddings and immersions from S^1 to $\mathbb{R}^3$ constitute two infinite-dimensional manifolds, *Emb* $\subset$ *Imm*. While the entire space *Imm* is connected, the subspace *Emb* is not, and its connected components are nothing but the topological types of knots. The type can only change when the point in the space *Imm* passes through a wall separating two connected components.

A knot invariant is the same thing as a function constant on the connected components of the space *Emb*. When a point in *Imm* representing a knot, goes through a wall, the invariant experiences a *jump*. Vassiliev's original idea was to prolong knot invariants from *Emb* to Σ assigning to a point $p \in \Sigma$ the value of the corresponding jump when one passes from the negative side to the negative side of Σ. This construction only makes sense for *generic* points of the discriminant which correspond to knots with one *simple double point*, and it relies on the fact that at these generic points the discriminant is naturally cooriented.

Definition. A point $p \in \mathrm{im}(f) \subset \mathbb{R}^3$ is a *simple double point* of f if its preimage under f consists of two values t_1 and t_2 and the two tangent vectors $f'(t_1)$ and $f'(t_2)$ are non-collinear. Geometrically, this means that, in a neighborhood of the point p, the curve consists of two branches with different tangents.

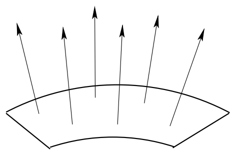


Simple double point

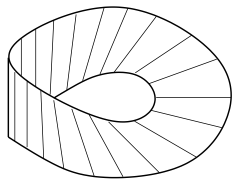
Smooth singular maps having one simple double point form a submanifold of codimension 1 in the space of all immersions, while for other types of singularities the codimension is greater. They form the principal stratum of the discriminant Σ.

Definition. A hypersurface in a real manifold is said to be *coorientable* if is has a non-zero section of its normal bundle, i.e. if there exists a continuous vector field which is not tangent to the hypersurface at any point and does not vanish anywhere. The only thing we want to know of this vector feld is the side of the hypersurface it points to. The *coorientation* of a coorientable hypersurface is a choice of one of the two such possibilities.

The figure below on the left illustrates this notion. The figure on the right shows the simplest example of a hypersurface that has no coorientation — the Möbius band in $\mathbb{R}^3$.



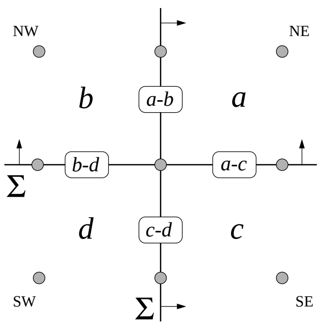
Coorientation

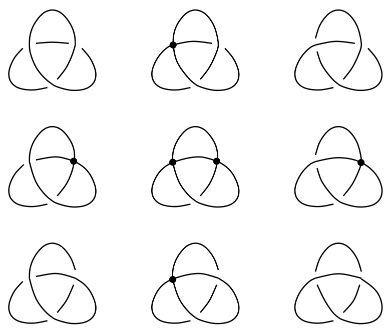


Möbius band

A crucial observation is that the (open) hypersurface in *Imm* consisting of all knots with exactly one simple double point is coorientable. A small shift off the discriminant is said to be positive if the local writhe at the point in question is $+1$, and negative, if it is -1 (see page 22). Each of these resolutions is well defined (does not depend on the plane projection used to express this relation). To make sure of this fact, the reader is invited to make a physical model of a crossing with the help of two sharpened (i.e. oriented) pencils and look at them from one side, then from the other.

A.4.2. While an arbitrary path from one knot to another can be effectuated by a path that goes only through the generic points of the discriminant, it turns out to be very important to understand what happens near the points where the discriminant has self-intersections. The simplest self-intersection occurs at the points where the corresponding singular knot has two simple double points:



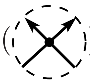
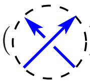
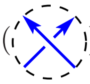


The figure on the left shows the neighborhood of a self-intersection in the functional space; Σ is the dicriminant, arrows show its coorientation, the fat grey points are typical points of the corresponding regions, and the letters a, b, c, d, as well as their differences are the values of some knot invariant in these regions. The figure on the right displays sample knots close to a knot with two self-intersections in accordance with the left-hand picture. Note that among the four non-singular knots which are present, one (NE) is a trefoil, while the other three (NW, SW, SE) are trivial knots and therefore belong to the same global component of the space $Imm \setminus Emb$: the local connected components shown in the figure are joined together somewhere far away.

We see that the difference of values of an invariant along one component of the discriminant $(a-b)-(c-d)$ is equal to the difference of its values on another component $(a-c)-(b-d)$. Therefore the invariant can be prolonged to the central point of the picture by the value $a-b-c+d$.

This observation is valid for knots with any number of distinct simple double points. Let $\mathcal{K}_n$ denote the set of equivalence classes of singular knots with n double points and no other singularities. By definition, $\mathcal{K}_0 = \mathcal{K}$ is the set of all non-singular knots. Put $\widetilde{\mathcal{K}} = \cup_{n\geqslant 0}\mathcal{K}_n$.

Definition. (Vassiliev's extension of knot invariants.) Let $\mathbb{G}$ be an Abelian group. Given a knot invariant $v : \mathcal{K} \to \mathbb{G}$, we define its extension $\widetilde{\mathcal{K}} \to \mathbb{G}$, denoted by the same letter v, according to the rule

(A.4.2.1) $v($  $) = v($  $) - v($  $),$

known as *Vassiliev's skein relation.*

The right hand side of Vassiliev's skein relation refers to the two resolutions of the double point — positive and negative, explained above. Let us stress that this definition does not appeal to knot diagrams, but directly to genuine knots embedded in $\mathbb{R}^3$.

It does not take long to understand that Vassiliev's extension from $\mathcal{K}_{n-1}$ to $\mathcal{K}_n$ is well defined, i.e. does not depend on the choice of a double point to resolve. Indeed, the calculation of $f(K)$, $K \in \mathcal{K}_n$, is in any case reduced to the complete resolution of the knot K which yields an alternating sum

(A.4.2.2)
$$v(K) = \sum_{\varepsilon_1=\pm 1} \cdots \sum_{\varepsilon_n=\pm 1} (-1)^{|\varepsilon|} v(K_{\varepsilon_1,\dots,\varepsilon_n}),$$

where $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$, $|\varepsilon|$ is the number of -1's in the sequence ε_1, ..., ε_n, and $K_{\varepsilon_1,\dots,\varepsilon_n}$ is the knot obtained from K by a positive or negative resolution of the double points according to the sign of ε_i for the point number i. The geometrical background of this phenomenon is that, in the vicinity of the singular knot K, the pair

$$(\text{space of immersions } S^1 \to \mathbb{R}^3,\ \text{discriminant})$$

is diffeomorphic to the pair

$$(\mathbb{R}^n,\ \text{union of coordinate hyperplanes})$$

multiplied by a vector subspace of codimension n, as shown above in a picture corresponding to the case $n = 2$. A locally constant function v defined in the complement of the coordinate cross in $\mathbb{R}^n$ is extended to the origin by the rule (3.1.2.2) and this number is equal to the difference of the two similar combinations of order $n-1$ corresponding to the two points on any coordinate axis in $\mathbb{R}^n$ lying on either side of the origin.

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Notations

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ — rings of integer, rational, real and complex numbers.

$\mathcal{A}$ — algebra of unframed chord diagrams on the circle, p.109.

$\mathcal{A}^{fr}$ — algebra of framed chord diagrams on the circle, p.106.

$\mathcal{A}_n$ — space of unframed chord diagrams of degree n, p.105.

$\mathcal{A}_n^{fr}$ — space of framed chord diagrams of degree n, p.105.

$\mathcal{A}(n)$ — algebra of chord diagrams on n lines, p.162.

$\mathcal{A}^h(n)$ — algebra of horizontal chord diagrams, p.160.

$\widehat{\mathcal{A}}$ — graded completion of the algebra of chord diagrams, p.226.

$\mathbf{A}_n$ — set of chord diagrams of degree n, p.79.

A — Alexander-Conway power series invariant, p.318.

α_n — map from $\mathcal{V}_n$ to $\mathcal{R}\mathbf{A}_n$, symbol of an invariant, p.80.

$\mathcal{B}$ — algebra of open Jacobi diagrams, p.142.

$\mathcal{B}(m)$ — space of m-colored open Jacobi diagrams, p.156.

$\mathcal{B}^\circ$ — enlarged algebra $\mathcal{B}$, p.324.

$\mathbf{B}_n$ — set of open Jacobi diagrams of degree n, p.142.

BNG — the Bar-Natan–Garoufalidis function, p.393.

$\mathcal{C}$ — space of closed diagrams, p.128.

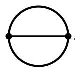
$\mathcal{C}_n$ — space of closed diagrams of degree n, p.128.

$\mathcal{C}_n$ — Goussarov–Habiro moves, p.413.

$\mathcal{C}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \,|\, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_m)$ — space of mixed Jacobi diagrams, p.156.

C — Conway polynomial, p.45.

$\mathfrak{C}_{2n}$ — Conway combination of Gauss diagrams, p.378.
$\mathbf{C}_n$ — set of closed diagrams of degree n, p.135.
c_n — n-th coefficient of the Conway polynomial, p.46.
$\mathbb{C}_n$ — n-th disconnected cabling of a knot, p.251.
$\hat{\mathbb{C}}_n$ — n-th connected cabling of a knot, p.251.
∂_C — diagrammatic differential operator on $\mathcal{B}$, p.320.
∂_C° — diagrammatic differential operator on $\mathcal{B}^\circ$, p.325.
∂_Ω — wheeling map, p.320.
δ — coproduct in $\mathcal{A}^{fr}$, p.108.
ε — counit in $\mathcal{A}^{fr}$, p.109.
$F(L)$ — unframed two-variable Kauffman polynomial, p.59.
$\mathcal{G}_n$ — Goussarov group, p.401.
$\mathfrak{G}$ — bialgebra of graphs, p.402.
Γ — algebra of 3-graphs, p.199.
$\Gamma(D)$ — intersection graph of a chord diagram D, p.116.
$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ — element of the Lie algebra $\mathfrak{sl}_2$, p.170.
H — hump unknot, p.229.
$\mathbf{I}_n$ — constant 1 weight system on $\mathcal{A}_n$, p.112.
I — a map of Gauss diagrams to arrow diagrams, p.363.
$I(K)$ — final Kontsevich integral, p.229.
$\mathcal{I}$ — algebra of knot invariants, p.49.
ι — unit in $\mathcal{A}^{fr}$, p.109.
j_n — n-th coefficient of the modified Jones polynomial, p.83.
$\mathcal{K}$ — set of knots, p.27.
Li_2 — Euler dilogarithm, p.247.
$\mathcal{L}$ — bialgebra Lando, p.405.
Λ — Vogel's algebra, p.210.
$\Lambda(L)$ — framed two-variable Kauffman polynomial, p.59.
∇ — difference operator for Vassiliev invariants, p.75.
$\mathcal{M}_n$ — Goussarov–Habiro moves, p.395.
MM — highest order part of the colored Jones polynomial, p.387.
M_T — mutation of a knot with respect to a tangle T, p.251.
μ — product in $\mathcal{A}^{fr}$, p.106.
$\mathcal{P}$ — Polyak algebra, p.374.

$\mathcal{P}_n$ — primitive subspace of the algebra of chord diagrams, p.113.
P — HOMFLY polynomial, p.57.
P^{fr} — framed HOMFLY polynomial, p.70.
$p_{k,l}(L)$ — k, l-th coefficient of the modified HOMFLY polynomial, , p.95.
ψ_n — n-th cabling of a knot, p.262.
$\mathcal{R}$ — ground ring (usually $\mathbb{Q}$ or $\mathbb{C}$), p.73.
$\mathcal{R}(\mathbf{A}_n)$ — $\mathcal{R}$-valued functions on chord diagrams, p.80.
R — R-matrix, p.51.
R, R^{-1} — Kontsevich integrals of two braided strings, p.232.
S_A — symbol of the Alexander-Conway invariant A, p.388.
S_{MM} — symbol of the Melvin-Morton invariant MM, p.388.
S_i — operation on tangle (chord) diagrams, p.249.
$\mathrm{symb}(v)$ — symbol of the Vasiliev invariant v, p.81.
σ — mirror reflection of knots, p.23.
τ — changing the orientation of a knot, p.23.
τ — inverse of $\chi : \mathcal{B} \to \mathcal{C}$, p.146.
Θ — the chord diagram with one chord, , p.109.
θ^{fr} —- quantum invariant, p.52.
θ^{fr} —- $\mathfrak{sl}_2$-quantum invariant, p.55.
$\theta^{fr,St}_{\mathfrak{sl}_N}$—- $\mathfrak{sl}_N$-quantum invariant, p.68.
$\mathcal{V}$ — space of Vassiliev (finite type) invariants, p.73
$\mathcal{V}_n$ — space of unframed Vassiliev knot invariants of degree $\leqslant n$, p.73.
$\mathcal{V}^{fr}_n$ — space of framed Vassiliev knot invariants of degree $\leqslant n$, p.81.
$\mathcal{V}_\bullet$ — space of polynomial Vassiliev invariants, p.77.
$\widehat{\mathcal{V}}_\bullet$ — space of power series invariants, graded completion of $\mathcal{V}_\bullet$, p.77.
$\mathcal{W}_n$ — space of unframed weight systems of degree n, p.100.
$\mathcal{W}^{fr}_n$ — space of framed weight systems of degree n, p.100.
$\widehat{\mathcal{W}}^{fr}$ — graded completion of the algebra of weight systems, p.111.
$\mathcal{Z}(\phi)$ — Kontsevich integral of ϕ in algebra $\mathcal{B}(\boldsymbol{y})$, p.321.
$\mathcal{Z}(\Phi)$ — Kontsevich integral of Φ in algebra $\mathcal{B}(\boldsymbol{y}_1, \boldsymbol{y}_2)$, p.322.
$\mathcal{Z}_i(\phi)$ — i-th part of the Kontsevich integral $\mathcal{Z}(\phi)$, p.323.
$Z(K)$ — Kontsevich integral, p.226.
$\mathbb{Z}\mathcal{K}$ — algebra of knots, p.27.

Δ_n

$\rho_{\mathfrak{g}}$, $\rho_{\mathfrak{g}}^V$

$\phi_{\mathfrak{g}}$, $\phi_{\mathfrak{g}}^V$

χ — symmetrization map $\mathcal{B} \to \mathcal{C}$, p.146.

$\chi_{\boldsymbol{y}_m}$ — map $\mathcal{C}(\boldsymbol{X} \,|\, \boldsymbol{y}_1, \dots, \boldsymbol{y}_m) \to \mathcal{C}(\boldsymbol{X}, \boldsymbol{y}_m \,|\, \boldsymbol{y}_1, \dots, \boldsymbol{y}_{m-1})$, p.156.

Φ — map $\mathcal{B}(\boldsymbol{y}) \to \mathcal{C}(\boldsymbol{x})$, p.321.

Φ_0 — map $\mathcal{B} \to \mathcal{C}$, p.324.

Φ_2 — map $\mathcal{B}(\boldsymbol{y}_1, \boldsymbol{y}_2) \to \mathcal{C}(\boldsymbol{x})$, p.322.

Ω' — part of $\mathcal{Z}_0($$)$ containing wheels, p.324.

$\langle\ ,\ \rangle_{\boldsymbol{y}}$ — pairing $\mathcal{C}(\boldsymbol{x} \,|\, \boldsymbol{y}) \otimes \mathcal{B}(\boldsymbol{y}) \to \mathcal{C}(\boldsymbol{x})$, p.158.

 — open Hopf link, p.321.

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